SHIMURA VARIETIES AND TAF

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1. Introduction

The primary source is chapter 6 of [**?**].

We've spent a long time learning generalities about abelian varieties. In this talk (or two), we'll assemble this knowledge into the spectrum TAF. To do this, we have to first construct a stack on which to apply Lurie's theorem, which is some sort of PEL Shimura variety $\mathcal{X}$. To prove representability, we'll actually end up constructing several different stacks. Then we'll need to check that the map $\mathcal{X} \to \mathcal{M}_p(n)$ is formally étale, which is to say that deformations of whatever objects $\mathcal{X}$ parametrizes are controlled by the deformations of the underlying p-divisible groups. We begin by discussing level structures, in a very linear-algebraic sort of setting.

2. Level structures

The Drinfel'd level structures used in [**?**] to study elliptic curves are essentially bases for finite subgroups of those elliptic curves. To make our lives easier, we'll just consider bases for the whole Tate module. Recall that this is defined as

$$T_\ell(A) = \varprojlim A[\ell^i].$$

The choice of ℓ being mostly irrelevant, we might as well consider

$$T^p(A) = \prod_{\ell \neq p} T_\ell(A).$$

We also have the vector spaces

$$V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}; \qquad V^p(A) = V^p(A) \otimes \mathbb{Q}.$$

Remark 2.1. This last object $V^p(A)$ is really a module over the "adèles away from p and ∞." Since we're homotopy theorists, I'm going to gloss over this: it's essentially just $\prod_{\ell \neq p} \mathbb{Q}_\ell$, with a certain topology making it a locally compact topological ring, which will sort of come into play later on.

Abstractly, we know what all these things are:

$$T^p(A) \cong L^p := \prod_{\ell \neq p} \mathbb{Z}_\ell^{2g}$$

and

$$V^p(A) \cong V^p := L^p \otimes \mathbb{Q}.$$

Let $K^p \subseteq \mathrm{Aut}(V^p)$. A good choice is K_0^p, the subgroup of automorphisms that preserve the lattice L^p.

Definition 2.2. An **integral** (resp. **rational**) **uniformization** of A is a choice of isomorphism above. A **integral** (resp. **rational**) K^p**-level structure** is an integral (resp. rational) uniformization, up to the action of K^p.

3. The ground data

We now fix a quadratic imaginary number field $F = \mathbb{Q}(\sqrt{-d})$ and a central simple F-algebra B, of dimension g^2, with a positive definite involution $*$ that restricts to complex conjugation on F. We'll assume that p splits as $u\overline{u}$ in F, and totally splits in B. F has its ring of integers $\mathcal{O}_F$; we'll also fix a maximal order $\mathcal{O}_B$ in B such that $\mathcal{O}_{B,(p)}$ is preserved by the involution $*$. This is the endomorphism data – that is, we're going to look at abelian varieties with an action of $\mathcal{O}_B$ (or at least $\mathcal{O}_{B,(p)}$).

We also fix an skew-Hermitian (for $*$) form

$$\langle \cdot, \cdot \rangle : B \times B \to \mathbb{Q}.$$

The maximality of the order $\mathcal{O}_B$ means that this pairing must send it to $\mathbb{Z}$. Some simple linear algebra, here obfuscated by the notation, gives the following.

Lemma 3.1. *There exists a unique $\beta \in B$ such that $\beta^* = -\beta$ and $\langle x, y \rangle = \mathrm{Tr}_{B/\mathbb{Q}}(x\beta y^*)$.*

Now, a polarization $\lambda : A \to A^\vee$ on an abelian variety gives a nondegenerate, skew-symmetric **Weil pairing**

$$e^\lambda : T^pA \times T^pA \overset{1\times T^p\lambda}{\to} T^pA \times T^pA^\vee \to \mathbb{G}_m.$$

This extends to V^pA in the obvious way. There's also a **Rosati involution** on $\mathrm{End}^0(A)$ given by

$$f^\dagger = \lambda^{-1} f^\vee \lambda.$$

The endomorphisms of A act on V^pA, and one can easily observe that the Rosati involution is the adjunction for the Weil pairing:

$$e^\lambda(x, f^\dagger y) = e^\lambda(fx, y).$$

With the additional endomorphism structure described above, we'll want our polarizations to be **compatible** with B in the sense that the Rosati involution restricts to the involution $*$ of B. This implies that the Weil pairing is skew-Hermitian for $*$. As part of the compatibility conditions, we'll require that the Weil pairing restricts to the given pairing on B.

Lemma 3.2. *Given $(B, *)$ as above, any B-linear abelian variety A admits a compatible polarization.*

4. Unitary and similitude transformations

Once we have a Hermitian form, we can talk about unitary transformations, which is to say the ones that preserve the form. In magical scheme land, it makes more sense to define this as a group scheme than just a group. For a $\mathbb{Q}$-algebra R, we define

$$U(R) = \{f \in (B \otimes_\mathbb{Q} R)^\times : f^* f = 1\},$$

which is equivalent to saying that f preserves the skew-Hermitian form on B. More generally, if V is a free left B-module admitting an involution compatible with $*$ ($V_\ell(A)$ being the obvious example), we define

$$U_V(R) = \{f \in (\mathrm{End}_B(V) \otimes_\mathbb{Q} R)^\times : f^* f = 1\}.$$

When dealing with polarizations, it helps to be able to scale them as well. Thus, we define the **similitude group** as

$$GU_V(R) = \{f \in (\mathrm{End}_B(V) \otimes_\mathbb{Q} R)^\times : f^* f \in R^\times\}.$$

That is to say, f only scales the skew-Hermitian form.

Finally, moving back from linear algebra and Tate modules to abelian varieties themselves, we define

$$GU_{(A,i,\lambda)}(R) = \{f \in (\mathrm{End}^0_B(A) \otimes_\mathbb{Q} R)^\times : f^\dagger f \in R^\times\}$$

where $\dagger$ is the λ-Rosati involution. We'll typically consider polarizations up to similitude. To make this perfectly clear, λ and λ' are **similar** if there are $\alpha \in \mathrm{End}^0_B(A)$ and $c \in \mathbb{Q}^\times$ such that

$$\lambda = c\alpha^*(\lambda') = c\alpha^\vee \lambda \alpha.$$

Remark 4.1. Behrens and Lawson quote some results from [1] parametrizing similitude classes of polarizations by Galois cohomology groups, which I am having a really hard time understanding, perhaps because I gave up on Hilbert around Theorem 89. If you'd like to talk about this, let me know.

As a final piece of our list of standing assumptions, note that $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$ and $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_g(\mathbb{C})$. Fixing one of the two complex embeddings of F, we'll require that this isomorphism makes

$$\beta = \begin{pmatrix} e_1 i & 0 & \dots & 0 \\ 0 & -e_2 i & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -e_g i \end{pmatrix}$$

with $e_i > 0$. In pithier terms, $U(\mathbb{R}) \cong U(1, g-1)$.

If we instead complete at the prime u of F lying over p, we get $\mathcal{O}_{B,u} \cong M_n(\mathcal{O}_{F,u}) \cong M_n(\mathbb{Z}_p)$, by the assumption that p totally splits in B. Define

$$\epsilon = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

This is an idempotent in $\mathcal{O}_{B,p}$ that will play the same role that u did.

5. The moduli problems

Fix a compact open subgroup $K^p \subseteq K_0^p \subseteq \mathrm{Aut}(V^p)$. We'll define two functors from formal $\mathbb{Z}_p$-schemes to groupoids. It suffices to define them on connected locally noetherian schemes S on which p is locally nilpotent, and take colimits.

First, $\mathcal{X}_{K^p}(S)$ is defined to be the groupoid of triples (A, i, λ) where

- A a g^2-dimensional abelian scheme over S,
- $i : \mathcal{O}_{B,(p)} \to \mathrm{End}(A)_{(p)}$ is an inclusion such that $\epsilon A(u)$ is a 1-dimensional p-divisible $\mathcal{O}_F$-module,
- $\lambda \in \mathrm{Hom}(A, A^\vee) \otimes \mathbb{Z}_{(p)}$ is a polarization compatible with the involution $*$,
- and for every geometric point $s \in S$, there exists an $\mathcal{O}_B$-linear integral K^p-level structure on $T^p(A_s)$ inducing e^λ, up to similitude. A choice of level structure is *not* part of the data. (It suffices to check this condition at just one geometric point, by some Galois cohomology computations I omitted.)

An isomorphism in this groupoid is a B-linear isomorphism of abelian schemes over S that induces a similitude on their polarizations (scaling by some element of $\mathbb{Z}_{(p)}^\times$).

Second, $\mathcal{X}'(S)$ is defined to be the groupoid of quadruples $(A, i, \lambda, [\eta])$ where

- A, i, λ are as above,
- and for some (and thus any) geometric point s, $[\eta]$ is a $\pi_1(S, s)$-invariant *rational $\mathcal{O}_{B,(p)}$-linear* K^p-level structure on $V^p(A_s)$ that induces e^λ, up to similitude.

An isomorphism in this groupoid is a B-linear *isogeny* of abelian schemes that preserves the level structure and inducing a $\mathbb{Z}_{(p)}^\times$-similitude on their polarizations.

Any object in $\mathcal{X}(S)$ can be given an integral uniformization, by hypothesis, and this is automatically $\pi_1(S)$-invariant; the similitude requirement uniquely determines this, up to an action by K^p; tensoring with $\mathbb{Q}$ gives a rational uniformization, whose K^p-orbit is well-defined; so we get a level structure on (A, i, λ). This whole process is a natural transformation $F : \mathcal{X} \to \mathcal{X}'$.

Theorem 5.1 (Equivalence theorem)**.** *The functor F is a natural equivalence.*

The beauty of this is that, while $\mathcal{X}$ is arguably the moduli problem we're interested in, integral level structures only make sense for $K^p \subseteq K_0^p$. But rational level structures can be defined with respect to any compact open subgroup. Moreover, as we're about to see, being able to vary the subgroup makes it easy to prove that these moduli problems are representable by Deligne-Mumford stacks.

Proof. I'm going to restrict to the case where $K^p = K_0^p$, but the general case is no harder. Suppose we're given an object $(A, i, \lambda, [\eta]) \in \mathcal{X}'(S)$, and choose a representative uniformization η for A_s. The image of $L^p \subseteq V^p$ under η is some lattice L_s in $V^p(A_s)$, with the property that it contains some integral multiple of $T^p(A_s)$. Such a lattice is called a **prime-to-p virtual subgroup of A_s**, since if it actually contains $T^p(A_s)$, it will correspond to some finite prime-to-p subgroup via the map

$$V^p(A_s) \twoheadrightarrow V^p(A_s)/T^p(A_s) \cong \{\text{prime-to-}p \text{ torsion of } A_s\}.$$

An honest subgroup is the kernel of a unique isogeny, and likewise virtual subgroups are the right notions of kernels for $\mathbb{Z}_{(p)}$-isogenies. In particular, the virtual subgroup L_s here induces a $\mathbb{Z}_{(p)}$-isogeny $A_s \to A'_s$ sending L_s to $T^p(A'_s)$. In particular, the rational uniformization η passes to an *integral* uniformization η' of A'_s. One likewise observes that A'_s carries an induced polarization and complex mutiplication by $\mathcal{O}_{B,(p)}$. We lastly need to check that the integral uniformization η' is $\mathcal{O}_B$-linear, which is equivalent to saying that we have a lifting

$$\begin{array}{ccc} \mathcal{O}_B & \dashrightarrow & \mathrm{End}(A'_s) \\ \downarrow & & \downarrow \\ \mathcal{O}_{B,(p)} & \longrightarrow & \mathrm{End}(A'_s)_{(p)}. \end{array}$$

But $\mathcal{O}_B$ maps into both $\mathrm{End}(T^pA'_s)$ and $\mathrm{End}(A'_s)_{(p)}$, whose intersection is precisely $\mathrm{End}(A'_s)$. Everything we've done is $\pi_1(S, s)$-invariant, so we can indeed lift the object $(A, i, \lambda, [\eta])$ to an object $(A, i, \lambda) \in \mathcal{X}(S)$, at least up to isogeny in $\mathcal{X}'(S)$.

This proves that F_S is essentially surjective. Faithfulness is obvious. For fullness, we're given a map $f : (A, \lambda, i, [\eta]) \to (A', \lambda', i', [\eta'])$ in $\mathcal{X}'(S)$, and we can assume that η and η' lift to integral level structures. This means that the induced isomorphism $f_* : V^p(A) \to V^p(A')$ lifts to an isomorphism of Tate modules. Thus the map $f : A \to A'$ is a prime-to-p quasi-isogeny that induces an isomorphism of Tate modules, so it's an isomorphism. ☐

6. The representability theorem

Theorem 6.1 (Representability theorem)**.**

1. *$\mathcal{X}'_{K^p}$ is representable by a Deligne-Mumford stack $Sh(K^p)$ over $\mathbb{Z}_p$.*
2. *For K^p sufficiently small, $Sh(K^p)$ is a quasi-projective scheme (and projective if B is a division algebra).*
3. *For $K'^p \subseteq K^p$ of finite index, the map*

$$f_{K'^p,K^p} : Sh(K'^p) \to Sh(K^p)$$

given on points by further quotienting the set of level structures by K^p is étale of degree $[K^p : K'^p]$.

Sketch of proof. Obviously, (2) and (3) (and checking some sheaf conditions will imply the hard part of (1): $Sh(K^p)$ has an étale cover by a scheme, so it's automatically a DM stack. We thus start by proving (2). Forgetting the complex multiplication gives a functor $G : \mathcal{X}'_{K^p} \to \mathcal{Y}_{K^p}$, the latter being the moduli of polarized abelian varieties with K^p-level structure. By the 'classical' Drinfel'dian theory of level structures, which we only briefly covered and might want to visit later in more detail, $\mathcal{Y}_{K^p}$ is representable by a quasi-projective scheme for sufficiently small K^p. This can be found in [3] though I haven't had time to look.

In any case, (2) reduces to showing that the functor G is representable, which is to say that there are 'a scheme's worth' of ways to attach $\mathcal{O}_{B,(p)}$-complex multiplication to a polarized abelian variety with level structure. The fiber over a point $(A, \lambda, [\eta]) \in Y_{K^p}(S)$ will be the groupoid of maps $i : \mathcal{O}_{B,(p)} \to \mathrm{End}_S(A)_{(p)}$ for which the polarization and level structure are B-linear. For K^p small, the endomorphism structure will be determined by how it interacts with the level structure, so that this is equivalent to a set. An element of this set is given by a section over S of the sheaf

$$T \mapsto \{i : \mathcal{O}_{B,(p)} \times_S T \to \mathrm{End}(A_T)_{(p)}.$$

But B splits completely at p, so for small T this will split as a disjoint union of the sheaves $\mathrm{End}(A_T)_{(p)}$; Grothendieck's theory of Hilbert schemes then tells us that these are in fact projective schemes over S.

To prove that this is projective when B is a division algebra, one essentially uses the valuative criterion of properness and the theory of Néron models, which allows one to push forward abelian varieties from fraction fields of DVRs to the DVRs themselves. One then has to show that PEL structure all pushes forward as well, which requires some cohomology computations. See [2].

For (3), when $K'^p \subseteq K^p$ is a normal subgroup, it's fairly clear that the map $f_{K'^p,K^p}$ is Galois with Galois group K^p/K'^p. In the general case, one could content oneself with this observation, or think about deformation theory: the level structures are only defined at the geometric points, so it stands to reason that an infinitesimal thickening of an abelian variety would introduce neither new K^p-level structures nor new K'^p-level structures. Moreover, a K^p-level structure can only be induced by a finite number of K'^p-level structures, which are of course parametrized by the cosets K^p/K'^p. □

References

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