# THE CONSTRUCTION OF TAF

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## 1. Introduction

The primary source for all of this is [1], chapters 7 and 8.

Last talk, we defined a family of Deligne-Mumford stacks $Sh(K^p)$ over $\mathbb{Z}_p$, which we called Shimura varieties. We were only able to define them with considerable time and notation, but the key points to be retained are as follows.

- The stacks parametrize $n^2$-dimensional abelian varieties with some additional structure on their endomorphism rings and a $\mathbb{Z}_{(p)}$-polarization, which can carry an integral $K^p$-level structure.
- There's a dependence on a compact open subgroup $K^p \subseteq GU_{2n^2}(\mathbb{A}^{p,\infty})$, where $\mathbb{A}^{p,\infty} = \mathbb{Q} \otimes \prod_{\ell \neq p} \mathbb{Z}_\ell$ is the adèles away from $p$ and $\infty$.
- Changing the subgroup $K^p$ changes the stack by an étale map. For small $K^p$, it's a smooth quasi-projective scheme.
- The endomorphism structure was designed in such a way as to pick a canonical 1-dimensional, height $n$ $p$-divisible group $\epsilon A(u)$, thus giving a map $Sh(K^p) \to \mathcal{M}_p(n)$.

In order to pull a spectrum out of this hat, we need to apply Lurie's realization theorem. We recall one version of the theorem.

**Theorem 1** (Lurie)**.** *Let $\mathcal{X}$ be a locally Noetherian separated Deligne-Mumford stack, complete over a local ring with perfect residue field, and let $\mathbb{G} : \mathcal{X} \to \mathcal{M}_p(n)$ be a formally étale map. Then there is a homotopy sheaf of weakly even periodic $E_\infty$-ring spectra $\mathcal{E}_\mathbb{G}$ on the étale site of $\mathcal{X}$, such that for any formal affine étale open* $\mathrm{Spf}(R) \to \mathcal{X}$, *we have $\pi_0 \mathcal{E}_\mathbb{G}(R) = R$, and the formal group law of $\mathcal{E}_\mathbb{G}(R)$ is precisely the formal part of $f^*\mathbb{G}$.*

Today, we'll apply this theorem to the map $Sh(K^p)^\wedge_p \to \mathcal{M}_p(n)$. The hypotheses on the stack are easy to check, given that it has an étale cover by a quasi-projective scheme. The hard part is showing that the map is formally étale. All this means is that deformations of abelian schemes with the PEL structure described above are the same as deformations of the underlying 1-dimensional $p$-divisible groups. More precisely:

**Proposition 2.** *Let $S$ be a scheme where $p$ is locally nilpotent, and let $j : S_0 \hookrightarrow S$ be a closed embedding with nilpotent ideal sheaf. Then the diagram of categories*

$$\begin{array}{ccc} \mathcal{X}'_{K^p}(S) & \longrightarrow & \mathcal{X}'_{K^p}(S_0) \\ \downarrow & & \downarrow \\ \mathcal{M}_p(n)(S) & \longrightarrow & \mathcal{M}_p(n)(S_0) \end{array}$$

*is a (2-)pullback diagram.*

Here $\mathcal{X}'_{K^p}$ is one of the models of $Sh(K^p)$ discussed last time, parametrizing abelian varieties with polarization, endomorphism structure, and rational $K^p$-level structure up to isogeny. $\mathcal{M}_p(n)$ is the moduli stack of 1-dimensional height $n$ $p$-divisible groups.

## 2. The Serre-Tate theorem

The key point of this statement is much older and interesting in its own right. This is the Serre-Tate theorem, which says that deformations of abelian varieties (forgetting the extra structure) are the same as deformations of their underlying $p$-divisible groups.

**Theorem 3** (Serre-Tate). *Let $S$ be a scheme where $p$ is locally nilpotent, and let $j : S_0 \hookrightarrow S$ be a closed embedding with nilpotent ideal sheaf. Then the diagram of categories*

$$\begin{array}{ccc} \{\textit{abelian schemes over } S\} & \longrightarrow & \{\textit{abelian schemes over } S_0\} \\ \downarrow & & \downarrow \\ \{p\textit{-divisible groups over } S\} & \longrightarrow & \{p\textit{-divisible groups over } S_0\} \end{array}$$

*is a (2-)pullback diagram.*

The following neat proof, due to Drinfeld and given in [2], relies on the formal relationship between abelian varieties and $p$-divisible groups in the more general setting of abelian sheaves for the fppf topology. Since the objects under consideration all satisfy Zariski descent, we can immediately reduce to the case where the ideal sheaf $\mathcal{I}$ is honestly nilpotent. For definiteness, we say that $p^N = 0$ on $S$ and that $\mathcal{I}^{k+1} = 0$.

**Lemma 4.** *Under the hypotheses above, let $\mathbb{G}$ and $\mathbb{H}$ be (fppf) abelian sheaves over $S$ that are $p$-divisible, formally smooth, and whose nilradicals are (fppf) locally representable by a formal group. Then the groups $\operatorname{Hom}(\mathbb{G}, \mathbb{H})$ and $\operatorname{Hom}(\mathbb{G}_0, \mathbb{H}_0)$ are $p$-torsion free, and the map*

$$\operatorname{Hom}(\mathbb{G}, \mathbb{H}) \to i_* i^* \operatorname{Hom}(\mathbb{G}, \mathbb{H}) = \operatorname{Hom}(\mathbb{G}_0, \mathbb{H}_0)$$

*is injective with $p^{Nk}$-torsion cokernel.*

*Proof.* The first claim follows from the $p$-divisibility of the groups involved. To understand the second, let

$$\mathbb{H}_{\mathcal{I}} = \ker(\mathbb{G} \to i_* i^* \mathbb{G}),$$

a subgroup scheme of $\mathbb{H}$. I claim that $\mathbb{H}_{\mathcal{I}}$ is $p^{Nk}$-torsion. Since $\mathcal{I}$ is nilpotent, $\mathbb{H}_{\mathcal{I}}$ is a subgroup of $\mathbb{H}_{\mathfrak{N}}$, where $\mathfrak{N}$ is the nilradical sheaf of $S$; but this is the nilradical of $\mathbb{H}$, which by hypothesis, is locally a formal group, of the form $\mathbf{Spf}\mathcal{O}_S[[X_1, \ldots, X_n]]$. In particular, for $T$ an $S$-scheme, we can locally represent $T$-points of $\mathbb{H}_{\mathcal{I}}$ as tuples $(x_1, \ldots, x_n) \in (\mathcal{I}\mathcal{O}_T)^n$. We then have

$$[p^N](x_1, \ldots, x_n)_i = p^N x_i + (\text{terms of degree } \geq 2) = (\text{terms of degree } \geq 2),$$

since $p^N = 0$ on $S$ and thus on $T$. Thus, $[p^{Nk}](\mathbb{H}_{\mathcal{I}}) \subseteq \mathbb{H}_{\mathcal{I}^2}$, and by induction, $[p^{Nk}](\mathbb{G}_{\mathcal{I}}) \subseteq \mathbb{G}_{\mathcal{I}^{k+1}} = 0$.

Through the remainder of this section, we let $M = p^{Nk}$. The irrelevance of this number cannot be overstated.

Now, the kernel of $\operatorname{Hom}(\mathbb{G}, \mathbb{H}) \to i_* i^* \operatorname{Hom}(\mathbb{G}, \mathbb{H})$ is evidently $\operatorname{Hom}(\mathbb{G}, \mathbb{H}_{\mathcal{I}})$, but $\mathbb{G}$ is $p$-divisible while $\mathbb{H}_{\mathcal{I}}$ is $M$-torsion, so this group is zero. This establishes injectivity.

Finally, we must show that for any $f_0 : \mathbb{G}_0 \to \mathbb{H}_0$, there is a lifting $\phi_M(f_0) : \mathbb{G} \to \mathbb{H}$ of $Mf_0$ (which will evidently be unique, by injectivity). Since $\mathbb{H}$ is smooth and $\mathcal{I}$ is nilpotent, $\mathbb{H} \to i_* \mathbb{H}_0$ is a surjection of sheaves. (This is just the lifting property of smooth morphisms.) There is thus an exact sequence

$$0 \to \mathbb{H}_{\mathcal{I}} \to \mathbb{H} \to i_* \mathbb{H}_0 \to 0,$$

and since $\mathbb{H}_{\mathcal{I}}$ is $M$-torsion, we get an isomorphism $M\mathbb{H} \cong M i_* \mathbb{H}_0$. Since $Mf_0$ has image in $M\mathbb{H}_0$, it then lifts to $\mathbb{G} \to \mathbb{H}$.

As a final point, we note that $f_0$ itself lifts to $f$, then $\phi_M(f_0) = Mf$, by injectivity; conversely, if $\phi_M(f_0)$ is of the form $MF$ for some $F : \mathbb{G} \to \mathbb{H}$, then since $\operatorname{Hom}(\mathbb{G}_0, \mathbb{H}_0)$ is $p$-torsion-free, we see that $F_0 = f_0$. $\square$

*Proof of the Serre-Tate theorem.* We can restate the problem as follows: we want to show that the obvious functor from the category of abelian schemes over $S$ to the category of abelian schemes over $S_0$ together with a deformation of their $p$-divisible groups over $S$ is an equivalence.

To prove essential surjectivity, let $A_0 \to S_0$ be an abelian scheme and let $\mathbb{G} \to S$ be a $p$-divisible group deforming $A_0[p^\infty]$; we will construct a deformation of $A_0$ realizing $\mathbb{G}$. Assume that there is some deformation $A_0$ to an abelian scheme $B$ over $S$, with $B_0 \cong A_0$ its reduction mod $\mathcal{I}$. There's then an isomorphism $f_0[p^\infty] : B_0[p^\infty] \cong A_0[p^\infty]$. By Lemma 4, $Mf_0[p^\infty]$ lifts to some isogeny

$$\phi_M(f_0[p^\infty]) : B[p^\infty] \to \mathbb{G}.$$

Since $f_0[p^\infty]$ is an isomorphism, there's also a $\phi_M(f_0[p^\infty]^{-1}) : \mathbb{G} \to B[p^\infty]$, and the composition $\phi_M(f_0[p^\infty]^{-1})\phi_M(f_0[p^\infty]) : B \to B$ is just $[M^2]$. It follows that $\phi_M(f_0[p^\infty])$ identifies $\mathbb{G}$ with $B[p^\infty]/K$, where $K$ is a subgroup scheme of the finite group scheme $B[M^2]$. Letting $A = B/K$, we evidently have $A[p^\infty] \cong \mathbb{G}$ and $i_*i^*A \cong A_0$.

To prove full faithfulness, let $f_0 : A_0 \to B_0$ be a homomorphism of abelian schemes and let $f[p^\infty] : A[p^\infty] \to B[p^\infty]$ be a homomorphism of $p$-divisible groups deforming $f_0[p^\infty]$. We will show that there is a unique homomorphism $f : A \to B$ inducing $f[p^\infty]$ and $f_0$. By Lemma 4 applied to $A$ and $B$, any lifting of $f_0$ to a map $A \to B$ is unique, so it remains to prove existence. Moreover, the lemma tells us that a lifting $\phi_M(f_0)$ of $Mf_0$ exists. But $\phi_M(f_0)[p^\infty]$ lifts $Mf_0[p^\infty]$, as does $Mf[p^\infty]$, so applying the lemma to $A[p^\infty]$ and $B[p^\infty]$, we see that they must be equal. In particular, $\phi_M(f_0)$ kills $A[p^\infty][M] = A[M]$, so it is actually of the form $MF$ for some $F : A \to B$ lifting $f_0$. Finally, $F[p^\infty]$ lifts $f_0[p^\infty]$, so by uniqueness, $F[p^\infty] = f[p^\infty]$, as desired. □

## 3. Deformations of abelian varieties

The attentive reader will have noticed a gap in the proof of essential surjectivity – we did not prove that deformations of abelian schemes over $S_0$ to abelian schemes over $S$ always exist! This is true, and its proof uses the full weight of classical deformation theory and the geometry of abelian varieties.

**Theorem 5.** *Let $S_0 \hookrightarrow S$ be a closed immersion of schemes defined by a nilpotent sheaf of ideals $\mathcal{I}$, and $A_0 \to S_0$ an abelian scheme. Then there is an abelian scheme $A \to S$ such that $A \times_S S_0 = A_0$.*

*Proof.* A very rough sketch of the proof:

(1) Reduce to the case where $S_0 = \operatorname{Spec} R_0$ for $R_0$ an artinian algebra over a field $k$, and $S = \operatorname{Spec} R$ is a thickening by an ideal $I$ with $I \cong k$ as an $R$-module.
(2) The existence of deformations of $A_0$ over $R$, *as a scheme*, are obstructed by a cohomology class $\theta \in H^2(A_0; T_{A_0} \otimes_k I) \cong H^2(A_0; T_{A_0})$, where $T_{A_0}$ is the tangent sheaf of $A_0$.
(3) Since $\overline{A} = A_0 \otimes_{R_0} k$ is an abelian variety, $T_{\overline{A}} \cong (T_{\overline{A}})_0 \otimes \mathcal{O}_{\overline{A}}$, and the cohomology of $\mathcal{O}_{\overline{A}}$ is an exterior algebra on $H^1(\overline{A}, \mathcal{O}_{\overline{A}})$. In particular, $[-1]^*$ acts as $-1$ on $H^1(\overline{A}, \mathcal{O}_{\overline{A}})$, thus trivially on $H^2(\overline{A}, \mathcal{O}_{\overline{A}})$, thus as $-1$ on $H^2(\overline{A}, T_{\overline{A}})$.
(4) We also have that $[-1]^*\theta = \theta$. So in nontwo characteristic, $\theta = 2\theta = 0$; in characteristic two, we must use a more complicated map than $[-1]$, but a similar argument.
(5) Thus $A_0$ deforms over $R$ as a scheme. To deform it as an abelian scheme, it suffices to deform the map $j_0 : A_0 \times_{R_0} A_0 \to A_0$ sending $(x, y) \mapsto x - y$, in such a way that the evident axioms are satisfied. Everything being separated, deforming this map is equivalent to deforming its graph, which is a closed subscheme $\Gamma_0$ of $A_0 \times_{R_0} A_0 \times_{R_0} A_0$.
(6) Deformations of this closed subscheme are now obstructed by a class $\theta' \in H^1(\overline{A} \times \overline{A}; N_{\overline{\Gamma}})$, where $N_{\overline{\Gamma}}$ is the normal bundle of the graph $\overline{\Gamma}$ of $\overline{j}$. but this vanishes when restricted to $\overline{A} \times \{0\}$, $\{0\} \times \overline{A}$, or the diagonal, so by the Künneth theorem, it vanishes over all of $\overline{A} \times \overline{A}$.
(7) This shows that a deformation of $j$ exists. The set of such deformations is a torsor for $H^0(\overline{A} \times \overline{A}; N_{\overline{\Gamma}})$, which is isomorphic to $(N_{\overline{\Gamma}})_{(0,0)}$. Thus, there is a unique deformation of $j$ that restricts to zero on the diagonal in $A \times_R \{0\}$, giving the identity axiom. From this uniqueness, one easily observes that the other group axioms are satisfied.

□

## 4. Deformations of PEL abelian varieties

We're now in good shape to prove Proposition 2. This is where all the seemingly random structure we introduced last time really comes into play. As usual, let $(A_0, i_0, \lambda_0, [\eta_0])$ be a $p$-divisible group over $S_0$, and $\mathbb{G}$ a $p$-divisible group over $S$ deforming $\epsilon A_0(u)$. First, we required a splitting $\mathcal{O}_{B,u} \cong M_n(\mathbb{Z}_p)$, with $\epsilon$ sent to the matrix that projects a vector onto its first coordinate. This means that over $\mathbb{Z}_p$, we have $A_0(u) \cong (\epsilon A_0(u))^n$. Thus, the height $n$ deformation $\mathbb{G}$ canonically determines an $\mathcal{O}_{B,u}$-linear height $n^2$ deformation $\mathbb{G}^n$ of $A_0(u)$.

We'd now like to bump this up to a deformation of $A_0(p)$. Since $p = u\overline{u}$ in $F$, we have

$$A_0(p) \cong A_0(u) \times A_0(\overline{u}).$$

Since the $\lambda_0$-Rosati involution extends complex conjugation on $F$, we get

$$A_0(p) \cong A_0(u) \times A_0(u)^\vee$$

via the isomorphism $A_0(\overline{u}) \to A_0(u)^\vee$ induced by $\lambda : A_0 \to A_0^\vee$. So there's a canonical $\mathcal{O}_{B,(p)}$-linear height $2n^2$ deformation $\mathbb{G}^n \times (\mathbb{G}^\vee)^n$ of the $p$-divisible group of $A_0$ to $S$. By the Serre-Tate theorem, this determines a deformation $A$ of $A_0$ to $S$, and the action of $\mathcal{O}_{B,(p)}$ on the $p$-divisible group deforms to an action $i : \mathcal{O}_{B,(p)} \to \mathrm{End}(A)_{(p)}$ extending $i_0$. (Importantly, the Serre-Tate theorem is a statement about categories, so we can deform homomorphisms of abelian varieties as well as the objects themselves.)

The polarization $\lambda$ should be a map $A \to A^\vee$, so by the Serre-Tate theorem, we can define it given maps $A_0 \to A_0^\vee$ and $\mathbb{G}^n \times (\mathbb{G}^\vee)^n \to (\mathbb{G}^\vee)^n \times \mathbb{G}^n$. The first should obviously be $\lambda_0$. For the second, we use the twist isomorphism, which is easily seen to deform the polarization isomorphism

$$A(u) \times A(\overline{u}) \to A^\vee(u) \times A^\vee(\overline{u}).$$

(Technically, our polarization $\lambda_0$ is only defined as a map after multiplying by some integer $N$ prime to $p$, but deforming $N\lambda_0$ and dividing by $N$ gives a deformation of $\lambda_0$, using the fact that endomorphism rings of abelian varieties are $\mathbb{Q}$-algebras.)

Anyway, both $\lambda_0$ and the twist isomorphism are Rosati-symmetric, so $\lambda$ will be as well. There's also a positivity condition, which can be checked at the fibers over geometric points; but $S_0$ and $S$ have the same geometric points. This also shows that the level structure has a unique extension.

Finally, it's clear that everything we've done is functorial for morphisms in $\mathcal{X}'_{K^p}(S_0)$ – any choices were dictated by the algebra of $B$ or the Serre-Tate theorem. This proves Proposition 2, as well as the following fun corollary:

**Corollary 6.** *$Sh(K^p) \to \mathbb{Z}_p$ is smooth of relative dimension $n-1$.*

## 5. The topological automorphic forms spectrum

**Theorem 7.** *There exists a homotopy sheaf of weakly even periodic $E_\infty$-ring spectra on the étale site of $Sh(K^p)^\wedge_p$, $\mathcal{E}(K^p)$, such that for any étale map*

$$(A, i, \lambda, [\eta]) : \mathrm{Spf}(R) \to Sh(K^p)^\wedge_p,$$

*the formal group of $\mathcal{E}(K^p)(R)$ is canonically isomorphic to $\epsilon A(u)^0$.*

**Definition 8.** The topological automorphic forms spectrum is the global sections of this homotopy sheaf:

$$TAF(K^p) = \mathcal{E}(K^p(Sh(K^p)^\vee_p).$$

As with the Goerss-Hopkins-Miller theorem, we can use the algebraic geometry of the stack to compute the homotopy groups of the TAF spectrum (or indeed, the sections of $\mathcal{E}(K^p)$ over any étale open).

**Definition 9.** If $\mathbb{G}$ is a formal group over a scheme $S$, $\omega_{\mathbb{G}} = \mathrm{Lie}(\mathbb{G})^\vee$ is the line bundle over $S$ of invariant 1-forms on $\mathbb{G}$. We'll let $\omega$ be the line bundle on $Sh(K^p)$ defined by

$$\omega(A, i, \lambda, [\eta]) = \omega_{\epsilon A(u)^0}.$$

**Proposition 10.** *If $E$ is a weakly even periodic ring spectrum with formal group $\mathbb{G}_E$, then there is a canonical isomorphism*

$$\pi_{2t}E \cong \Gamma(\mathrm{Spec}\, E^0, \omega_{\mathbb{G}_E}^{\otimes t}).$$

*(Of course, $\pi_k E = 0$ for $k$ odd.)*

Thus, if $f : U \to Sh(K^p)_p^\wedge$ is a formal affine étale open, then we have

$$\pi_{2t}(\mathcal{E}_{\mathbb{G}}(U)) = \Gamma(U, \omega_{f^*\mathbb{G}^0}^{\otimes t}).$$

If $f : U \to Sh(K^p)_p^\wedge$ is a general étale open, then take an étale cover $U' \to U$ by a formal affine scheme. We get a cosimplicial object by evaluating $\mathcal{E}_{\mathbb{G}}$ on the successive iterated pullbacks

$$U'^{n+1} = \underbrace{U' \times_U \cdots \times_U U'}_{n+1},$$

all of which are affine formal schemes since our stack is separated. By homotopy descent,

$$\mathcal{E}_{\mathbb{G}}(U) \simeq \mathrm{holim}\, \mathcal{E}_{\mathbb{G}}(U'^{\bullet+1}).$$

The Bousfield-Kan spectral sequence of this homotopy limit then takes the following form.

**Theorem 11.** *For $f : U \to Sh(K^p)_p^\wedge$ an étale map from a scheme, there is a conditionally convergent **descent spectral sequence***

$$E_2^{s,2t} = H^s_{\mathrm{zar}}(U, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(\mathcal{E}(K^p)(U)).$$

*In particular, on global sections this is a spectral sequence*

$$E_2^{s,2t} = H^s_{\mathrm{zar}}(Sh(K^p)_p^\wedge, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(TAF(K^p)).$$

## References

[1] Mark Behrens and Tyler Lawson. Topological automorphic forms. *Memoirs of the American Mathematical Society*, 204(958), 2010.

[2] Nicholas Katz. Serre-tate local moduli. In *Surfaces algébriques*, pages 138–202. Springer, 1981.