K3 cohomology theories

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1 Introduction

This is the first talk in this quarter's seminar, on K3 surfaces and their possible homotopy-theoretic applications. Today, we're going to start right off with the homotopy-theoretic applications. First, let me say a few words about chromatic homotopy theory.

Chromatic homotopy theory has learned how to associate to any space a quasi-coherent sheaf over the moduli of formal groups, $\mathcal{M}_{\mathrm{fg}}$. Localized at a prime p, this moduli space is stratified by the height of the formal group, an integer between 0 and ∞. Using the chromatic convergence theorem and the Adams-Novikov spectral sequence, we could immediately turn around and calculate homotopy groups, but this is opaque and computationally intense, so the game has long been to find geometric substitutes for the formal groups themselves. Typically, this comes in the form of a stack with a flat map to $\mathcal{M}_{\mathrm{fg}}$, with the associated 'sheaf' of Landweber exact cohomology theories enriched to a sheaf of even periodic E_∞ ring spectra. The main example is the moduli $\mathcal{M}_{\mathrm{ell}}$ of elliptic curves, from which we get the topological modular forms spectrum, TMF, which detects height 2 phenomena; the height 1 story analogously retcons into a story about constructing KO from the stack $B\mathbb{Z}/(2)$, classifying $\mathbb{G}_m$-torsors.

Now we ask where to go next. One idea is to move from elliptic curves to higher-dimensional abelian varieties with extra structure; this has been fully worked out by [BL10], but as yet has not been amenable to calculation. The idea of using K3 surfaces apparently is due to [Tho00], but the first attempt at working it out, using the stacky techniques of the new millennium, is in Szymik's [Szy10] and [Szy09]. [Szy10] defines K3 cohomology theories and proves their Landweber exactness, while [Szy09] does an obstruction theory calculation, enriching this to a $K(1)$-local sheaf of E_∞ ring spectra. To obtain the full strength of the theory associated to TMF, we need to at least also do this $K(n)$-locally for $2 \le n \le 10$, 10 being the largest height (less than ∞) of the formal groupa ssociated to a K3 surface.

I'll quickly discuss K3 surfaces and the formal group construction we'll be using, define K3 cohomology theories, and finally walk us through the obstruction theory calculation. Anything below labeled 'black box' or 'question' is to be proved later in the quarter.

2 K3 surfaces

Definition 1. A **K3 surface** is a smooth proper surface X with trivial canonical bundle $\mathscr{O}_X \cong \omega_X := \bigwedge^2 \Omega_X$, and with $H^1(X, \mathscr{O}_X) = 0$.

Let's unpack this for a second. The name 'K3,' one of the silliest in math, is due to Weil, and is 'in honor of Kummer, Kähler, Kodaira, and the beautiful K2 mountain in Kashmir' [Wei79][p. 546]. Any abelian variety has trivial canonical bundle; indeed, it has trivial tangent sheaf, since a trivialization at the identity section can be translated around the whole variety using the group structure. This is the connection with elliptic curves; generally, a smooth proper scheme with trivial canonical bundle is called a **Calabi-Yau scheme**, and the homotopy theory we'll discuss today is hypothesized to generalize to higher-dimensional Calabi-Yaus as well. The only Calabi-Yau surfaces are K3 surfaces and abelian surfaces, and the last condition, triviality of H^1, is just used to exclude abelian surfaces, as well as to define the formal group below.

Definition 2. A **polarization** of a K3 surface X is an ample line bundle $\mathcal{L}$ on X. Some power of $\mathcal{L}$ will then give a projective embedding of X. The **degree** of $\mathcal{L}$ is the self-intersection of an associated Cartier divisor; if $\mathcal{L}$ is very ample, this is the degree of the projective embedding associated to $\mathcal{L}$.

Proposition 3 (Black box)**.** *Every K3 surface admits a polarization, and its degree is always even. Not every ample line bundle on a K3 surface is very ample.*

Proposition 4 (Black box)**.** *Every K3 surface has Hodge diamond*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 22 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and thus Euler characteristic $\chi(\mathscr{O}_X) = 24$*.*

Example 5 (Black box). Some examples of K3 surfaces should be given: in particular, a smooth quartic hypersurface in $\mathbb{P}^3$ is a K3 surface (polarized of degree 4); the **Kummer surface** of an abelian surface, given by blowing up at the 16 2-torsion points and then quotienting by the automorphism $[-1]$, is another example. We should talk about the formal Brauer groups of these objects.

We can now define the moduli stack we're interested in. Define

$$\mathcal{M}_{2d}^{\mathrm{K3}}(\operatorname{Spec} R) = \{\text{K3 surfaces over } R \text{ with a polarization of degree } 2d\}.$$

Theorem 6 (Black box)**.** $\mathcal{M}_{2d}^{\mathrm{K3}}$ *is a separated Deligne-Mumford stack that is smooth of dimension 19 over* $\mathbb{Z}\left[\frac{1}{2d}\right]$*.*

As motivating example, consider the moduli of smooth quartics in $\mathbb{P}^3$. A quartic is given by a homogeneous degree 4 polynomial in 4 variables, which is observed to have $\binom{4+4-1}{4} = 35$ coefficients; thus, the moduli of all quartic equations, up to scaling, is just $\mathbb{P}^{34}$, and the moduli of smooth quartic equations is obtained by throwing out the closed hypersurface Ω where the discriminant vanishes, leaving behind a 34-dimensional affine variety. Of course, two distinct equations can cut out isomorphic surfaces, so we must quotient out by $PGL(4)$, an affine algebraic group of dimension $4^2 - 1 = 15$. The resulting stack,

$$(\mathbb{P}^{34} - \Omega)//PGL(4),$$

has the required dimension 15. Unfortunately, this is not all of $\mathcal{M}_4^{\mathrm{K3}}$, since not every degree-4 ample line bundle is very ample.

Question 7. *What does* $\mathcal{M}_4^{\mathrm{K3}}$ *look like outside the image of this substack?*

3 The formal Brauer group

This is the biggest black box of all, since it's described in the hard paper [AM77] and everyone else just quotes it.

Definition 8. The **Brauer group** of a smooth proper scheme X over a field k is

$$\mathrm{Br}_X(k) = H^2_{\text{ét}}(X; \mathbb{G}_m).$$

The **formal Brauer group** of X is the functor from Artin local k-algebras to abelian groups given by

$$\widehat{\mathrm{Br}}_X : R \mapsto \ker(H^2_{\text{ét}}(X \times \operatorname{Spec} R; \mathbb{G}_m) \to H^2_{\text{ét}}(X; \mathbb{G}_m).$$

Theorem 9 (Black box)**.** *If X is a K3 surface, then* $\widehat{\mathrm{Br}}_X$ *is a 1-dimensional formal group over k. If in addition* $\operatorname{char} k = p > 0$*, then the height of* $\widehat{\mathrm{Br}}_X$ *is between 1 and 10, or* ∞*.*

The triviality of H^1 is required for the above theorem. More generally, we can define formal H^n in the same way, and thus get a formal group out of a Calabi-Yau scheme of dimension n with vanishing H^{n-1}.

When $n = 1$, we get the **formal Picard group**. For an elliptic curve, one can and should show that this is canonically isomorphic to the dual of its formal group. For a higher genus curve, we get instead the formal group of its Jacobian, which has dimension higher than 1. This might explain the 'trivial canonical bundle' condition.

Proposition 10 (Black box)**.** *Let $\mathcal{M}^{\mathrm{K3}}_{2d,p} = \mathcal{M}^{\mathrm{K3}}_{2d} \times \operatorname{Spec} \mathbb{F}_p$. Then $\mathcal{M}^{\mathrm{K3}}_{2d,p}$ admits a stratification by closed substacks*

$$\mathcal{M}^{\mathrm{K3}}_{2d,p} = \mathcal{M}^{\mathrm{K3}}_{2d,p,\geq 1} \supseteq \mathcal{M}^{\mathrm{K3}}_{2d,p,\geq 2} \subseteq \cdots \subseteq \mathcal{M}^{\mathrm{K3}}_{2d,p,\geq 11} = \mathcal{M}^{\mathrm{K3}}_{2d,p,\infty}$$

where $\mathcal{M}^{\mathrm{K3}}_{2d,p,\geq h}$ parametrizes degree $2d$ polarized K3 surfaces with formal Brauer group of height at least h. The stacks $\mathcal{M}^{\mathrm{K3}}_{2d,p,\geq h} - \mathcal{M}^{\mathrm{K3}}_{2d,p,\infty}$ is smooth of dimension $20-h$, for $1 \leq h < 11$. Each closed substack is cut out from the last by the vanishing of a section of an invertible sheaf of ideals.

The existence of this stratification is a corollary of the existence of the formal Brauer group, viewed as a map $\mathcal{M}^{\mathrm{K3}}_{2d,p} \to \mathcal{M}_{\mathrm{fg}}$. The hard work here seems to be the last assertion, which importantly implies that the stratification is locally defined by a regular sequence, and thus implies the dimension assertion. Szymik directs us to [Ogu01] and [vdGK00], which look very exciting. A hint of the proof: one starts by showing that the Dieudonné module of the formal Brauer group of a K3 surface X is the Witt vector cohomology $H^2(X; \mathbb{W}(\mathscr{O}_X))$. The height $\geq n$ locus is then defined by the vanishing of Frobenius on the truncated version $H^2(X; \mathbb{W}_n(\mathscr{O}_X))$. With a little work, one gets a line bundle out of this. In Ogus' words, 'it is not only possible, but even easy, to extract geometric information from crystalline periods.'

4 K3 spectra

Definition 11. A **K3 spectrum** is a triple (E, X, π) where

- E is an even periodic ring spectrum,
- X is a K3 surface over $\pi_0 E$,
- and π is an isomorphism $\pi : \mathbb{G}_E \xrightarrow{\sim} \widehat{\mathrm{Br}}_X$.

Theorem 12. *Let R be a noetherian local $\mathbb{Z}_{(p)}$-algebra, with $p \nmid 2d$, and X a degree $2d$ polarized K3 surface over $\operatorname{Spec} R$ that is classified by a flat map $\operatorname{Spec} R \to \mathcal{M}^{\mathrm{K3}}_{2d}$, with finite height h at the closed point of $\operatorname{Spec} R$. Then $\widehat{\mathrm{Br}}_X$ is Landweber exact.*

Proof. We want to show that the height filtration of $\operatorname{Spec} R$ is cut out by a regular sequence, and stabilizes at some finite height. First we deal with height 0: R is flat over the moduli of K3 surfaces, so it is flat over $\mathbb{Z}\left[\frac{1}{2d}\right]$, and thus p is a non-zero-divisor by the condition on p. We can now safely mod out by p, and thus assume that R is a noetherian local $\mathbb{F}_p$-algebra, X classified by a flat map to $\mathcal{M}^{\mathrm{K3}}_{2d,p}$.

Since X has finite height h at the closed point of $\operatorname{Spec} R$, and since $\mathcal{M}^{\mathrm{K3}}_{2d,p,\geq h+1}$ is a closed substack of $\mathcal{M}^{\mathrm{K3}}_{2d,p,\geq h}$, the height $\geq h+1$ locus is a closed subscheme of $\operatorname{Spec} R$ that doesn't contain its closed point. By Nakayama's lemma, it's empty. Thus the filtration terminates.

Finally, regularity of the filtration after height 0 is forced by regularity of the height filtration on $\mathcal{M}^{\mathrm{K3}}_{2d,p}$ and flatness of the map classifying X. $\square$

Question 13. *This proof is a little confusing. It seems like we should be able to show that $\widehat{\mathrm{Br}} : \mathcal{M}^{\mathrm{K3}}_{2d,(p),<\infty} \to \mathcal{M}_{\mathrm{fg},(p)}$ is flat (where the $<\infty$ in the first stack denotes finite height mod p. I'm too tired to figure this out, so is it true? In the case of elliptic curves, even to prove this map is representable seems to require the Weierstrass parametrization.*

In any case, we have the much-vaunted 'sheaf of Landweber exact cohomology theories' on $\mathcal{M}^{\mathrm{K3}}_{2d}$. Since $\mathcal{M}^{\mathrm{K3}}_{2d}$ is algebraic, and since K3 surfaces lift to characteristic 0 (see [DI81]), we can get a K3 spectrum at any geometric point of $\mathcal{M}^{\mathrm{K3}}_{2d}$: just lift it to a smooth cover by a scheme, take the local ring at the point, lift to characteristic 0 if need be, and use the theorem.

The issue is that this sheaf is not truly a sheaf in the sense of derived algebraic geometry. In order to take global sections in any sort of homotopy-invariant sense, we need the sections of the sheaf to be more than just homotopy types. We need them to be E_∞ ring spectra. The next part of the work does this for the ordinary locus $\mathcal{M}^{\mathrm{K3}}_{2d,p,\leq 1}$ – that is, $K(1)$-locally.

5 $K(1)$-local K3 spectra

I'll only discuss this briefly, since we need harder techniques of homotopy theory to do it. It's a series of black boxes.

First, let's say we have a diagram of cohomology theories we want to enrich to a diagram of E-local E_∞ ring spectra, uniquely up to homotopy. Goerss-Hopkins obstruction theory tells us when we can do this: in general, the obstruction groups are groups of derivations over the monad of power operations of E. When $E = K(1) = \text{mod } p$ K-theory, this monad is very well understood, and its algebras are **θ-algebras**, or p-complete λ-rings; in the torsion-free case, these correspond to algebras with a p-power Adams operation.

Next we need to apply this to the diagram of height 1 K3 cohomology theories over $\mathcal{M}^{\mathrm{K3}}_{2d,p,\leq 1}$. If we can show that each cohomology theory has a unique E_∞ structure, and the restriction maps of the sheaf have unique E_∞ realizations, we'll be done. This comes down to constructing θ-algebra structures on the formal local rings of the moduli stack. This is easily done once we know what these formal neighborhoods look like: they're just formal completions of $\mathbb{A}^{20}$, and the Adams operation is given by a certain lift of the Frobenius map to this formal neighborhood.

6 The future

After this, we're going to leave homotopy theory behind and spend a while talking about algebraic surfaces.

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