Stable homotopy theory and geometry

Paul VanKoughnett

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- Even simpler: $S^{n-1} \to X^{(n-1)}/X^{(n-2)}$, a bouquet of $(n-1)$-spheres; or $S^{n-1} \to X^{(n-2)}/X^{(n-3)}$, a bouquet of $(n-2)$-spheres; or . . .

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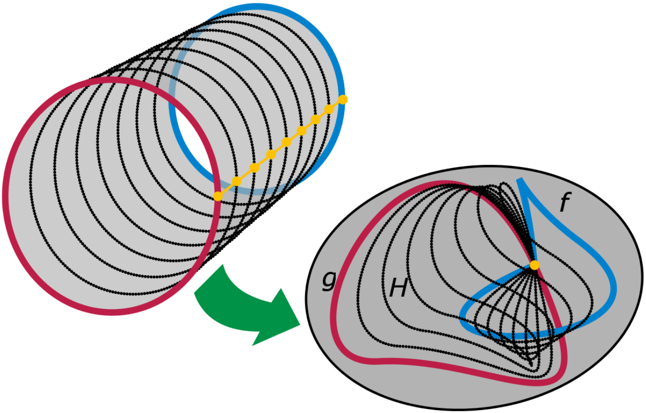
Given two maps $f, g : X \to Y$, a **homotopy** $f \sim g$ is map $H : X \times [0, 1] \to Y$ with $H|_{X\times\{0\}} = f$, and $H|_{X\times\{1\}} = Y$.

$$[X, Y] = \{\text{maps } X \to Y\}/\text{homotopy}$$

Homotopy



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Remark

Always take spaces to come equipped with a fixed basepoint; maps preserve basepoint; homotopies don't move basepoint.

An example

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- Is it? How many other complexes like this are there?

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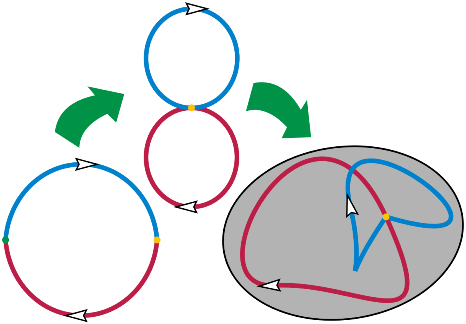

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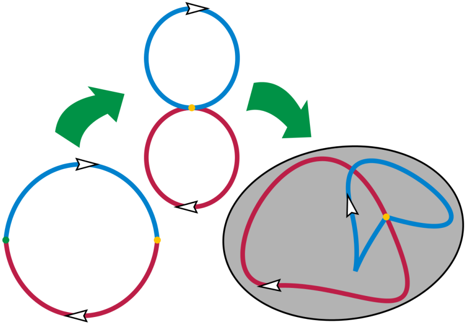

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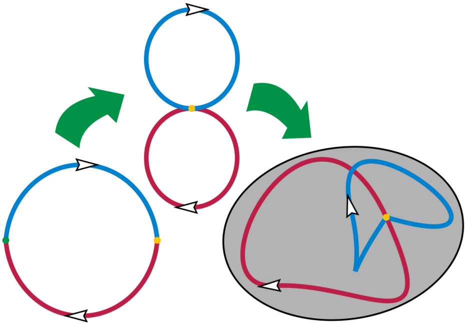

π_0 is just a set. π_n is abelian for $n \geq 2$ (why?).

Homotopy groups of spheres

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$\pi_k S^n$	1	2	3	4	5	6	7	8	9	k

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Theorem (Freudenthal suspension theorem)

The suspension map

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is a surjection for $k = 2n-1$ and an isomorphism for $k < 2n-1$.

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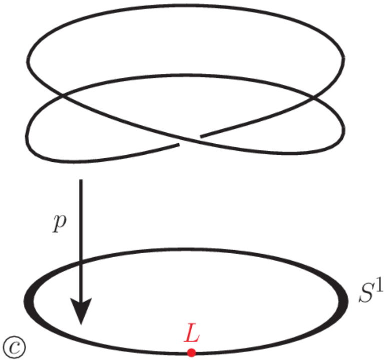
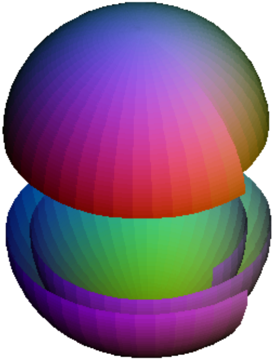
$\pi_n S^n$ is cyclic. . . and must be $\mathbb{Z}$, because the *degree* of a map is homotopy invariant.

$\pi_n S^n = \mathbb{Z}$

Degree two maps:





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Is everything else zero? NO!

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- The fiber over two points are two linked circles.

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4	0	0	0	$\mathbb{Z}$	2	2	$\mathbb{Z} \oplus 12$	$2 \oplus 2$	$2 \oplus 2$	
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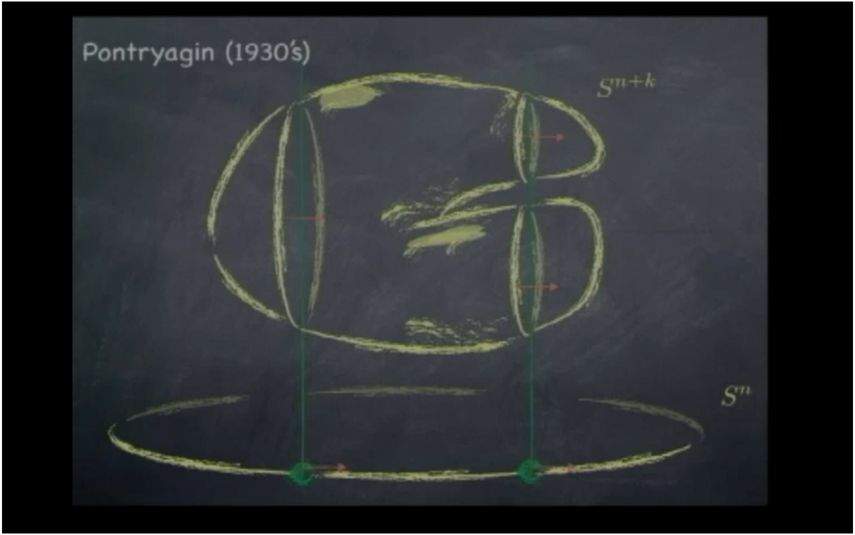
- The degree of a map $S^n \to S^n$ is an integer, because its fibers are signed 0-manifolds.
- The 'degree' of a map $S^k \to S^n$ should just be its fiber – a **stably framed** $(n-k)$**-manifold**.

Generalizing the degree



Stably framed manifolds

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A **stably framed manifold** is a manifold M^n with an embedding $i : M^n \hookrightarrow \mathbb{R}^{n+k}$, $k \gg 0$, and a trivialization of the normal bundle $N_i M \cong M \times \mathbb{R}^k$.

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We identify $M \hookrightarrow \mathbb{R}^{n+k}$ with $M \hookrightarrow \mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1}$ the inclusion into the first $n+k$ coordinates, together with the larger framing given by adding an upward-pointing normal vector everywhere.

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Definition

A **framed cobordism** of n-dimensional stably framed manifolds M, N is an $(n+1)$-manifold W with $\partial W \cong M \sqcup N$, together with a stable framing on W extending those on M and N.

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Homotopy elements can be described as framed n-manifolds!

0-manifolds: $\pi_n S^n$

A 0-manifold in $\mathbb{R}^n$ is a set of points. A framing is a sign attached to each point: orient their normal bundles either with or against the orientation of $\mathbb{R}^n$.

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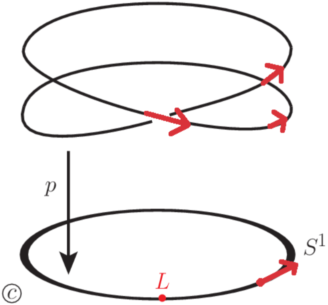

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- Framings are classified by $\pi_1 SO(n) = \mathbb{Z}/2$ for $n \geq 3$ (and $\mathbb{Z}$ for $n \geq 2$).
- The Hopf map $S^3 \to S^2$ corresponds to $S^1 \to S^3$ together with a basis for its normal bundle that twists once.

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- But $H_1(M; \mathbb{Z}/2)$ is positive-dimensional if M has genus ≥ 1, so $\ker \phi$ is always nonzero, so any M is framed-cobordant to a sphere. $\therefore \pi_{n+2}S^n = 0,\ n \gg 0$.

Pontryagin's mistake

- $$\phi : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2$$

  is not a linear map, but a *quadratic* map. Even if we can do surgery on two-cycles, we might not be able to on their sum.

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- Whether or not we can do framed surgery on M depends on the nature of this quadratic map. We can conclude that

  $$\pi_{n+2}S^2 \cong \mathbb{Z}/2, n \gg 0.$$

  A representative for the nontrivial class is given by the product of two nontrivially framed circles.

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n										
9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	
8	0	0	0	0	0	0	0	$\mathbb{Z}$	2	
7	0	0	0	0	0	0	$\mathbb{Z}$	2	2	
6	0	0	0	0	0	$\mathbb{Z}$	2	2	24	
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