Stable homotopy theories

Paul VanKoughnett

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Introduction

This talk is a Mayer-Vietoris square:

$$\begin{array}{ccc} \text{Spaces} & \longrightarrow & \text{Spectra} \\ \downarrow & & \downarrow \\ G\text{−Spaces} & \longrightarrow & G\text{−Spectra} \end{array}$$

I'll conduct it from the ∞-categorical perspective, discussing each of the above categories in terms of the properties we want them to satisfy. When I say ∞-categories, I mean quasi-categories, i. e., simplicial sets in which every inner horn has a filler. Most of what I have to say about spaces, spectra, and stabilization comes from Chapter 1 of Lurie's *Higher Algebra*. Of course, everything should be translatable into the other ∞-categorical languages Dylan defined, simplicial categories or complete Segal spaces.

Spaces

What is a space? At some point in our lives, we've all seen a very old definition:

Definition 0.1. A **topological space** is a set X with an algebra of open subsets $\mathcal{O}(X) \subseteq \mathcal{P}(X)$, satisfying certain conditions.

This is a little too broad for the purposes of algebraic topology, which has historically been interested in things like manifolds and simplicial complexes. Typically, we restrict to a full subcategory of the category of topological spaces, containing these nice spaces we're interested in, and with good categorical properties. One such is the following:

Definition 0.2. A **space** is a Hausdorff topological space X which is:

- **compactly generated**: a set $U \subseteq X$ is open if and only if $p^{-1}(U)$ is open in C, for every continuous map $p : C \to X$ from a compact Hausdorff space C.

The category Top is the category of spaces and continuous maps.

Among the nice properties this category has is the existence of internal mapping objects. Given two spaces X and Y, define $F(X, Y)$ to be the set of continuous maps $X \to Y$ with the **compactly generated compact-open topology**. This is the smallest compactly generated topology in which the sets

$$\{f : X \to Y \mid f(C) \subseteq U\},$$

for given compact $C \subseteq X$ and open $U \subseteq Y$, are open. This is an internal mapping object because there are natural isomorphisms

$$\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(X, F(Y, Z)).$$

In fact, there are natural homeomorphisms of spaces

$$F(X \times Y, Z) \cong F(X, F(Y, Z)).$$

Moreover, the composition map

$$\mathsf{Top}(Y,Z) \times \mathsf{Top}(X,Y) \to \mathsf{Top}(X,Z)$$

gives a continuous map of spaces

$$F(Y,Z) \times F(X,Y) \to F(X,Z).$$

This gives Top the structure of a **topological category**. It's enriched over itself!

Remark 0.3. If you care about this stuff, I couldn't just take the compact-open topology, because that might not be compactly generated. I had to make sure that all the sets which looked open inside every compact set were compact in the whole mapping space. Fortunately, there's a canonical way to do this. Likewise, the products above are products in the category Top. The product $X \times Y$ has the cartesian product of X and Y as its underlying set, but not the standard product topology – again, we have to throw in more open sets to make it compactly generated.

Alternatively, we could define a simplicial object

$$F_n(X,Y) = \mathsf{Top}(X \times \Delta^n, Y)$$

where Δ^n is the standard n-simplex. The geometric realization of $F_\bullet(X,Y)$ is canonically weakly equivalent to the space $F(X,Y)$, and the simplicial mapping objects $F_\bullet(X,Y)$ make Top into a simplicial category.

More recently, people have avoided the point-set technicalities above by using simplicial sets. There's a category sSet of simplicial sets, which also has internal mapping objects, and is thus a simplicial category. There's an adjunction of simplicial categories

$$|\cdot| : \mathsf{sSet} \leftrightarrows \mathsf{Top} : \mathrm{Sing}.$$

It would be too much to expect this adjunction to be an equivalence of categories, or of simplicial categories. If this were true, then the canonical map $X \to |\mathrm{Sing}(X)|$ would be a homeomorphism, but clearly the second space is much bigger than the first. Instead, we have the following:

Theorem 0.4. *There are simplicial model structures on* sSet *and* Sing *such that the above adjunction is a Quillen equivalence of model categories.*

Remark 0.5. In case you haven't seen this before: the weak equivalences on sSet and Top are weak homotopy equivalences, i. e., maps that induce equivalences on homotopy groups. The cofibrations on sSet are inclusions of simplicial sets. The fibrations are Kan fibrations, which are maps that satisfy the right lifting property with respect to inclusions of horns $\Lambda^n_k \to \Delta_n$. The fibrations on Top are Serre fibrations, which are maps that have the right lifting property with respect to the standard inclusions of cubes, $[0,1]^{n-1} \to [0,1]^n$. Cofibrations on Top are a little harder to describe: they're retracts of maps given by attaching cells.

Both of these model categories are what's called **combinatorial**, so they present ∞-categories. To be a bit more precise, given a combinatorial model category $\mathcal{C}$, the full subcategory $\mathcal{C}^{cf}$ on the fibrant-cofibrant objects has the property that all mapping spaces are Kan complexes. As we've seen, these mapping spaces have the right homotopy types: the derived space of maps from X to Y is homotopy equivalent to the actual space of maps from a cofibrant replacement of X to a fibrant replacement of Y. So $\mathcal{C}^{cf}$ is a simplicial category, and a construction called the simplicial nerve turns it into an ∞-category. We now have:

Corollary 0.6. *There is an equivalence of* ∞*-categories* $\mathsf{sSet} \simeq \mathsf{Top}$.

Let's now move without too much commotion to an ∞-category of pointed spaces, $\mathsf{sSet}_* \simeq \mathsf{Top}_*$. An n-simplex in the ∞-category of pointed spaces is an $(n+1)$-simplex in the ∞-category of spaces whose zeroth vertex goes to the one-point space.

Now, as we saw, one of the advantages of ∞-categories is they give us an easy way of talking about homotopy limits and homotopy colimits.

Definition 0.7. Let X be a pointed space. A **suspension** of X is a colimit of a diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \ulcorner\Sigma X \end{array}$$

A **loopspace** of X is a limit of a diagram of the form

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & X \end{array}$$

A square of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z \end{array}$$

is a **fiber sequence** if it's a limit diagram, and a **cofiber sequence** if it's a colimit diagram.

Now, as we know, spaces all have the weak homotopy type of CW-complexes. What's a CW-complex? Well, it's a colimit of finite CW-complexes along cofibrations. What's a finite CW-complex? Well, this is something we can build by a finite sequence of cell attachments, each of which is the cofiber of a map from a sphere. And what's a sphere? It's an iterated suspension of S^0. What we've shown is that every finite complex is a colimit of a finite diagram built out of S^0, and every space is (up to homotopy) a filtered colimit of such diagrams. With a little more work, we can prove the following:

Theorem 0.8. 1. *Let $\mathcal{D}$ be a pointed ∞-category with all finite colimits, and write* $\mathrm{Fun}^{\mathrm{Rex}}(\mathsf{Top}_*^{\mathrm{fin}}, \mathcal{D})$ *for the ∞-category of pointed, finite colimit preserving functors from the ∞-category of pointed finite spaces to $\mathcal{D}$. Then there is an equivalence*

$$\mathrm{Fun}^{\mathrm{Rex}}(\mathsf{Top}_*^{\mathrm{fin}}, \mathcal{D}) \simeq \mathcal{D}.$$

2. *Let $\mathcal{D}$ be a pointed ∞-category with all colimits, and write* $\mathrm{Fun}^{L}(\mathsf{Top}_*, \mathcal{D})$ *for the ∞-category of all pointed, colimit preserving functors from* Top_* *to $\mathcal{D}$. Then there is an equivalence*

$$\mathrm{Fun}^{L}(\mathsf{Top}_*, \mathcal{D}) \simeq \mathcal{D}.$$

Proof sketch. In both cases, the equivalences in one direction send a functor F to $F(S^0)$. In the other direction, they're given by left Kan extension. As said above, objects of Top_* and $\mathsf{Top}_*^{\mathrm{fin}}$ have CW-decompositions, which are colimit diagrams built out of spheres that are preserved by F. So the idea is that if you have a CW-decomposition of X and you know what $F(S^0)$ is, then $F(X)$ is forced to be the colimit in $\mathcal{D}$ of F of the CW-decomposition. More invariantly, given an object $F(S^0) \in \mathcal{D}$, we define

$$F(X) = \mathrm{colim}_{\mathrm{map}(S^0, X)} F(S^0)$$

where the colimit is taken over the full sub-∞-category of Top_* or $\mathsf{Top}_*^{\mathrm{fin}}$ generated by the arrows with source S^0 and target X.

There's a trick here that's ubiquitous in *Higher Topos Theory*. The formula above doesn't define a single functor called the left Kan extension, because colimits are only defined up to a contractible space of equivalences. In particular, it's not immediately clear that picking a colimit for each F in one of the functor categories defines an actual map from the functor category back to $\mathcal{D}$. So instead, we'll say that F is *a* left Kan extension of $F(S^0)$ if it fits into an appropriate colimit diagram. There's then a diagram of ∞-categories

$$\mathrm{Fun}^{\mathrm{Rex}}(\mathsf{Top}_*^{\mathrm{fin}}, \mathcal{D}) \leftarrow \{\text{left Kan extensions of } F(S^0)\} \rightarrow \mathcal{D}$$

and both arrows have contractible fibers. (Likewise for Top_*.) One then shows that both arrows are trivial Kan fibrations, which is a slightly stronger condition than having contractible fibers. As a consequence, they admit sections (since all simplicial sets, and in particular all ∞-categories, are cofibrant). Of course, there's a space of such sections, which is again contractible. □

The upshot of all this is that the category of spaces is, in the ∞-category sense, the free cocomplete ∞-category on one generator (and the category of finite spaces is the free finitely cocomplete ∞-category on one generator). More generally, given an ordinary category $\mathcal{C}$, there's an ∞-category $\widehat{\mathcal{C}}$ obtained from $\mathcal{C}$ by freely adjoining (homotopy) colimits. This means that there's a map of ∞-categories $\mathcal{C} \to \widehat{\mathcal{C}}$ that induces, for cocomplete ∞-categories $\mathcal{D}$, there's a natural equivalence

$$\mathrm{Fun}^L(\widehat{\mathcal{C}}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

On the right we're viewing $\mathcal{C}$ as an ∞-category with discrete mapping spaces. The ∞-category $\widehat{\mathcal{C}}$ is pretty easy to construct: it's just

$$\widehat{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{sSet}),$$

which comes with a Yoneda embedding, just as in ordinary category theory. What the above theorem comes down to is that the free homotopy cocompletion of a point – which is presheaves of simplicial sets on a point – is the same as the ∞-category of simplicial sets – which is the ∞-category of spaces.

Spectra

We'd now like to describe the construction of spectra in a way that generalizes to other ∞-categories. First, we might ask: why even construct spectra? There are a number of different and differently related reasons to do so.

1. There's a map $\Sigma^\infty : \mathsf{Top}_* \to \mathsf{Sp}$ such that

$$\mathrm{ho}\mathsf{Sp}(\Sigma^\infty X, \Sigma^\infty Y) \simeq \mathrm{colim}_{k\to\infty}\, \mathrm{ho}\mathsf{Top}_*(\Sigma^k X, \Sigma^k Y).$$

   In other words, homotopy classes of maps of spectra are stable homotopy classes of maps of spaces. This is only really compelling if you already have a good reason to do stable homotopy theory, though.

2. A little better, as Dylan said last time, spectra have canonical presentations of the form

$$\mathrm{colim}\, \Sigma^{-n}\Sigma^\infty K_n$$

   where K_n is a finite CW-complex. We can think of spectra as what we get by starting with $\mathsf{Top}_*^{\mathrm{fin}}$, inverting the suspension functor, and cocompleting.

3. Cohomological Brown representability.

   Definition 0.9. A **cohomology theory** is a set of functors $\{E^n : n \in \mathbb{Z}\}$ from $\mathrm{ho}\mathsf{Top}_*^{\mathrm{op}}$ to abelian groups, together with natural isomorphisms $\sigma^n : E^n \cong E^{n+1} \circ \Sigma$, such that

   - each E^n sends coproducts to products (and in particular, $E^n(*) = 0$);
   - E^* sends each cofiber sequence $A \to X \to X/A$ to a long exact sequence

$$\cdots \to E^n(X/A) \to E^n(X) \to E^n(A) \xrightarrow{\delta} E^{n+1}(X/A) \to \cdots$$

     where the maps δ are given by

$$E^n(A) \xrightarrow{\sigma} E^{n+1}(\Sigma A) \to E^{n+1}(X/A).$$

   For any cohomology theory E^* of spaces, there's an object E in Sp such that

$$E^*(X) \cong \pi_*\mathsf{Sp}(\Sigma^\infty X, E).$$

   As we'll see in a moment, if we're interested in representing E^n for a single n, there's no reason to leave the world of spaces: we can always find a space $E(n)$ such that

$$E^n(X) \cong \mathrm{ho}\mathsf{Top}_*(X, E(n)).$$

   The suspension isomorphisms σ^n translate to homotopy equivalences $E(n) \simeq \Omega E(n+1)$, which give each $E(n)$ the structure of an **infinite loop space**. Thus, the theory of spectra can be thought of as an enrichment of the theory of infinite loop spaces.

4. Homological Brown representability. Here we give an analogous definition of 'homology functor', and show that for every homology functor of spaces E_*, there's a spectrum E such that

$$E_n(X) = \pi_n(\Sigma^\infty X \wedge E).$$

   Unlike the cohomological version, here we are forced to enter the world of spectra to even state the claim.

5. Spanier-Whitehead duality. Let X be a finite complex and embed X into a sphere S^n, with open complement Y. Define $DX = \Sigma^{-n}Y$ in the category of spectra. For any cohomology theory E, there's a natural isomorphism

$$E_*(X) \cong E^*(DX).$$

6. Since every spectrum is a twofold suspension, every set of homotopy classes $\mathrm{ho}\mathsf{Sp}(X,Y)$ has the natural structure of an abelian group. Better yet, the category $\mathrm{ho}\mathsf{Sp}$ is naturally a triangulated category. (Define this.)

We'll start from the perspective of the last point. The most familiar triangulated category is the homotopy category of chain complexes over a ring R, $\mathrm{ho}\mathsf{Ch}(R)$. There's a shift operator defined by

$$A[1]_n = A_{n-1}.$$

We can figure out what fiber and cofiber sequences are by thinking that a cofibration is a degreewise injection, and a fibration is a degreewise surjection. Given a map $f : A_* \to B_*$, we can replace with a chain homotopy equivalent injection, the **mapping cylinder**. This is

$$\mathrm{Cyl}(f)_n = B_n \oplus A_{n-1}, \quad d(b,a) = (d_B(b) + f(a), d_A(a)).$$

There's then a short exact sequence

$$0 \to A_* \to \mathrm{Cyl}(f)_* \to C(f)_* \to 0;$$

the cofiber, the mapping cone, is by construction the homotopy cofiber of f. However, since $\mathrm{Cyl}(f)_* \to C(f)_*$ is surjective, this homotopy cofiber sequence is also a homotopy fiber sequence. More generally, any short exact sequence of chain complexes is both a homotopy cofiber and a homotopy fiber sequence. The images of these sequences in $\mathrm{ho}\mathsf{Ch}(R)$ are the **distinguished triangles**. Since they're both fiber and cofiber sequences, a distinguished triangle can be shifted backwards or forwards. That is, if $A_* \to B_* \to C_* \to A_*[1]$ is a distinguished triangle, so are

$$B_* \to C_* \to A_*[1] \to B_*[1] \quad \text{and} \quad C_*[-1] \to A_* \to B_* \to C_*.$$

Contrast this with spaces, in which cofiber sequences can be shifted rightwards by suspending, and fiber sequences can be shifted leftwards by desuspending.

Let's now lift this idea from homotopy categories to ∞-categories.

Definition 0.10. Let $\mathcal{C}$ be a pointed ∞-category. A **triangle** in $\mathcal{C}$ is a diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$

This is a **cofiber sequence** if it's a pushout diagram (in which case Z is the **cofiber** of $X \to Y$), and a **fiber sequence** if it's a pullback diagram (in which case X is the **fiber** of $Y \to Z$).

Definition 0.11. A **stable ∞-category** is a pointed ∞-category $\mathcal{C}$ in which:

- every morphism has a fiber and a cofiber;

- a triangle is a cofiber sequence if and only if it's a fiber sequence.

As a more or less immediate consequence of the definition:

Theorem 0.12. *The homotopy category of a stable ∞-category $\mathcal{C}$ is naturally triangulated. In particular, every set* $\mathrm{ho}\mathcal{C}(X,Y)$ *has the natural structure of an abelian group.*

(Show octahedral axiom?)

Another consequence is that stable ∞-categories are finitely complete and cocomplete, and pushout squares are the same as pullback squares. The point is that any pushout square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z \end{array}$$

can be replaced by a cofiber sequence

$$\begin{array}{ccc} X & \longrightarrow & Y \oplus Y' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$

A third consequence is that the loops and suspension functors are inverse equivalences. Conversely:

Proposition 0.13. *A pointed ∞-category is stable if and only if it has finite limits and colimits, and the loops and suspension functors are inverse equivalences.*

(Prove?)

Definition 0.14. An **exact functor** $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories, is a pointed functor that preserves cofiber sequences.

Likewise, such functors automatically preserve finite colimits and limits.

We'd now like to have a way to canonically turn a pointed ∞-category into a stable ∞-category. That is, we want an adjunction

$$\Sigma^\infty : \mathcal{C} \leftrightarrows \mathsf{Sp}(C) : \Omega^\infty$$

such that, for example, any finite colimit-preserving functor $\mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is stable factors canonically as $F \circ \Sigma^\infty$, where $F : \mathsf{Sp}(\mathcal{C}) \to \mathcal{D}$ is an exact functor. Note that if $\mathcal{C} \to \mathcal{D}$ is colimit-preserving, then in particular it sends cofiber sequences to cofiber sequences in $\mathcal{D}$. However, cofiber sequences in $\mathcal{D}$ are also fiber sequences. One should think here of the Mayer-Vietoris theorem: a cofibration of spaces

$$U \cap V \to U \sqcup V \to X$$

(where U and V are an open cover of X) gets sent to a *short exact sequence* of singular chain complexes

$$0 \to C_*(U \cap V) \to C_*(U) \oplus C_*(V) \to C_*(X) \to 0,$$

whose associated long exact sequence is the Mayer-Vietoris sequence. And this is both a cofiber and a fiber sequence.

Definition 0.15. A functor $F : \mathcal{C} \to \mathcal{D}$ of pointed ∞-categories is **excisive** if it sends pushout squares in $\mathcal{C}$ to pullbacks squares in $\mathcal{D}$. We write $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ for the ∞-category of pointed excisive functors.

Note that pointed excisive functors between two stable ∞-categories are just exact functors.

Definition 0.16. The **spectrum objects** of $\mathcal{C}$ are the pointed excisive functors $\mathsf{Sp}(\mathcal{C}) = \mathrm{Exc}_*(\mathsf{Top}_*^{\mathrm{fin}}, \mathcal{C})$.

Theorem 0.17. *For any pointed ∞-category $\mathcal{C}$ which has finite limits,* $\mathsf{Sp}(\mathcal{C})$ *is a stable ∞-category.*

Proof. We can compute limits objectwise, and it's not hard to see that the finite limit of a diagram of pointed excisive functors is also pointed and excisive. (Colimits?)

Now define the suspension and loops functors on $\mathsf{Sp}(\mathcal{C})$ as $\Sigma(F) = F \circ \Sigma$, $\Omega F = \Omega \circ F$. These are adjoint, and we have only to observe that $F \to \Omega F \Sigma F$ is an equivalence. But for any $X \in \mathcal{C}$, ΣX is defined by a pushout square

As F is pointed and excisive, it sends this to a pullback square in $\mathcal{D}$:

$$\begin{array}{ccc} F(X) & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & F(\Sigma X). \end{array}$$

Clearly, this square exhibits $F(X) \simeq \Omega F(\Sigma X)$. ☐

There's a map $\Omega^\infty : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C}$ given by evaluating excisive functors at the zero-sphere. (Likewise, evaluating at the n-sphere gives the 'nth space' of a spectrum object.)

Proposition 0.18. *If $\mathcal{C}$ is a pointed ∞-category with finite colimits, and $\mathcal{D}$ is a pointed ∞-category with finite limits, there's a canonical equivalence*

$$\mathrm{Exc}_*(\mathcal{C}, \mathsf{Sp}(\mathcal{D})) \simeq \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}).$$

Proof. We have

$$\mathrm{Exc}_*(\mathcal{C}, \mathsf{Sp}(\mathcal{D})) \simeq \mathrm{Exc}_*(\mathcal{C} \times \mathsf{Top}_*^{\mathrm{fin}}, \mathcal{D}) \simeq \mathsf{Sp}(\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})).$$

We can apply Ω^∞ to get a map to the ∞-category of pointed excisive functors. But by the same arguments as before, this is a stable ∞-category, so its Ω^∞ is an equivalence. ☐

Proposition 0.19. *If $\mathcal{C}$ is pointed and has finite limits and colimits, there is a natural equivalence from $\mathsf{Sp}(\mathcal{C})$ to the limit of the tower*

$$\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \cdots.$$

Proof. (Assume prsentable). Write $\overline{\mathcal{C}}$ for this limit. First, $\overline{\mathcal{C}}$ is stable. This follows from the fact that it also has finite limits and colimits and its loops functor is patently an equivalence. Thus, the map $\overline{\mathcal{C}} \to \mathcal{C}$ factors through $\mathsf{Sp}(\mathcal{C})$. If $\mathcal{D}$ is another stable ∞-category, then we have a map

$$\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{D}, \overline{\mathcal{C}}) \to \mathrm{Fun}^{\mathrm{Rex}}(\mathcal{D}, \mathsf{Sp}(\mathcal{C})) \to \mathrm{Fun}^{\mathrm{Rex}}(\mathcal{D}, \mathcal{C})$$

where the right-hand map is an equivalence. But the composition is an equivalence, too, because $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{D}, \mathcal{C})$ is stable, so its loops functor is an equivalence. ☐

Proposition 0.20. *If $\mathcal{D}$ is a presentable stable ∞-category, there's an equivalence*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathsf{Sp}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{Rex}}(\mathsf{Sp}, \mathcal{D}) \simeq \mathcal{D}$$

with the second map given by evaluating at S^0. Thus, Sp is the free stable ∞-category on one object.

Homological and cohomological Brown representability.

G-spaces

Let G be a finite group. There's an obvious candidate for the ∞-category of G-spaces: the functor category $\mathrm{Fun}(G, \mathsf{Top}_*)$. Equivalently, this is the category of presheaves on the discrete category G. However, this doesn't have the right equivalences! The most obvious example is the map

$$EG \to *.$$

This is a G-equivariant map and an equivalence in spaces, if we forget the G-action. However, it's not an equivalence of G-spaces, since it doesn't have a G-equivariant homotopy inverse. For example, if we take the G-quotients, we get $BG \to *$, which is not an equivalence if G is nontrivial. Moreover, the map doesn't even have any sections. The reason why is that the fixed points EG^G are empty, while $(*)^G = *$.

One of the key realizations was that if you fix the second problem, you also fix the first.

Theorem 0.21 (Equivariant Whitehead theorem). *A map $X \to Y$ of G-spaces is a G-weak equivalence if and only if, for every subgroup $H \leq G$, the map of spaces $X^H \to Y^H$ is a weak equivalence of spaces.*

As a corollary, equivalences of G-spaces are detected by the equivariant homotopy groups

$$\pi_n^H(X) = \pi_n(X^H).$$

Thus, the correct homotopy theory for G-spaces is one that takes into account the entire diagram of fixed point spaces. Define the **orbit category** of G to be the category of finite transitive G-sets (of the form G/H) and G-equivariant maps.

Theorem 0.22 (Elmendorf). *There's an equivalence of ∞-categories between*

$$\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathsf{sSet}^G) \simeq \mathsf{Top}_*^G.$$

G-spectra

. Hill-Hopkins-Ravenel state requirements on a symmetric monoidal model category of G-spectra. I've translated those requirements into ∞-categorical language. We return to our list of properties of spectra: one of them was that mapping spaces accurately computed stable homotopy groups of finite spaces. Likewise, we can define:

Definition 0.23. The G-**Spanier-Whitehead category**, $\mathcal{SW}^G$, is the category whose objects are finite G-CW-complexes, and with

$$\{X, Y\}^G = \mathrm{colim}_V \, \mathrm{ho}\mathsf{Top}_*^G(S^V \wedge X, S^V \wedge Y).$$

Here the colimit runs over the partially ordered set of finite-dimensional G-representations embedded in some fixed infinite-dimensional G-representation with the property that it contains every irreducible G-representation infinitely many times.

1. There is an adjunction $\Sigma^\infty : \mathsf{Top}^G \leftrightarrows \mathsf{Sp}^G : \Omega^\infty$ of ∞-categories.
2. Sp^G is symmetric monoidal, and Σ^∞ is a symmetric monoidal functor.
3. Σ^∞ induces a functor $\mathrm{ho}\mathsf{Top}^G \to \mathrm{ho}\mathsf{Sp}^G$ that factors through a fully faithful, SM embedding $\mathcal{SW}^G \hookrightarrow \mathsf{Sp}^G$.
4. S^V is invertible in Sp^G.
5. Infinite wedges exist in Sp^G and compute coproducts in the homotopy category.
6. Every X has a presentation $\mathrm{colim}\, S^{-V_n} \wedge X_n$.

Stabilizing Top^G as above gives us a stable ∞-category. This is symmetric monoidal, and has a Ω^∞ functor, and a Σ^∞ functor since Top^G is presentable. However, it does not satisfy the rest of the properties above. The problem is that inverting suspension does not automatically invert equivariant suspensions – we can't smash a representation sphere S^V with something else and get back a G-fixed sphere.

The quickest way to fix this is to localize with $\mathsf{Sp}(\mathsf{Top}^G)$ with respect to maps of the form

$$X \to \Omega^V \Sigma^V(X).$$

Localization for ∞-categories works just like it does for model categories. An object of Sp^G will be an object Y of $\mathsf{Sp}(\mathsf{Top}^G)$ with the property that, for any map f of this form,

$$\mathsf{Sp}(\mathsf{Top}^G)(\Omega^V \Sigma^V X, Y) \to \mathsf{Sp}(\mathsf{Top}^G)(X, Y)$$

is an equivalence. Since $\mathsf{Sp}(\mathsf{Top}^G)$ is presentable, the inclusion of Sp^G into $\mathsf{Sp}(\mathsf{Top}^G)$ has a left adjoint. Now it should be clear that V-loops and V-suspension are adjoint equivalences on this new category.

An object of Sp^G is then built from a sequence of 'spaces'

$$X_V = \mathsf{Sp}^G(S^V, X).$$

In particular, we automatically have homotopy groups which are graded by the representation ring of G. We likewise have Spanier-Whitehead duality, etc.