The Steenrod algebra

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January 25, 2016

References are the first few chapters of Mosher and Tangora, and if you can read French, Serre's 'Cohomologie modulo 2 des complexes d'Eilenberg-MacLane' (1953). Hatcher §4.L is OK.

1 Eilenberg-Mac Lane spaces

Definition 1.1. Let $n \geq 1$ and let A be a group, which is abelian if $n \geq 2$. An **Eilenberg-Mac Lane space** $K(A, n)$ is a space whose only nonzero homotopy group is $\pi_n K(A, n) = A$.

We've seen, but not proved, a few important properties of these things:

1. They exist.
2. A little more interesting: they're unique up to homotopy equivalence, so we're justified in speaking of 'the Eilenberg-Mac Lane space $K(A, n)$'.
3. We have $\Omega K(A, n) \simeq K(A, n-1)$.
4. Most importantly today, $K(A, n)$ represents the functor degree n cohomology with coefficients in A on the homotopy category. This means that for any space X, there's an isomorphism

   $$[X, K(A, n)] \cong H^n(X; A)$$

   (and the source has a natural abelian group structure). This isomorphism is natural in X, meaning that a map $f : X \to Y$ induces a diagram

   $$\begin{array}{ccc} [Y, K(A,n)] & \xrightarrow{\cong} & H^n(Y; A) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ [X, K(A,n)] & \xrightarrow[\cong]{} & H^n(X; A) \end{array}$$

   and this square commutes.

Representing a functor by maps into an object is a really useful technique that I think originated in algebraic topology (and is in turn the starting point of category theory). We've seen one example of this already: we realized we could study real vector bundles through the object BO that represented them, and in particular, define cohomology classes of vector bundles as cohomology elements of BO. Naturality of the Stiefel-Whitney classes came for free from this definition.

Today, we'll study the properties of cohomology through the lens of representability. I'll spend the rest of the talk on mod 2 cohomology, and just write this H^*, though similar things work for mod p cohomology. The mod 2 cohomology of a space is a graded commutative ring, which is a lot of structure already. Is there more structure floating around? One way to make this question precise is to ask: what are all the natural transformations

$$H^n X \to H^m X,$$

also known as **cohomology operations**?

Example 1.2. The cup-product square, $H^nX \to H^{2n}X$, is a cohomology operation. (It's a group homomorphism because we're mod 2!)

By the Yoneda lemma, natural transformations between these two functors are the same as maps between their representing objects. So I'm asking for the set of homotopy classes of maps

$$[K(\mathbb{F}_2, n), K(\mathbb{F}_2, m)]$$

which is the same as the cohomology group

$$H^m(K(\mathbb{F}_2, n)).$$

We could compute this right now, as a matter of fact, using the Serre spectral sequence, Borel's theorem on simple systems of generators, and the fact that $K(\mathbb{F}_2, 1) = \mathbb{R}P^\infty$, whose cohomology we know. It's a nice exercise to compute it for the first few values of n. As you'll see, it's a little hard to write it down without better notation.

So instead, let's remember another piece of structure on homology: there's a suspension isomorphism

$$H^nX \cong H^{n+1}\Sigma X.$$

In terms of the representing object, this is an isomorphism

$$[X, K(\mathbb{F}_2, n)] \cong [\Sigma X, K(\mathbb{F}_2, n+1)].$$

The right-hand side is the same as $[X, \Omega K(\mathbb{F}_2, n+1)]$, so the suspension isomorphism is induced by the homotopy equivalence $K(\mathbb{F}_2, n) \simeq \Omega K(\mathbb{F}_2, n+1)$. Now instead of asking for all cohomology operations, let's ask for stable cohomology operations which are **stable**. A degree k stable cohomology operation is a natural transformation

$$H^nX \to H^{n+k}X$$

for all n that commutes with the suspension isomorphisms.

Example 1.3. The cup-product square for varying n is not stable. Suspending the map $H^nX \to H^{2n}X$ gives a map $H^{n+1}\Sigma X \to H^{2n+1}\Sigma X$, which isn't even the right degree. In fact, on $H^*\Sigma X$, all cup products vanish! (Exercise, using Mayer-Vietoris, if you haven't seen this.) A natural question at this point is: is there a stable cohomology operation, $H^nX \to H^{n+k}X$ for varying n, that is the cup product square on H^k?

Again using representability, the set of stable cohomology operations will be the colimit of the groups $[K(\mathbb{F}_2, n), K(\mathbb{F}_2, n+k)]$ along the maps

$$\begin{aligned}[K(\mathbb{F}_2, n), K(\mathbb{F}_2, n+k)] &\cong [\Omega K(\mathbb{F}_2, n+1), \Omega K(\mathbb{F}_2, n+k+1)] \\ &\cong [\Sigma\Omega K(\mathbb{F}_2, n+1), K(\mathbb{F}_2, n+k+1)] \to [K(\mathbb{F}_2, n+1), K(\mathbb{F}_2, n+k+1)].\end{aligned}$$

Equivalently, I want the colimit

$$\mathrm{colim}_n\ H^{n+k}K(\mathbb{F}_2, n)$$

along the maps

$$K(\mathbb{F}_2, n+1) \to \Sigma\Omega K(\mathbb{F}_2, n+1) \simeq \Sigma K(\mathbb{F}_2, n).$$

This colimit is the group of degree-n stable cohomology operations. Taking this for all n, we get a graded ring (I can compose two stable cohomology operations to get another one). This ring is the Steenrod algebra.

Remark 1.4. I've written this down as a colimit of cohomologies, but it might be better to think of it as the cohomology of some kind of *limit* of suspensions of Eilenberg-Mac Lane spaces along the maps above. This limit is what we'll later call an 'Eilenberg-Mac Lane spectrum'.

2 Constructing the Steenrod operations

The way I've just described it, the natural next thing to do would be to compute this cohomology ring. AgaIn, it's possible to do this blind, but knowing the Steenrod operations beforehand makes it practically much, much easier. Historically, people instead constructed the Steenrod operations geometrically and then used them to find cohomology of Eilenberg-Mac Lane spaces. I want to give you an idea of how this works because the same method is still useful today to give analogous operations, so-called 'power operations', in other cohomology theories. For references, see chapter 2 of Mosher-Tangora and this MathOverflow answer by Charles Rezk.

I'll write $K = \prod_n K(\mathbb{F}_2, n)$, so an arbitrary formal sum a of cohomology classes of X is represented by an element of $[X, K]$. The cup-product square of a is represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{a} K \times K \to K.$$

Here the last map is some kind of multiplication map on K that represents the cup product. Now, since the cup product is associative and graded-commutative on cohomology, this multiplication is associative and commutative *up to homotopy.* If it were commutative on the nose, we'd have a factorization

$$\begin{array}{ccccccc} X & \longrightarrow & X \times X & \longrightarrow & K \times K & \longrightarrow & K. \\ & & \downarrow & & & \nearrow & \\ & & (X \times X)/\Sigma_2 & & & & \end{array}$$

Since it's only commutative up to homotopy, we can factor down to the quotient space only after we've added extra coordinates that allow us to make a Σ_2-equivariant homotopy from (x, y) to (y, x). If you think about this for a while, you realize that you need to multiply $X \times X$ by a contractible space with a free Σ_2-action. Such spaces are often coyly written $E\Sigma_2$, but in this case, there's a really simple geometric model: the space S^∞, the colimit of the spheres S^n along the maps that include each sphere as the equator of the next. (Exercise: why is this contractible?) The antipodal map is a free Σ_2-action. So there's a diagram

$$\begin{array}{ccccccc} X & \longrightarrow & X \times X & \longrightarrow & K \times K & \longrightarrow & K, \\ & & \downarrow & & & \nearrow & \\ & & (X \times X \times S^\infty)/\Sigma_2 & & & & \end{array}$$

and the space on the bottom is $X \times \mathbb{R}P^\infty$. Thus, from a cohomology class of X, we've produced an element of

$$H^*(X \times \mathbb{R}P^\infty) = H^*X \otimes H^*\mathbb{R}P^\infty = H^*X[t],$$

and the degree zero term, which is what you get by pulling back along the inclusion

$$X \to X \times X \times S^\infty \to X \times \mathbb{R}P^\infty,$$

is the cup-product square of our original class. The higher-degree coefficients are the Steenrod squares of the original class.

The upshot of all this is that the cup product, which is commutative on cohomology, is only commutative up to homotopy on the chain level. These homotopies produce artifacts in lower-degree cohomology, and these artifacts are the lower Steenrod squares.

3 The Steenrod algebra

Unfortunately, I'm not going to start calculating this, either, though Philip might do some on Wednesday. Instead, I'll write down a bunch of algebraic properties of the Steenrod algebra.

Theorem 3.1. *There are a unique set of operations* $\mathrm{Sq}^i : H^n(X, Y) \to H^{n+i}(X, Y)$*, called the* ***Steenrod squares****, that satisfy the following properties:*

1. *The squares are homomorphisms of abelian groups, natural in X, and commute with suspension. They also commute with long exact sequences of cohomology.*
2. Sq^0 *is the identity,* $\mathrm{Sq}^{|x|}(x) = x^2$, *and* $\mathrm{Sq}^i(x) = 0$ *for* $i > |x|$.
3. $\mathrm{Sq}^1 : H^n(X;\mathbb{F}_2) \to H^{n+1}(X;\mathbb{F}_2)$ *is the connecting homomorphism in the long exact sequence*

$$\cdots \to H^n(X;\mathbb{F}_2) \to H^n(X;\mathbb{Z}/4) \to H^n(X;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^1} H^{n+1}(X;\mathbb{F}_2) \to \cdots$$

induced by the short exact sequence of coefficients

$$0 \to \mathbb{F}_2 \to \mathbb{Z}/4 \to \mathbb{F}_2 \to 0.$$

(This map is also called a ***Bockstein homomorphism****.)*

4. *The squares satisfy a Cartan formula:*

$$\mathrm{Sq}^k(xy) = \sum_{i+j=k} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(x).$$

Equivalently, the 'total square' $\mathrm{Sq} = \sum_k \mathrm{Sq}^k$ *is a ring homomorphism.*

5. *There are* ***Adem relations*** *describing the composition of two squares: if* $a < 2b$, *then*

$$\mathrm{Sq}^a\,\mathrm{Sq}^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c}\,\mathrm{Sq}^c.$$

Finally, the Steenrod algebra $\mathcal{A}$ *is the quotient of the free (noncommutative)* $\mathbb{F}_2$*-algebra on generators* Sq^i *by the Adem relations.*

Remark 3.2. At odd primes, we write $\mathcal{P}^i$ for the ith 'Steenrod pth power', and there's a very similar theorem. The biggest difference is that if x has odd degree, then $x^p = 0$, so we should only be trying to reduce the pth power map on even degree classes. People generally write $\mathcal{P}^n$ for an operation that increases degrees by $2p(n-1)$ – so it's the pth power on degree $2n$ – and β for the Bockstein, which increases degrees by 1 and gives all the odd degree operations. This makes the formulas much more complicated. If you read about this you should think about β as the analogue of Sq^1 and $\mathcal{P}^n$ as the analogue of Sq^{2n}.

Remark 3.3. We've seen three ways to think about the Steenrod algebra. There's this axiomatic definition of the squares, which is all you need if you're most interested in messing around with cohomology classes, as we'll do in a second. These squares generate an algebra of cohomology operations, which turn out to be all of the stable cohomology operations – the way we started the discussion. (In fact, as we might see from Philip tomorrow, the algebras of unstable operations are all quotients of the Steenrod algebra.) Finally, the geometric approach had us finding Steenrod squares as artifacts of the homotopy-commutativity of the cup-product square, i. e. as power operations. In general, power operations and cohomology operations are not the same concept!

This big, infinite-dimensional algebra acting on cohomology is a huge amount of structure, and can be used to distinguish homotopy types, obstruct various maps and homotopy types, and exhume hidden relations.

Example 3.4. $\mathbb{C}P^2$ has integral cohomology $\mathbb{Z}[x]/(x^3)$ where $|x| = 2$. Suspending it gives a space with a 3-cell and a 5-cell and thus cohomology in degrees 3 and 5, and all cup products vanish. Is this space $S^3 \vee S^5$? We can't tell from the cohomology ring alone. However, the mod 2 cohomology groups are generated by elements Σx and $\Sigma(x^2)$, and stability of the Steenrod squares shows that $\mathrm{Sq}^2(\Sigma x) = \Sigma(\mathrm{Sq}^2 x) = \Sigma(x^2)$. This is not true in $S^3 \vee S^5$: there's a map $S^3 \vee S^5 \to S^3$ inducing $H^*S^3 \to H^*(S^3 \vee S^5)$, which is an isomorphism on H^3, and all Steenrod squares vanish on the generator of H^3S^3.

For the same reason, no suspension of $\mathbb{C}P^2$ is a wedge of spheres. Another way of saying this is that the map attaching the 4-cell of $\mathbb{C}P^2$ to the 2-cell doesn't become nullhomotopic after suspending. This is a map $S^3 \to S^2$ – in fact, it's the Hopf fibration. So we've just shown that $\pi_{n+1}S^n \neq 0$ for all $n \geq 2$, which is to say that the first stable homotopy group of spheres is nonzero.

Example 3.5. Here's an example of how we can use the Adem relations. Let n be a number that's not a power of 2, let $b = 2^k$ be the largest power of 2 less than n, and let $a = n - b$. Then $a < 2b$, so there's an Adem relation

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum \binom{b-c-1}{a-2c} \mathrm{Sq}^{n-c} \mathrm{Sq}^c .$$

The coefficient when $c = 0$ is

$$\binom{1 + 2 + \cdots + 2^{k-1}}{a_0 + 2a_1 + \cdots + 2^{k-1}a_{k-1}}$$

where the bottom is the base 2 expansion of a. Exercise: the mod 2 binomial coefficient

$$\binom{b_0 + 2b_1 + \cdots + 2^i b_i}{a_0 + 2a_1 + \cdots + 2^i a_i} = \binom{b_0}{a_0}\binom{b_1}{a_1}\cdots\binom{b_i}{a_i}.$$

So this coefficient is 1. This means that there's a relation

$$\mathrm{Sq}^n = \sum \mathrm{Sq}^i \mathrm{Sq}^j,$$

i. e., if n isn't a power of 2, the nth square is decomposable into smaller squares.

In particular:

Theorem 3.6. *The Steenrod algebra is generated by the elements* Sq^{2^k}.

Corollary 3.7. *If X is a space with $H^*X \cong \mathbb{F}_2[x]$ as a ring, then $|x|$ is a power of 2.*

Proof. If $|x| = n$, then $\mathrm{Sq}^n(x) = x^2 \neq 0$, but all lower squares are zero. So Sq^n can't be decomposable, meaning n must be a power of 2. □

Remark 3.8. By making the same argument at odd primes, you can show that if $H^*(X, \mathbb{Z}) \cong \mathbb{Z}[x]$, then $|x| = 2$ or 4. In fact, if $H^*X \cong \mathbb{F}_2[x]$, then $|x|$ is 1, 2, 4, or 8. This follows from Adams and Atiyah's proof of the Hopf invariant one theorem ('K-theory and the Hopf invariant', 1960), and 8 is ruled out from the integral case by work of Toda ('Note on the cohomology ring of certain spaces', 1963). Similar results hold mod odd primes. I don't know if the obvious projective spaces are the only spaces with these cohomology rings.