Stability, Consistency and Convergence

We wish to consider the 'stability' of the numerical method with respect to 'small perturbations' in the starting conditions.

I. Zero-stability

We start with the abstract concept of 0-stability (0 refers to the stability when $\Delta t \to 0$)

Example. Consider the LMM

$$U^{n+2} - 3U^{n+1} + 2U^{n} = -\Delta t\, f(U^{n})$$

LTE

$$\tau^{n} = \frac{1}{\Delta t}\left[u(t_{n+2}) - 3u(t_{n+1}) + 2u(t_n) + \Delta t\, u'(t_n) \right]$$

$$= \frac{5}{2}\Delta t\, u''(t_n) + O(\Delta t^2).$$

Trivial IVP

$$u'(t) = 0, \quad u(0) = 0.$$

$$\Rightarrow U^{n+2} - 3U^{n+1} + 2U^{n} = 0$$

① take $U^0 = U^1 = 0$. then $U^n = 0$

② $U^0 = 0$, $U^1 = \Delta t$

$$U^n = 2U^0 - U^1 + 2^n (U^1 - U^0) \to \infty \text{ as } n \to \infty.$$

Def. A linear r-step method for the ODE $u' = f(u,t)$ is said to be <u>zero-stable</u> if there exists a constant K such that, for any two sequences $\{U^n\}$ and $\{\hat{U}^n\}$ which have been generated by the same scheme but different initial data $\{U^0, \dots, U^{r-1}\}$, $\{\hat{U}^0, \dots, \hat{U}^{r-1}\}$ respectively, we have

$$|U^n - \hat{U}^n| \le K \max\{|U^0 - \hat{U}^0|, \dots, |U^{r-1} - \hat{U}^{r-1}|\}$$

for $t_n = n\Delta t \le T$ and as $\Delta t \to 0$.

It would be a very tedious exercise to verify 0-stability of a LMM using the above definition.

Examples. 1. Apply the consistent LMM

$$U^{n+2} - 2U^{n+1} + U^n = \frac{\Delta t}{2}\left(f(U^{n+2}) - f(U^n)\right)$$

to $u' = 0$.

$$U^{n+2} - 2U^{n+1} + U^n = 0$$

characteristic poly.

$$\rho(x) = x^2 - 2x + 1 = (x-1)^2$$

$$\Rightarrow \quad x_1 = x_2 = 1.$$

The general soln.

$$U^n = C_1 + C_2 n = U^0 + (U^1 - U^0) n.$$

$$\lim_{\substack{\Delta t \to 0 \\ N\Delta t = T}} U^N = \Delta t N = T \quad , \quad \text{with } U^0 = 0, \ U^1 = \Delta t.$$

If $f(x)$ has repeated roots of modulus 1, the method cannot be convergent.

2. Consider the LMM

$$U^{n+3} - 2U^{n+2} + \frac{5}{4} U^{n+1} - \frac{1}{4} U^n = \frac{\Delta t}{4} f(U^n)$$

the characteristic poly.

$$f(x) = x^3 - 2x^2 + \frac{5}{4} x - \frac{1}{4} = (x-1)(x-\frac{1}{2})^2$$

$$\Rightarrow x_1 = 1, \quad x_2 = x_3 = \frac{1}{2}.$$

(repeated roots with modulus less than 1)

general soln.

$$U^n = C_1 + C_2 (\frac{1}{2})^n + C_3 n (\frac{1}{2})^n$$

C_1, C_2, C_3 will be linear combinations of U^0, U^1, U^2.

Then $\lim_{\substack{\Delta t \to 0 \\ N\Delta t = T}} U^N = 0.$

Theorem. (root condition) An r-step LMM is zero-stable for any ODE $u' = f$ where f satisfies the Lipschitz condition if and only if the roots of the characteristic poly. $\rho(x)$ satisfies the condition

$$|x_i| \le 1, \quad \text{for } i = 1, 2, \dots, r$$

$$\text{if } x_i \text{ is a repeated root, then } |x_i| < 1.$$

Proof. '$\Rightarrow$'

Consider the r-step LMM method, applied to $u' = 0$

$$a_r U^{n+r} + a_{r-1} U^{n+r-1} + \dots + a_1 U^{n+1} + a_0 U^n = 0$$

The general soln has the form

$$U^n = \sum_s p_s(n) x_s^n$$

where x_s is a root of $\rho(x)$.

- If $|x_s| > 1$, then there is a starting value s.t. the soln. grows like $|x_s|^n$.
- If $|x_s| = 1$ and multiplicity $m_s > 1$, then there is soln. grows like $n^{m_s - 1}$.

To summarize, if the root condition is violated then the method is not zero-stable.

'<=' The proof is long. See Refs.

Example. The Adams methods

$$U^{n+r} = U^{n+r-1} + \Delta t \sum_{j=1}^{r} \beta_j f(U^{n+j})$$

$$\rho(x) = x^r - x^{r-1} = (x-1)x^{r-1}$$

The root condition is satisfied and all Adams-Bashforth & Adams-Moulton methods are zero-stable.

II. Consistency

The LTE

$$\tau^n = \frac{\sum_{j=0}^{r} [\alpha_j u(t_{n+j}) - \Delta t \beta_j u'(t_{n+j})]}{\Delta t \sum_{j=0}^{r} \beta_j}$$

Def. The LMM scheme is said to be consistent if the truncation error τ^n is such that for any $\varepsilon > 0$ there exists $h(\varepsilon)$ for which

$$|\tau^n| < \varepsilon \quad \text{for } 0 < \Delta t < h(\varepsilon)$$

and any $(k+1)$-points $(t_n, u(t_n)), \dots, (t_{n+r}, u(t_{n+r}))$ on any solution curve of the IVP.

Remind for consistency $\tau^n \to 0$ as $\Delta t \to 0$, one requires

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1) \neq 0$$

($x=1$ must be a root !)

Def. The LMM scheme is said to have order of p if p is the longest pos. integer such that for any sufficiently smooth solution curve of the IVP, there exists constants K and h_0 s.t.

$$|\tau^n| \le K \Delta t^p \quad \text{for } 0 < \Delta t < h_0$$

for any $(k+1)$-points $(t_n, u(t_n)), \dots, (t_{n+r}, u(t_{n+r}))$ on any solution curve.

$$\tau^n = \frac{1}{\Delta t\, \sigma(1)} \left[C_0 u(t_n) + C_1 \Delta t\, u'(t_n) + C_2 \Delta t^2 u''(t_n) + \cdots \right]$$

$$C_0 = \sum_{j=0}^{r} \alpha_j, \quad C_1 = \sum_{j=1}^{r} j\alpha_j - \sum_{j=0}^{r} \beta_j$$

$$C_2 = \sum_{j=1}^{r} \frac{j^2}{2!}\alpha_j - \sum_{j=0}^{r} j\beta_j$$

...

$$C_q = \sum_{j=1}^{r} \frac{j^q}{q!}\alpha_j - \sum_{j=0}^{r} \frac{j^{q-1}}{(q-1)!}\beta_j$$

The LMM is of order of accuracy p if and only if

$$C_0 = C_1 = \cdots = C_p = 0 \quad \text{and} \quad C_{p+1} \neq 0$$

$$\tau^n = \frac{C_{p+1}}{\sigma(1)} \Delta t^p u^{(p+1)}(t_n) + O(\Delta t^{p+1})$$

C_{p+1} $(\neq 0)$ is called the error constant.

Example. Construct an implicit linear 2-step method of maximum order.

The method has the form

$$U^{n+2} + \alpha_1 U^{n+1} + a U^n = \Delta t (\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n)$$

with $\alpha_0 = a$.

$$C_0 = 1 + \alpha_1 + a \qquad = 0$$

$$C_1 = \alpha_1 + 2 - (\beta_0 + \beta_1 + \beta_2) \qquad = 0$$

$$C_2 = \frac{1}{2}(\alpha_1 + 4) - (\beta_1 + 2\beta_2) \qquad = 0$$

$$C_3 = \frac{1}{3!}(\alpha_1 + 8) - \frac{1}{2}(\beta_1 + 4\beta_2) \quad = 0$$

Hence $\alpha_1 = -1 - a$

$$\beta_0 = -\frac{1}{12}(1 + 5a) \quad \beta_1 = \frac{2}{3}(1 - a) \quad \beta_2 = \frac{1}{12}(5 + a)$$

$$U^{n+2} - (a+1)U^{n+1} + aU^n = \frac{\Delta t}{12}\left[(5+a)f_{n+2} + 8(1-a)f_{n+1} - (1+5a)f_n\right]$$

$$C_4 = -\frac{1}{4!}(1+a) \qquad C_5 = -\frac{4}{3\cdot 5!} \quad (a=-1)$$

If $a \neq -1$, $C_4 \neq 0 \Rightarrow$ 3rd-order accuracy

If $a = -1$, $C_4 = 0$, $C_5 \neq 0 \Rightarrow$ 4th-order accuracy

(Simpson's rule)

III. Convergence

Def. The LMM scheme is said to be convergent if we have for the IVP

$$\lim_{\substack{\Delta t \to 0 \\ n\Delta t = t}} U^n = u(t)$$

holds for all $t \in [0, T]$ and for all solutions $\{U^n\}_{n=0}^{N}$ with consistent starting condition.

The starting condition

$$U^s = \eta_s(\Delta t) \quad s = 0, 1, \cdots, r-1 \quad \text{with}$$

$$\lim_{\Delta t \to 0} \eta_s(\Delta t) = U^0, \quad s = 0, \cdots, r-1$$

a) Necessary Conditions

Lemma 1. A necessary condition for the convergence of LMM is that it be zero-stable.

Proof.

Consider the IVP

$$u'(t) = 0, \quad u(0) = 0 \quad \text{on } t \in [0, T] \quad T > 0$$

The problem yields the difference equation

$$a_r U^{n+r} + a_{r-1} U^{n+r-1} + \cdots + a_0 U^n = 0 \qquad (1)$$

The method is assumed to be convergent, then

$$\lim_{\substack{\Delta t \to 0 \\ n\Delta t = t}} U^n = 0 \qquad (2)$$

for all $U^s = \eta_s(\Delta t)$, $s = 0, \dots, r-1$ with

$$\lim_{\Delta t \to 0} \eta_s(\Delta t) = 0, \quad s = 0, \dots, r-1.$$

Let $x = re^{i\varphi}$ be a root of the characteristic poly. $f(x)$

then $U^n = \Delta t\, r^n \cos n\varphi$ defines a soln. to (1)

and satisfies (2).

If $\varphi \neq 0, \pi$

$$\frac{(U^n)^2 - U^{n+1}U^{n-1}}{\sin^2\varphi} = \Delta t^2 \gamma^{2n}$$

LHS $\to 0$ as $\Delta t \to 0$, $n \to \infty$, $n\Delta t = t$

$\Rightarrow$ RHS $= (\frac{t}{n})^2 \gamma^{2n} \to 0$

$\Rightarrow$ $\gamma \le 1$.

Next, assume $x = re^{i\varphi}$ is a multiple root of $f(x)$ with $|r| = 1$. Again

$$U^n = (\Delta t)^{\frac{1}{2}} n \gamma^n \cos(n\varphi)$$

defines a soln. to (1) and satisfies (2).

($|\eta_s(\Delta t)| = |U^s| \le (\Delta t)^{\frac{1}{2}} s \le (\Delta t)^{\frac{1}{2}} (r-1)$)

i) $\varphi = 0, \pi$

$$|U^n| = t^{\frac{1}{2}} n^{\frac{1}{2}} \gamma^n \to \infty \quad \text{as} \quad n \to \infty$$

Contradiction!

ii) $\varphi \neq 0, \pi$

$$\frac{(V^n)^2 - V^{n+1}V^{n-1}}{\sin^2\varphi} = \gamma^{2n}$$

where $V^n = n^{-1} \Delta t^{-1} U^n = \Delta t^{\frac{1}{2}} t^{-1} U^n$

$\lim\limits_{n\to\infty} V^n = 0 \Rightarrow$ LHS $\to 0$ as $n \to \infty$.

$\Rightarrow$ RHS $\to 0$ as $n \to \infty$.

Contradiction!

Lemma 2. A necessary condition for the convergence of LMM is that it be consistent.

(It is to show $\rho(1) = 0$, i.e. $C_0 = 0$
& $\rho'(1) = \sigma(1)$, i.e. $C_1 = 0$)

Proof.

i) $C_0 = 0$:

Consider the IVP $u'(t) = 0$, $u(0) = 1$ (that is, $u \equiv 1$)

The problem yields the difference equation

$$a_r U^{n+r} + a_{r-1} U^{n+r-1} + \cdots + a_0 U^n = 0 \qquad (1)$$

Suppose 'exact' starting values $U^s = 1$, $s = 0, \cdots, r-1$

Given the method is convergent

$$\lim_{\substack{\Delta t \to 0 \\ n\Delta t = t}} U^n = 1$$

and in this case U^n is indepent of Δt

$$\lim_{n \to \infty} U^n = 1.$$

Passing the limit $n \to \infty$ to (1) gives

$$\alpha_r + \alpha_{r-1} + \cdots + \alpha_0 = 0$$

$$\Rightarrow \quad f(1) = 0 .$$

ii) Consider the IVP

$$u'(t) = 1, \quad u(0) = 0 \qquad (u(t) = t)$$

The difference equation becomes

$$\alpha_r U^{n+r} + \cdots + \alpha_0 U^0 = \Delta t (\beta_r + \cdots + \beta_0) \qquad (2)$$

with $N\Delta t = T$ and $1 \le n \le N-r$.

For a Convergent method

$$\lim_{\Delta t \to 0} \eta_s(\Delta t) = 0 \qquad s=0, \ldots, r-1 \qquad (3)$$

and $\displaystyle\lim_{\substack{\Delta t \to 0 \\ n\Delta t = t}} U^n = t$.

Since 0-stability is necessary for Convergence, $f(x)$ must not have a multiple root of $|x| = 1$.

$$f'(x) = r\alpha_r + \cdots + 2\alpha_2 + \alpha_1 \neq 0$$

Let $U^n = Rn\Delta t$ where

$$R = \frac{\beta_r + \cdots + \beta_1}{r\alpha_r + \cdots + \alpha_1}$$

The solution $\{U^n\}_{n=0}^{N}$ Satisfies (2) & (3)

Then $\displaystyle t = u(t) = \lim_{\substack{\Delta t \to 0 \\ n\Delta t = t}} U^n = \lim_{\substack{\Delta t \to 0 \\ n\Delta t = t}} Rn\Delta t = Rt$

$\Rightarrow \quad R=1 \quad \Rightarrow \quad C_1 = (r\alpha_r + \cdots + \alpha_1) - (\beta_r + \cdots + \beta_1) = 0$

$\Rightarrow \quad \rho'(1) = \sigma(1)$.

b) Sufficient Conditions

Fact. Define γ_ℓ, $\ell = 0, 1, 2, \cdots$

$$\frac{1}{\alpha_r + \cdots + \alpha_1 x^{r-1} + \alpha_0 x^r} = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \cdots$$

Then $\quad \Gamma \equiv \sup_{\ell \geq 0} |\gamma_\ell| < \infty$.

We apply the above fact to estimate soln. of the linear difference eqn.

$$\alpha_r e^{m+r} + \cdots + \alpha_0 e^m = \Delta t (\beta_{r,m} e^{m+r} + \cdots \beta_{0,m} e^m) + \lambda_m \quad (*)$$

Lemma 1. Suppose all roots of $\rho(x)$ lie in the closed unit disk $|x| \leq 1$ and those on the unit circle $|x| = 1$ are simple. Let B and Λ be non-negative constants and β a positive constant s.t.

$$|\beta_{r,n}| + \cdots + |\beta_{0,n}| \leq B$$

$$|\beta_{r,n}| \leq \beta, \quad |\lambda_n| \leq \Lambda$$

$n = 0, \cdots, N$.

and $0 \le \Delta t < |\alpha_r| \beta^{-1}$. Then every soln. of (*) for which

$$|e^s| \le E, \quad s = 0, \cdots, r-1$$

satisfies

$$|e^n| \le K \exp(n \Delta t L), \quad n = 0, \cdots, N$$

where

$$L = \tilde{\Gamma} B, \quad K = \tilde{\Gamma}(N\Lambda + \Lambda E r)$$

$$\tilde{\Gamma} = \Gamma / (1 - \Delta t |\alpha_r|^{-1} \beta),$$

Γ is defined before.

Proof. Fix r.

Multiply both sides of (*) by γ_ℓ for $m = n - r - \ell$ with $\ell = 0, \cdots, n-r$ and denote the summing equ. by S_n

$$\begin{aligned}
S_n &= (\alpha_r e^n + \alpha_{r-1} e^{n-1} + \cdots + \alpha_0 e^{n-r}) \gamma_0 && \ell=0 \\
&+ (\alpha_r e^{n-1} + \alpha_{r-1} e^{n-2} + \cdots + \alpha_0 e^{n-r-1}) \gamma_1 && \ell=1 \\
&+ \cdots \\
&+ (\alpha_r e^r + \alpha_{r-1} e^{r-1} + \cdots + \alpha_0 e^0) \gamma_\ell && \ell = n-r \\
&= \alpha_r \gamma_0 e^n + (\alpha_r \gamma_1 + \alpha_{r-1} \gamma_0) e^{n-1} + \cdots \\
&+ (\alpha_r \gamma_{n-r} + \alpha_{r-1} \gamma_{n-r-1} + \cdots + \alpha_0 \gamma_{n-2r}) e^r \\
&+ (\alpha_{r-1} \gamma_{n-r} + \cdots + \alpha_0 \gamma_{n-2r+1}) e^{r-1} \\
&+ \cdots \\
&+ \alpha_0 \gamma_{n-r} e^0 .
\end{aligned}$$

Define $\gamma_l = 0$ for $l < 0$ and note

$$\alpha_r \gamma_l + \alpha_{r-1}\gamma_{l-1} + \cdots + \alpha_0 \gamma_{l-k} = \begin{cases} 1, & l=0 \\ 0, & l>0 \end{cases}$$

thus

$$S_n = e^n + (\alpha_{r-1}\gamma_{n-r} + \cdots + \alpha_0 \gamma_{n-2r+1}) e^{r-1}$$
$$+ \cdots + \alpha_0 \gamma_{n-r} e^0$$

$$= \Delta t \, [\, \beta_{r,n-r} \gamma_0 e^n + (\beta_{r-1,n-r}\gamma_0 + \beta_{r,n-r-1}\gamma_1) e^{n-1}$$
$$+ \cdots$$
$$+ (\beta_{0,n-r}\gamma_0 + \cdots + \beta_{r,n-2r}\gamma_r) e^{n-r} + \cdots + \beta_{0,0}\gamma_{n-r} e^0 \,]$$

$$+ (\lambda_{n-r}\gamma_0 + \lambda_{n-r-1}\gamma_1 + \cdots + \lambda_0 \gamma_{n-r}).$$

Noting that $\gamma_0 = \alpha_r^{-1}$

$$|e^n| \le \Delta t \beta |\alpha_r^{-1}| |e^n| + \Delta t \Gamma B \sum_{m=0}^{n-1} |e_m|$$
$$+ N\Gamma\Lambda + A\Gamma E_r$$

with $A = |\alpha_r| + |\alpha_{r-1}| + \cdots + |\alpha_0|$.

$$\Rightarrow (1 - \Delta t \beta |\alpha_r^{-1}|) |e^n| \le \Delta t \Gamma B \sum_{m=0}^{n-1} |e^m| + N\Gamma\Lambda + A\Gamma E_r$$

$$\Rightarrow |e^n| \le K + \Delta t L \sum_{m=0}^{n-1} |e^m|$$

(discrete Grönwall)

$$\Rightarrow |e^n| \le K + \Delta t L K \frac{(1+\Delta t L)^n - 1}{\Delta t L} = K(1+\Delta t L)^n$$

$$\Rightarrow \ |e^n| \le K e^{\Delta t L n}, \quad n=0,\dots,N. \qquad \#$$

Lemma 2. For a LMM that is consistent with the ODE $u'=f$ where f is Lipschitz, and starting with consistent starting conditions, zero-stability is sufficient for convergence.

Proof.

Define

$$\delta = \delta(\Delta t) = \max_{0\le s\le r-1} |\eta_s(\Delta t) - u(s\Delta t)|$$

given $U^s = \eta_s(\Delta t)$, $s=0,\dots,r-1$ and consistency $\lim_{\Delta t\to 0} \delta(\Delta t) = 0$.

Need to prove

$$\lim_{\substack{n\to\infty \\ n\Delta t = t}} U^n = u(t) \quad \text{for all } t\in[0,T].$$

LTE

$$\tau^n = \frac{1}{\Delta t\,\sigma(1)}\left[\sum_{j=0}^{r} \alpha_j u(t_{n+j}) - \Delta t \beta_j u'(t_{n+j})\right]$$

Define

$$\chi(\varepsilon) = \max_{\substack{|t^*-t|\le\varepsilon \\ t^*,t\in[0,T]}} |u'(t^*) - u'(t)|$$

(above well-defined since $u\in C[0,T]$)

$$u'(t_{n+s}) = u'(t_n) + \theta_s \chi(s\Delta t) \qquad |\theta_s|\le 1.$$

By MVT, there exists $\xi_s \in [t_n, t_{n+s}]$

$$u(t_{n+s}) = u(t_n) + s\Delta t\, u'(\xi_s)$$

$$\Rightarrow \quad u(t_{m+s}) = u(t_m) + s\Delta t\,[\, u'(t_m) + \theta_s' \chi(s\Delta t)\,]$$

Then

$$|\,\sigma(1)\,\tau^n| \le \left.\begin{aligned} &|\,\Delta t^{-1}(\alpha_1 + \cdots + \alpha_r)u(t_n) \\ &+ (\alpha_1 + 2\alpha_2 + \cdots + r\alpha_r)u'(t_n) \\ &- (\beta_0 + \cdots + \beta_r)u'(t_n)\,| \end{aligned}\right\} = 0 \quad \text{Consistency}$$

$$+ (|\alpha_1| + 2|\alpha_2| + \cdots + r|\alpha_r|)|\chi(r\Delta t)|$$

$$+ (|\beta_0| + |\beta_1| + \cdots + |\beta_r|)\,|\chi(r\Delta t)|.$$

$$\Rightarrow \quad |\,\sigma(1)\,\tau^n| \le K\chi(r\Delta t)$$

$$K = |\alpha_1| + \cdots + r|\alpha_r| + |\beta_0| + \cdots + |\beta_r|.$$

Global error

$$e^m = u(t_m) - U^m$$

$$\alpha_r e^{m+r} + \cdots + \alpha_0 e^m = \Delta t\,(\beta_r g_{m+r} e^{m+r} + \cdots + \beta_0 g_m e^m)$$

$$+ \sigma(1)\tau^n \Delta t$$

$$= \Delta t\,(\beta_r g_{m+r} e^{m+r} + \cdots + \beta_0 g_m e^m)$$

$$+ \theta K \chi(r\Delta t)\,\Delta t.$$

$$g_m = \begin{cases} [f(u(t_m), t_m) - f(U^m, t_m)]/e^m, & e^m \neq 0 \\ 0, & e^m = 0. \end{cases}$$

(Lipschitz) $\Rightarrow$ $|g_m| \le L$, $m = 0, 1, \ldots$

(Lemma 1)

$$\Rightarrow \quad |e^n| \le \hat{\Gamma}\,[\,A r \delta(h) + T K \chi(r\Delta t)\,]\, e^{t_n L \hat{\Gamma} B}$$

with $E = \delta(h)$, $\Lambda = K\chi(r\Delta t)\Delta t$, $N = T/\Delta t$

$B = |\beta_0| + \cdots + |\beta_r|$.

u uniformly continuous on $[0, T]$

$\Rightarrow$ $\chi(r\Delta t) \to 0$ as $\Delta t \to 0$

Passing to the limit $h \to 0$ ($\delta(h) \to 0$ by consistency in starting value)

$$\lim_{\substack{n \to \infty \\ n\Delta t = t}} |e^n| = 0$$

#

Theorem (Dahlquist) For a LMM that is consistent with the ODE $u' = f$ where f is Lipschitz and starting with consistent initial data, zero-stability is necessary and sufficient for convergence.

Remarks.

* Consistency + 0-stability = Convergence.

* if $u(t)$ has continuous derivative of order $(p+1)$ and LTE $O(\Delta t^p)$, the global error

$$e^n = u(t_n) - U^n = O(\Delta t^p).$$

Maximum order of 0-stable LMM

We state the theorems describing the maximum order by choosing the coefficients β_j, $j=0,\dots,r$ given α_j, $j=0,\dots,r$ in an r-step method.

Theorem Let $\rho(x)$ the characteristic poly. of degree r such that $\rho(1)=0$, $\rho'(1)\neq 0$, and let $\hat{r}$ be an integer $0 \le \hat{r} \le r$. Then there exists a unique poly. $\sigma(x)$ of degree $\hat{r}$ such that $\rho'(1) - \sigma(1) = 0$ and the order of the LMM associated with $\rho(x)$ and $\sigma(x)$ is $\geq \hat{r}+1$.

Theorem There is no 0-stable linear r-step method whose order exceeds $r+1$ if r is odd or $r+2$ if r is even.

Examples. 0-stable LMM

Adams $\rho(x) = x^{r} - x^{r-1}$

- Adams-Bashforth : explicit
- Adams-Moulton : implicit

Nyström $\rho(x) = x^{r} - x^{r-2}$: explicit

Milne-Simpson : implicit.