Numerical PDE:

Continuous and Discrete Fourier Series and Numerical Discretization

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4 Continuous and Discrete Fourier Series and Numerical Discretization

In this chapter, we briefly review the basic ideas involved in continuous and discrete Fourier series such as aliasing. After we have the tools which we need from continuous and discrete Fourier series, we will use them to solve differential and difference equations as well as to analyze properties of good difference equations for numerical approximations.

4.1 Continuous and Discrete Fourier Series

Fourier series are a mathematical tool that represents continuous periodic functions as linear combinations of trigonometric functions $\{e^{i\ell x}\}$. Throughout this chapter, we only consider 2π-periodic function $f(x) \in L^2(0, 2\pi)$, that is, $f(x) = f(x + 2\pi)$ for all x and $\int_0^{2\pi} |f|^2 < \infty$. Consequently, we can restrict our attention to a domain of $0 \leq x \leq 2\pi$. The corresponding discrete case has a domain of finite grid points $x_j = jh, j = 0, 1, \ldots, 2N$, where $(2N + 1)h = 2\pi$. In this setup, a 2π-periodic discretized function satisfies $f_j = f_{j+(2N+1)}$ for $f_j \equiv f(x_j), j \in \mathbb{Z}$. For functions with different periods, we can always normalize them to be 2π-periodic. To define the discrete equivalent to the integral in the continuous case, we consider a simple example, $f(x) = 1$. A proper normalization for the discrete case chooses h and N such that

$$\int_0^{2\pi} f(x)dx = \sum_{j=0}^{2N} f(x_j)h \qquad \int_0^{2\pi} 1 \quad dx = \sum_{j=0}^{2N} 1h,$$

which explains the choice of $2\pi = (2N + 1)h$. Thus, we can define the discrete complex inner product and the discrete norm, following the respective continuous case (see Table 4.1). One can easily see that the dimension of the discrete vector space is $2N + 1$ since we sum over $2N + 1$ points. On the other hand we need to prove that the dimension of the continuous vector space is infinite.

	Continuous	Discrete
Complex inner product	$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x)g^*(x)dx$	$(\vec{f}, \vec{g})_h = \frac{h}{2\pi} \sum_{j=0}^{2N} f_j g_j^*$
Norm	$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx$	$\|\vec{f}\|_h^2 = \frac{h}{2\pi} \sum_{j=0}^{2N} |f_j|^2$

4.1 Continuous and Discrete Fourier Series

Proposition 4.1. *The trigonometric functions* $\{e^{i\ell x} = \cos(\ell x) + i\sin(\ell x), \ell \in \mathbb{Z}\}$ *form a basis for the vector space* $L^2(0, 2\pi)$. *Consequently, this vector space is infinite dimensional.*
Proof: To show that they form a basis, we check that they are orthogonal.

$$\left(e^{i\ell x}, e^{imx}\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\ell x}e^{-imx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(\ell-m)x}\,dx.$$

Note that $\left(e^{i\ell x}, e^{imx}\right) = 1$ when $\ell = m$ and $\left(e^{i\ell x}, e^{imx}\right) = 0$ when $\ell \neq m$ since $\ell - m$ is an integer and e^{ix} is a 2π-periodic function. Thus, $\{e^{i\ell x}, \ell \in \mathbb{Z}\}$ form an infinite dimensional basis for our vector space.

Now, let us construct a basis for the discrete vector space with dimension $2N+1$. Based on the continuous basis, our first guess would be $\vec{e}^{\ell} = (e^{i\ell x_j})$ where $|\ell| \leq N$ and $x_j = jh$ since we only evaluate our discrete function at lattice points.

Now, let us construct a basis for the discrete vector space with dimension $2N+1$. Based on the continuous basis, our first guess would be $\vec{e}^{\ell} = (e^{\mathrm{i}\ell x_j})$ where $|\ell| \leq N$ and $x_j = jh$ since we only evaluate our discrete function at lattice points.

Proposition 4.2. $\{\vec{e}^{\ell}\}_{\ell=0,\pm1,\ldots,\pm N}$, *form an orthonormal basis for the vector space* $\mathbb{C}^{2N+1}$ *with discrete inner product.*
Proof: Let $\ell, m = 0, \pm 1, \ldots, \pm N$. The discrete inner product

$$\left(\vec{e}^{\ell}, \vec{e}^{m}\right)_h = \frac{h}{2\pi} \sum_{j=0}^{2N} e^{\mathrm{i}(\ell-m)jh}$$

is exactly 1 when $\ell = m$ since $(2N+1)h = 2\pi$. When $\ell \neq m$, set $\omega = e^{\mathrm{i}(\ell-m)h}$ such that

$$\left(\vec{e}^{\ell}, \vec{e}^{m}\right)_h = \frac{h}{2\pi} \sum_{j=0}^{2N} \omega^j = \frac{h}{2\pi} \frac{1-\omega^{2N+1}}{1-\omega} = \frac{h}{2\pi} \frac{1-e^{\mathrm{i}(\ell-m)h(2N+1)}}{1-e^{\mathrm{i}(\ell-m)h}}$$

through geometric series. Notice that the denominator is nonzero but the numerator is always zero since $(2N+1)h = 2\pi$. Therefore, $\vec{e}^{\ell}$ is a basis.

We end this section by stating the main results for the continuous and discrete expansion theory (consult [44] for proofs of these facts).

4.1 Continuous and Discrete Fourier Series

Continuous Expansion Theory: For the continuous 2π-periodic function $f(x) \in L^2(0, 2\pi)$, if the Fourier coefficients

$$\hat{f}(\ell) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-i\ell x}dx, \quad \ell = 0, \pm 1, \pm 2, \ldots$$

satisfy

$$\sum_{\ell=-\infty}^{\infty} |\hat{f}(\ell)| < \infty, \tag{4.1}$$

then

$$f(x) = \sum_{\ell=-\infty}^{\infty} \hat{f}(\ell)e^{i\ell x}$$

uniformly in x. Note that if we don't have condition (4.1), the convergence is only in the L^2 sense. What is true when condition (4.1) is not satisfied is

$$\|f\|^2 = \sum_{\ell=-\infty}^{\infty} |\hat{f}(\ell)|^2,$$

which is known as the Parseval's identity.

Discrete Expansion Theory: Let $\vec{f} \in \mathbb{C}^{2N+1}$, where $\vec{f} = (f_0, f_1, \ldots, f_{2N})$. Since $\vec{e}^{\ell}$ is an orthonormal basis for the vector space $\mathbb{C}^{2N+1}$, we have

$$\vec{f} = \sum_{\ell=-N}^{N} (\vec{f}, \vec{e}^{\ell})_h \vec{e}^{\ell},$$

where the discrete Fourier coefficients are given by

$$\hat{f}_h(\ell) = (\vec{f}, \vec{e}^{\ell})_h = \frac{h}{2\pi} \sum_{j=0}^{2N} f_j e^{-i\ell jh}.$$

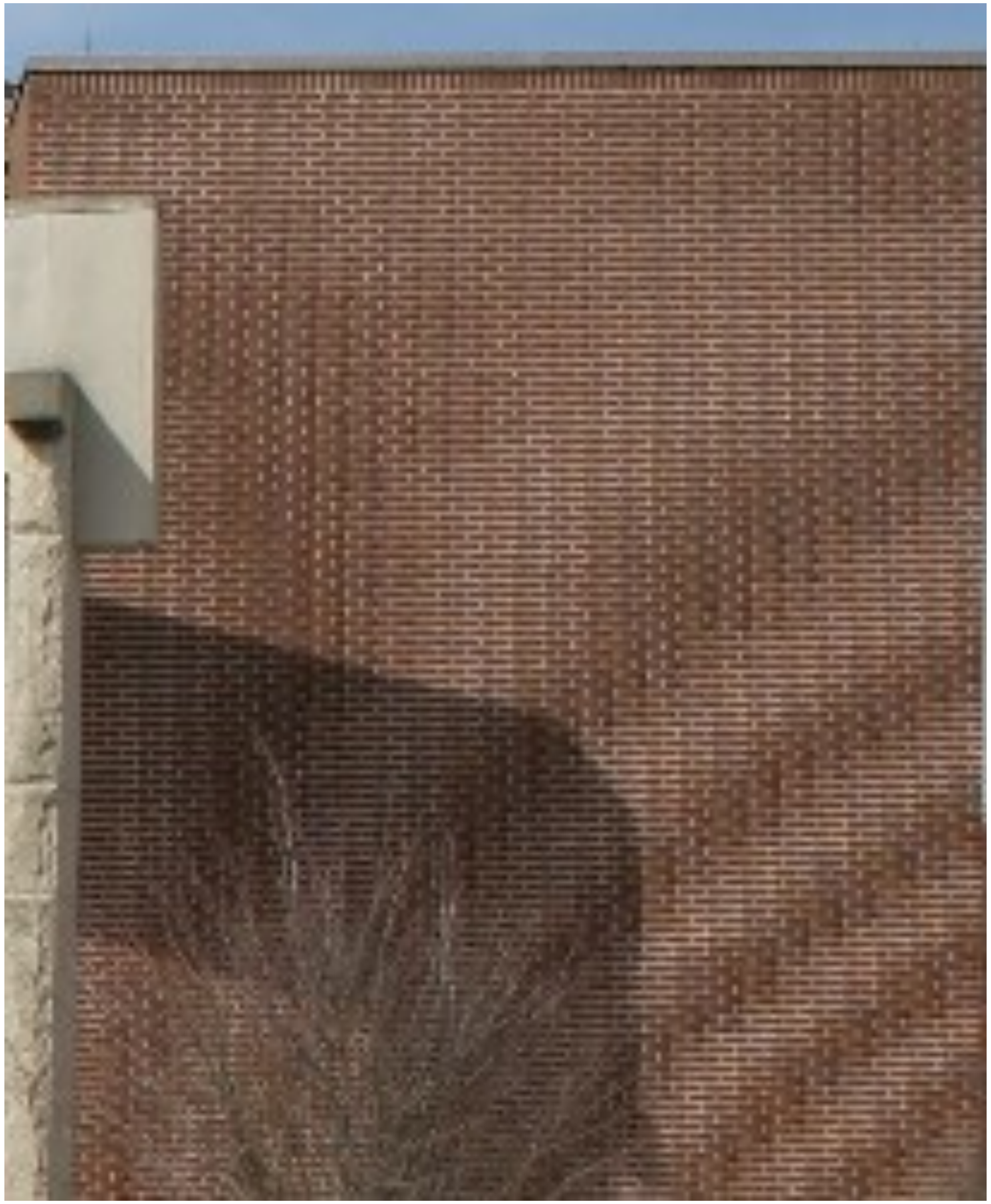
We can also write $\vec{f}$ componentwise

$$f_j = \sum_{|\ell| \le N} \hat{f}_h(\ell) e^{i\ell jh}. \tag{4.2}$$

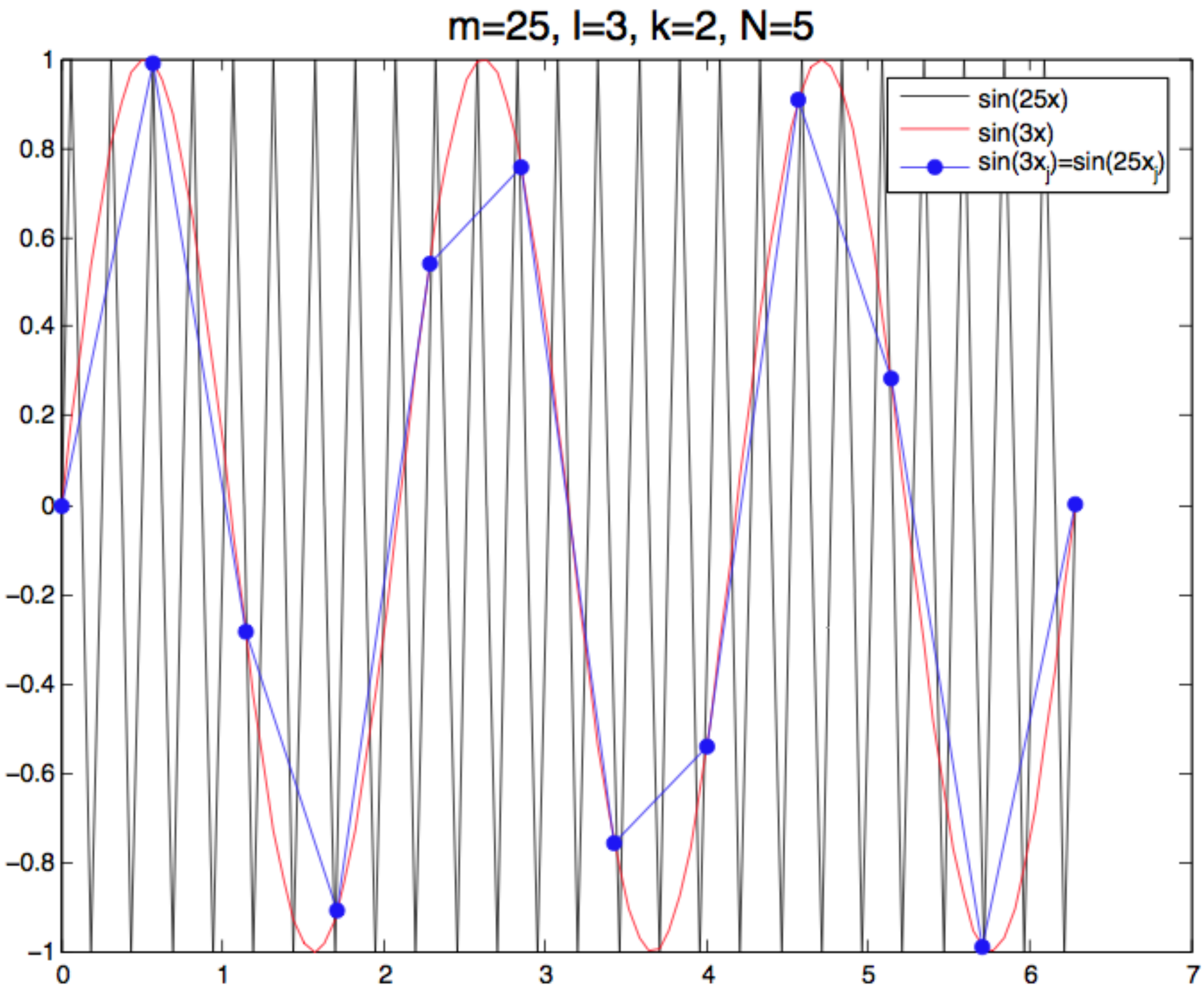
The natural notion of length (Parseval's identity) follows from Table 4.1

$$\|\vec{f}\|_h^2 = (\vec{f}, \vec{f})_h = \sum_{|\ell| \le N} |\hat{f}_h(\ell)|^2.$$

4.2 Aliasing



4.2 Aliasing



m=25, l=3, k=2, N=5
sin(25x)
sin(3x)
$\sin(3x_j)=\sin(25x_j)$
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
1
2
3
4
5
6
7

4.2 Aliasing

Aliasing is an artifact of the discretization process. When we try to represent a continuous function by a discrete set of points we lose information. In particular we cannot distinguish high harmonics from their low counterpart (e.g., see Figure 4.1). Mathematically, we can describe this fact by considering the discrete oscillatory basis $\vec{e}^{\ell} = (e^{\mathrm{i}\ell jh})$ for $|\ell| \leq N$ with the following harmonics

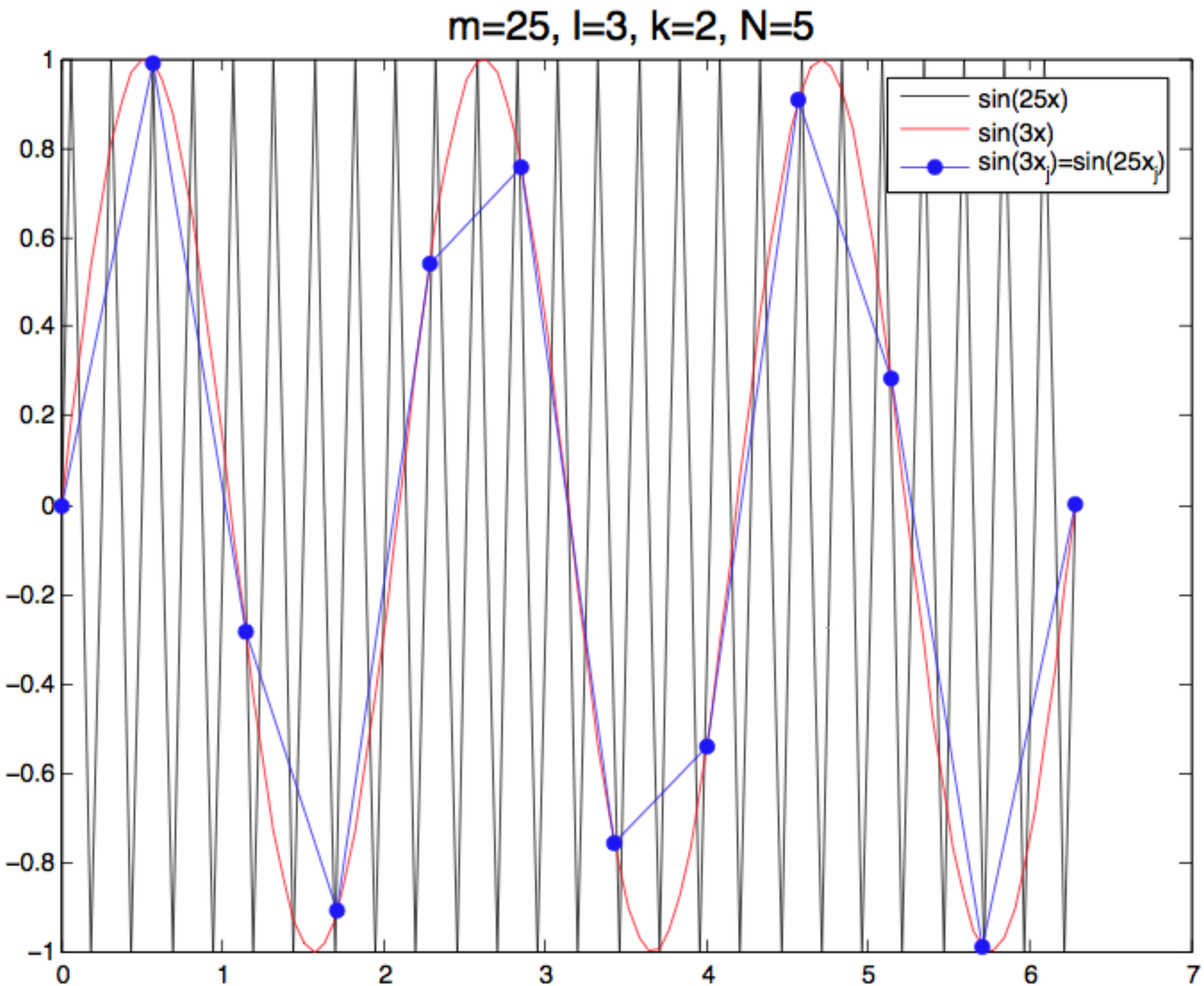
$$m = \ell + (2N+1)k, \quad k = 0, \pm 1, \pm 2, \ldots .$$

Since $(2N+1)h = 2\pi$, it is clear that

$$e^{\mathrm{i}mjh} = e^{\mathrm{i}(\ell+(2N+1)k)jh} = e^{\mathrm{i}\ell jh} e^{\mathrm{i}2\pi kj} = e^{\mathrm{i}\ell jh}.$$

We have shown that when we sample e^{imx} with $2N+1$ grid points, $x = x_j = jh$, such that $(2N+1)h = 2\pi$, we cannot distinguish it from its lower harmonics, $e^{\mathrm{i}\ell x}$, at the corresponding grid points. In Figure 4.1, we show an example where aliasing occurs with two sinusoidal functions, one with higher frequency, $m = 25$, and another with lower frequency, $\ell = 3$. Here, both functions are sampled with identically $2N+1 = 11$ uniformly distributed grid points.

4.2 Aliasing



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7

4.2 Aliasing

Next, we would like to know how the coefficients of the continuous Fourier series, $\hat{f}(\ell), |\ell| \leq N$, compare to the discrete coefficients, $\hat{f}_h(\ell)$. To understand the difference, let us consider

$$f(x) = e^{imx}$$

for any m. No matter what m is, it can be written in the following form

$$m = \ell + (2N+1)k$$

for some $\ell, |\ell| \leq N$ and $k \in \mathbb{Z}$. Since $f(x)$ is sufficiently simple, we can calculate $\hat{f}(\ell)$ and $\hat{f}_h(\ell)$. In particular,

$$\begin{aligned} \hat{f}(\ell) &= \frac{1}{2\pi}\int_0^{2\pi} e^{imx}e^{-i\ell x}dx = \begin{cases} 1, & \text{if } \ell = m \\ 0, & \text{else} \end{cases}, \\ \hat{f}_h(\ell) &= \frac{h}{2\pi}\sum_{j=0}^{2N} e^{imjh}e^{-i\ell jh} = 1. \end{aligned}$$

The last equality is due to the aliasing $e^{imjh} = e^{i\ell jh}$. If $\ell = m$, both $\hat{f}(\ell) = \hat{f}_h(\ell) = 1$ and we do not have aliasing since $k = 0, |\ell| \leq N$. For $\ell \neq m$, there is a difference between $\hat{f}(\ell)$ and $\hat{f}_h(\ell)$, and this difference is the result of aliasing.

For a more general $f(x)$, we can look at it as a superposition of exponentials.

Proposition 4.3: *Consider*

$$f(x) = \sum_{m=-\infty}^{\infty} \hat{f}(m) e^{imx}.$$

For $|\ell| \leq N$, *we can write down an expression for* $\hat{f}_h(\ell)$ *in terms of* $\hat{f}(\ell)$,

$$\hat{f}_h(\ell) = \hat{f}(\ell) + \sum_{k \neq 0} \hat{f}(\ell + k(2N+1)),$$

where the second term on the right hand side is the aliasing error.

Proof: Compute $\vec{f} = (f(0), f(h), \ldots, f(2\pi))$. We want to reorganize the following sum

$$\begin{aligned}
f_j &= \sum_{m=-\infty}^{\infty} \hat{f}(m) e^{imjh}, \quad j = 0, 1, \ldots, 2N, \\
&= \ldots + \sum_{\ell=-3N-1}^{-N-1} \hat{f}(\ell) e^{i\ell jh} + \sum_{\ell=-N}^{N} \hat{f}(\ell) e^{i\ell jh} + \sum_{\ell=N+1}^{3N+1} \hat{f}(\ell) e^{i\ell jh} + \ldots \\
&= \ldots + \sum_{|\ell| \leq N} \hat{f}(\ell - (2N+1)) e^{i(\ell-(2N+1))jh} + \sum_{\ell=-N}^{N} \hat{f}(\ell) e^{i\ell jh} \\
&\quad + \sum_{|\ell| \leq N} \hat{f}(\ell + 2N + 1) e^{i(\ell+2N+1)jh} + \ldots \\
&= \sum_{|\ell| \leq N} \sum_{k=-\infty}^{\infty} \hat{f}(\ell + k(2N+1)) e^{i(\ell+(2N+1)k)jh} \\
&= \sum_{|\ell| \leq N} \sum_{k=-\infty}^{\infty} \hat{f}(\ell + k(2N+1)) e^{i\ell jh}.
\end{aligned}$$

$$f_j = \sum_{|\ell|\le N} \sum_{k=-\infty}^{\infty} \hat{f}(\ell + k(2N+1)) e^{i\ell jh}.$$

$$f_j = \sum_{|\ell|\le N} \hat{f}_h(\ell) e^{i\ell jh}. \tag{4.2}$$

The last equality uses the fact that $(2N+1)h = 2\pi$. By the discrete expansion theory in (4.2), we have

$$\hat{f}_h(\ell) = \sum_{k=-\infty}^{\infty} \hat{f}(\ell + k(2N+1))$$

and the proof is completed.

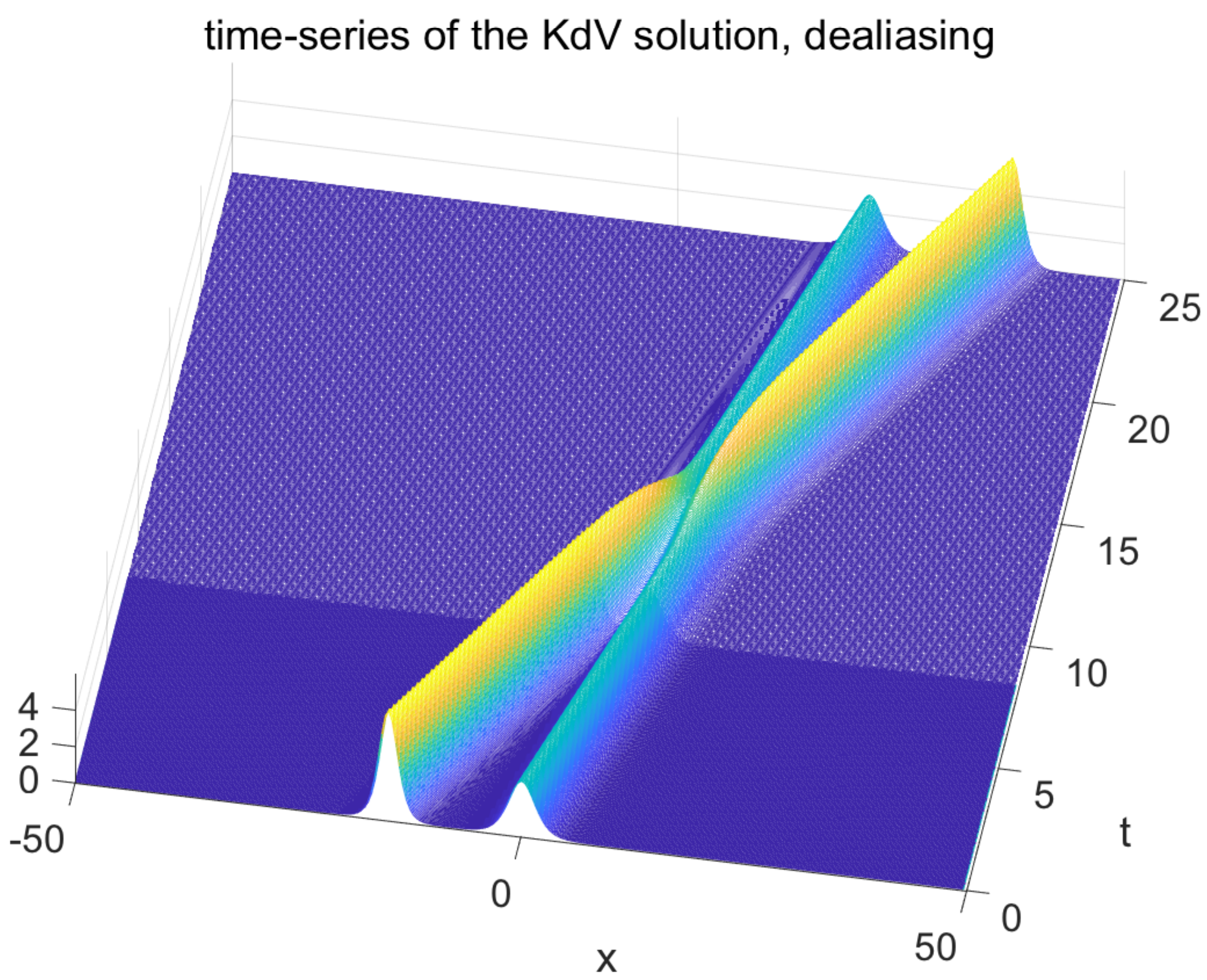
Aliasing and nonlinearities

- assume we have a non-linear term uv in our PDE, and $u(x) = \sin(k_1 x)$, $v(x) = \sin(k_2 x)$, with k_1, k_2 from our set of available wave-vectors k_j
- now uv ~ $-\cos[(k_1+k_2) z] + \cos[(k2-k_1)z]$, and k_1+k_2 may lie outside our range of k's, and the available Fourier amplitudes might get aliased !
- k_1+k_2 outside range if $k_1+k_2 > \pi$, and the amplitude appears wrongly in the range of k's at $k_1+k_2-2\pi$ ($l=-1$, j_1+j_2-N), the DFT is aliased

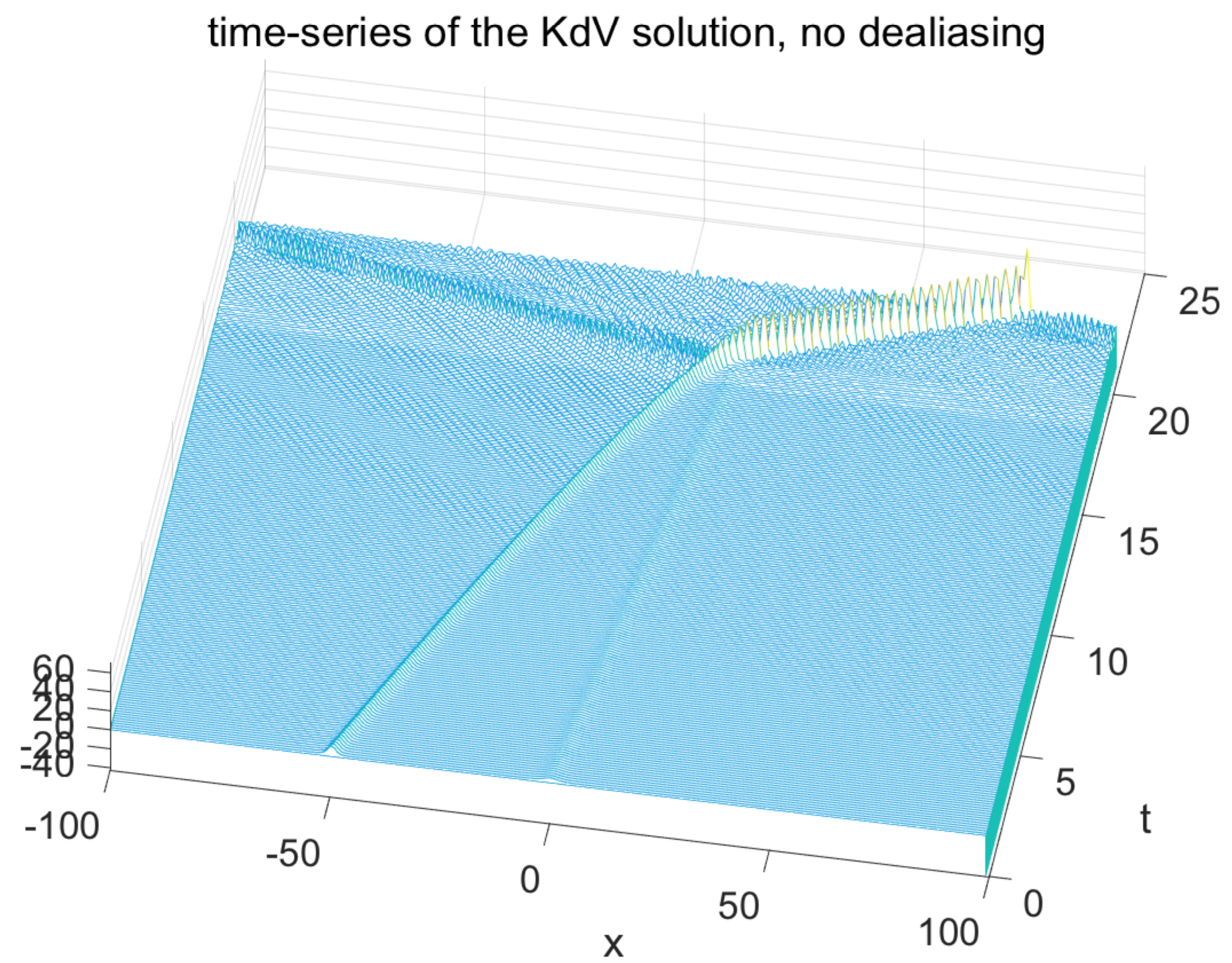
Example: Korteweg de Vries equation (KdV)

$$\partial_t u + \frac{1}{2}\partial_x u^2 + \partial_{xxx} u = 0$$

Numerically, two colliding solitons



De-aliased solution



Not de-aliased solution

4.3 Differential and Difference Operators

In Section 4.1, we showed that both the continuous and discrete basis involving oscillations were made up of exponentials. In this section, we are going to see that these exponentials are indeed eigenfunctions of both differential and difference operators, respectively. For the continuous case, let $u(x) \in \mathcal{C}^K(0, 2\pi)$ be a 2π-periodic K-times continuously differentiable function. For known constant p_m, we define a differential operator

$$\mathcal{P}\left(\frac{\partial}{\partial x}\right)u(x) = \sum_{m=0}^{K} p_m \frac{d^m u(x)}{dx^m}.$$

Proposition 4.4: *Every harmonic $e^{i\ell x}$ is a 2π-periodic eigenfunction of $\mathcal{P}(\partial_x)$.*
Proof: Since

$$\begin{aligned}\frac{d}{dx}e^{i\ell x} &= (i\ell)e^{i\ell x},\\ &\vdots\\ \frac{d^m}{dx^m}e^{i\ell x} &= (i\ell)^m e^{i\ell x},\end{aligned}$$

we have

$$\mathcal{P}\left(\frac{\partial}{\partial x}\right)e^{i\ell x} = \sum_{m=0}^{K} p_m \frac{d^m e^{i\ell x}}{dx^m} = \sum_{m=0}^{K} p_m (i\ell)^m e^{i\ell x}.$$

Thus, we have shown that $e^{i\ell x}$ is an eigenfunction of $\mathcal{P}(\partial_x)$ with eigenvalue

$$\tilde{p}(i\ell) = \sum_{m=0}^{K} p_m (i\ell)^m.$$

The equivalent discrete case is for $\vec{u} \in \mathbb{C}^{2N+1}$ with components $u_j = u(x_j)$, $x_j = jh$, where $j \in \mathbb{Z}$ and $(2N+1)h = 2\pi$. Each component satisfies $u_j = u_{j+2N+1}$. We define the difference operator $\vec{G}$ with component

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$$(G(u_j))_k = \sum_{m=-K}^{K} a_m u_{k+m},$$

where a_m are known constant.

Proposition 4.5: *Every discrete harmonics $e^{i\ell x_j}$ with $x_j = jh$ is an eigenfunction of the difference operator $\vec{G}$.*
Proof: Again the proof follows from te definition of $(G(u_j))_k$. Let $u_j = e^{i\ell jh}$ and

$$(G(e^{i\ell jh}))_k = \sum_{m=-K}^{K} a_m e^{i\ell(k+m)h} = \sum_{m=-K}^{K} a_m e^{i\ell mh} e^{i\ell kh}.$$

Here, the corresponding eigenvalue is

$$\tilde{g}_h(\ell) = \sum_{m=-K}^{K} a_m e^{i\ell mh}, \tag{4.3}$$

which is sometimes called the amplification factor.

4.4 Solving Initial Value Problems

In this section, our goal is to understand when we can solve the Initial Value Problem (IVP):

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mathcal{P}\Big(\frac{\partial}{\partial x}\Big)u, \\ u(x,0) &= f(x),\end{aligned}$$

for $t > 0$ and $u(x+2\pi, t) = u(x,t)$ and what the general solution is. In particular, we would like to find explicit algebraic criteria which would tell us when we have a solution for the IVP. Some examples of differential operators which arise in applications are the free space Schrodinger operator, $\mathcal{P}(\partial_x)u = \mathrm{i}u_{xx}$, the simple wave equation, $\mathcal{P}(\partial_x)u = Cu_x$, and the heat/diffusion equation, $\mathcal{P}(\partial_x)u = u_{xx}$.

Consider a set of trigonometric function, T_N, defined as follows

$$T_N = \Big\{ \sum_{|\ell|\leq N} C_\ell e^{\mathrm{i}\ell x}, C_\ell \in \mathbb{C} \Big\}. \tag{4.4}$$

Since we have shown that $e^{\mathrm{i}\ell x}$ is an eigenfunction of the differential operator $\mathcal{P}(\partial_x)$, we would not be surprised to see that for functions from T_N, the IVP can always be solved. Formally, this is given by the next proposition.

Proposition 4.6: *For any operator $\mathcal{P}(\partial_x)$, any function $f(x) \in T_N$ and any N, we can always solve the IVP.*
Proof: Solve the IVP

$$\begin{aligned} \frac{\partial u^\ell}{\partial t} &= \mathcal{P}\Big(\frac{\partial}{\partial x}\Big) u^\ell, \\ u^\ell(x,0) &= e^{\mathrm{i}\ell x}. \end{aligned} \tag{4.5}$$

We want to use separation of variables so we assume

$$u^\ell(x,t) = a^\ell(t) e^{\mathrm{i}\ell x}. \tag{4.6}$$

Recall that since $e^{\mathrm{i}\ell x}$ is an eigenfunction of $\mathcal{P}(\partial_x)$, substituting (4.6) into the IVP in (4.5) yields

$$\frac{\partial a^\ell}{\partial t} e^{\mathrm{i}\ell x} = a^\ell \tilde{p}(\mathrm{i}\ell) e^{\mathrm{i}\ell x} \Rightarrow \frac{\partial a^\ell}{\partial t} = \tilde{p}(\mathrm{i}\ell) a^\ell, \text{ where, } a^\ell(0) = 1.$$

This equation has the solution $a^\ell(t) = e^{\tilde{p}(\mathrm{i}\ell)t}$ and therefore $u^\ell(t) = e^{\tilde{p}(\mathrm{i}\ell)t} e^{\mathrm{i}\ell x}$ solves (4.5).

Now we want to solve the same PDE but with more general initial data, i.e.,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{P}\Big(\frac{\partial}{\partial x}\Big)u, \\ u(x,0) &= \sum_{|\ell|\le N} C_\ell e^{\mathrm{i}\ell x}. \end{aligned}$$

Since this equation is linear we know that sum of the solutions is also a solution. From the above we know that we can solve the IVP for initial data which consists of one harmonic. In order to get the solution matches the general initial data we just need to add up all the individual solutions,

$$u(x,t) = \sum_{|\ell|\le N} C_\ell e^{\tilde{p}(\mathrm{i}\ell)t} e^{\mathrm{i}\ell x}.$$

In discrete setting, consider $\vec{u} = (u_0, u_1, \ldots, u_{2N}) \in \mathbb{C}^{2N+1}$ such that $u_j = u_{j+2N+1}$ and $(2N+1)h = 2\pi$. The corresponding IVP is given as follows

$$\begin{aligned} \vec{u}^{M+1} &= \vec{G}\vec{u}^{M}, \\ \vec{u}^{0} &= \vec{f}, \end{aligned}$$

where superscript M denotes the discrete time step, the difference operator $\vec{G} : \mathbb{C}^{2N+1} \to \mathbb{C}^{2N+1}$ is defined as follows

$$(G(u_j^M))_k = \sum_{m=-K}^{K} a_m u_{k+m}^M.$$

Based on the Fourier discrete expansion theory, we can write any initial conditions, $\vec{f} \in \mathbb{C}^{2N+1}$, as a linear combination of the exponential basis $\vec{e}^{\ell}$ (see Eqn. (4.2)). By superposition, the general solution for the discrete IVP is always given by

$$\vec{u}^M = \sum_{|\ell| \leq N} \tilde{g}_h(\ell)^M \hat{f}_h(\ell) \vec{e}^{\ell},$$

where $\tilde{g}_h(\ell)$ is the eigenvalue of the difference operator $\vec{G}$ as defined in (4.3).

4.5 Convergence of the Difference Operator

Naively one might think that any numerical scheme to discretize in x and t would produce a convergence solution to the IVP. We will show that is not true even in a simple context. To be more explicit, we consider the linear wave equation as an example since it constitutes the simplest prototype model for turbulent systems as we will discuss in the next chapter.

Linear wave equation:

$$\begin{aligned} u_t &= C u_x \\ u(x,0) &= f(x). \end{aligned} \tag{4.7}$$

We know that this equation has the solution

$$u(x,t) = f(x + Ct),$$

which is a wave that propagates to the left if $C > 0$ and to the right if $C < 0$.

The fundamental theory in the analysis of finite difference methods for the numerical solutions of partial differential equations is the **Lax Equivalence Theorem**, which states the following: *Provided that differential equation $u_t = \mathcal{P}(\partial_x)u$ is stable. The convergence of the difference scheme is guaranteed only when it is stable and consistent.*

The strength of this theorem is that it is quite often easier to check the stability and the consistency relative to directly showing the convergence since the numerical method is defined by recurrence relation while the differential equation involves differentiable functions. To confirm the stability of the differential and difference operators, it suffices to check the following algebraic conditions.

Proposition 4.7: IVP is stable for a given $\mathcal{P}(\partial_x)$ if and only if

$$\max_{|\ell|\leq\infty, 0\leq t\leq T} |e^{\tilde{p}(\mathrm{i}\ell)t}|^2 \leq C(T).$$

Proposition 4.8: The difference scheme is stable for a strategy $\Delta t \leq S(h)$, where h is the discrete spatial mesh size, if and only if

$$\max_{|\ell|\leq\infty, 0\leq M\Delta t\leq T} |\tilde{g}_h(\Delta t, \ell)^M|^2 \leq C(T).$$

In our example, the stability of the PDE is clearly satisfied since the differential operator of the wave equation in (4.7) is bounded from above, $|e^{\tilde{p}(i\ell)t}|^2 = |e^{Ci\ell}|^2 = 1$. Let us consider the forward Euler time discretization (as described in Chapter 2) as well as the symmetric difference to approximate the spatial derivative,

$$\frac{u_{j+1}^M - u_{j-1}^M}{2h} = u_x + \mathcal{O}(h^2),$$

where $u_j^M \simeq u(jh, M\Delta t)$. This second order accurate approximation can be easily deduced by substracting the Taylor expansions of u_{j+1}^M and u_{j-1}^M about their mid point, $x_j = jh$. With these approximations, the numerical estimate of the wave equation is given by the following recurrence relation

$$\begin{aligned} u_j^{M+1} &= u_j^M + \frac{C\Delta t}{2h}(u_{j+1}^M - u_{j-1}^M), \\ u_j^0 &= f_j. \end{aligned} \tag{4.8}$$

We will show that this difference scheme is indeed not stable. The amplification factor of (4.8),

$$\tilde{g}_h(\Delta t, \ell) = 1 + \frac{C\Delta t}{2h}(e^{\mathrm{i}\ell h} - e^{-\mathrm{i}\ell h}) = 1 + \mathrm{i}\frac{C\Delta t}{h}\sin \ell h,$$

satisfies

$$|\tilde{g}_h(\Delta t, \ell)|^2 = 1 + \left(\frac{C\Delta t}{h}\right)^2 \sin^2(\ell h) > 1,$$

when $\ell \neq 0, |\ell| \leq N$ and for any constant $C\Delta t/h \neq 0$. For a fixed time T where $0 \leq M\Delta t \leq T$, whenever Δt is small with $\Delta t/h$ constant, we need to increase the time step M, therefore $|\tilde{g}_h(\Delta t, \ell)^M|^2$ keeps growing as M increases, and the algebraic condition in Proposition 4.8 is not satisfied.

Now, let us reduce the accuracy in the spatial derivative approximation by considering a first order forward difference method,

$$\frac{u_{j+1}^M - u_j^M}{h} = u_x + \mathcal{O}(h).$$

With the forward Euler time discretization, we called the following approximation the upwind difference scheme,

$$\begin{aligned} u_j^{M+1} &= u_j^M + \frac{C\Delta t}{h}(u_{j+1}^M - u_j^M), \\ u_j^0 &= f_j. \end{aligned} \tag{4.9}$$

Before we check the stability and consistency of this scheme, let us intuitively give a conjecture for this scheme. We know that when $C > 0$ the exact solution propagates to the left and when $C < 0$ it propagates to the right. The difference equation in (4.9) uses the values at the right (grid point $j+1$) to calculate a value at the left (grid point j). Thus, we expect the difference scheme will work for $C > 0$. On the other hand, for $C < 0$, the difference scheme (4.9) still uses the values at the right side to calculate values at the left side. Since the difference scheme uses information from the wrong side, we do not expect it to work well in this case. When $C < 0$, one needs to consider the backward difference scheme to approximate the spatial derivative since it uses information from the left to calculate the value at the right.

To check the stability, let us denote $\lambda = \Delta t/h$ such that the amplification factor of (4.9) can be written as follows

$$\tilde{g}_h(\Delta t, \ell) = 1 + \lambda C(e^{\mathrm{i}\ell h} - 1). \tag{4.10}$$

The algebraic condition in Proposition 4.8 is satisfied when $|\tilde{g}_h|^2 \leq 1$ for all $|\ell| \leq N$.

$$0 \leq |\tilde{g}_h(\Delta t, \ell)|^2 = 1 - 2C\lambda(1 - C\lambda)(1 - \cos(\ell h)).$$

For $|\tilde{g}_h|^2 \leq 1$, we need

$$2C\lambda(1 - C\lambda)(1 - \cos(\ell h)) \geq 0.$$

Since $(1 - \cos(\ell h)) \geq 0$ for all h, ℓ, the stability holds when $C\lambda(1 - C\lambda) \geq 0$. That is, for $C, \lambda > 0, \lambda C \leq 1$. This is the well-known Courant-Friedrichs-Lewy (CFL) criterion for stability.

Consistency is a condition which guarantees that the discrete problem approximates the correct continuous problem. To verify this, let $\tilde{u}_j^M = u(jh, M\Delta t)$ be the exact solution of the wave equation in (4.7), evaluated at grid point jh and time $M\Delta t$. The Taylor expansions about the grid spacing h and discrete time Δt are given as follows

$$\begin{aligned} \tilde{u}_{j+1}^M &= \tilde{u}_j^M + h\tilde{u}_x + h^2\tilde{u}_{xx} + \mathcal{O}(h^3), \\ \tilde{u}_j^{M+1} &= \tilde{u}_j^M + \Delta t\tilde{u}_t + \mathcal{O}(\Delta t^2). \end{aligned}$$

Substituting these expansions to the finite difference scheme in (4.9), we obtain

$$\tilde{u}_t + \mathcal{O}(\Delta t) = C(\tilde{u}_x + h\tilde{u}_{xx} + \mathcal{O}(h^2)).$$

Taking the limit $h, \Delta t \to 0$, we obtain the continuous wave equation in (4.7) and the consistency is satisfied. Therefore, the upwind difference scheme is a convergent method whenever the CFL condition holds.