1

Introduction

There are many different types of partial differential equations. A good choice of numerical schemes is often dependent on the type of equations, which is the key difficulty of studying numerical methods.

1.1 Partial differential equations

Most of classical PDEs originate from modeling physical phenomenon, used in science and engineering problems. One thing we should always keep in mind is that these equations are chosen *models*, which are supposed to be valid, suitable or acceptable only under certain assumptions or only within certain context. For instance, compressible Navier-Stokes equations is a good continuum description of gas dynamics, if gas is not as rarefied as in a space shuttle entering the outer atmosphere.

In many applications, a PDE is a simplified approximated continuum modeling, as opposed to alternative particle models, e.g., the Boltzmann equation describes the statistical behaviour of a thermodynamic system, which can also be descirbed via molecular dynamics. PDEs have also been used for an efficient surrogate modeling of pedestrian flows or a flock of birds for which a particle model might seem more reasonable at least intuitively.

For beginners, equations can be assumed as given and *well-posed*, which roughly means that the equation has a unique nice solution. For a better understanding of the numerical methods, eventually one must understand the origin of the equation, which often plays a critical role in designing numerical schemes. Classical equations were mostly derived from *physical principles* (e.g., compressible Euler equations were derived from conservation of mass, momentum and energy) along with some empirical formula (e.g., equation of state for descrbing pressure dependence on mass, momentum and energy). On the other hand, in practical applications, many *ad hoc* equations have been proposed and used. For example, if we know $u_t = u_x$ represents *convection*, $u_t = u_{xx}$ represents *dissipation* and $u_t = u_{xxx}$

represents *dispersion*, then it makes sense, at least seemingly, to use $u_t = au_x + bu_{xx} + cu_{xxx}$ as a model equation for modeling a system of convection-dissipation-dispersion. Nonetheless, a common practice does not necessarily mean that it is the right way.

Examples of PDEs

- Wave equation
$$u_{tt} = u_{xx}$$
Sound waves in air and water, acoustics

- Heat equation
$$u_t = u_{xx}$$
Diffusion of heat, solutes, probability

- Laplace equation
$$u_{xx} + u_{yy} = 0$$
Steady-state potentials in heat conduction, electromagnetics, fluids

- Biharmonic equation
$$(\partial_{xx} + \partial_{yy})^2 u = 0$$
Solid mechanics, viscous fluids

- Poisson equation
$$\Delta u = f$$
Steady-state potentials in the presence of sources

- Elastic wave equation
$$u_{tt} = -u_{xxxx}$$
Sound waves in solids, seismology, structural mechanics

- Helmholtz equation
$$\Delta u + k^2 u = 0$$
Sound waves at prescribed frequency, scattering theory

- Schrödinger equation
$$iu_t = -u_{xx} + Vu$$
Quantum mechanics

- Korteweg-de Vries (KdV) equation
$$u_t + uu_x + u_{xxx} = 0$$
Solitons, water waves

- Klein-Gordon equation
$$u_{tt} = u_{xx} - u$$
Relativistic quantum mechanics

- Burgers equation
$$u_t + uu_x = \epsilon u_{xx}$$
Shock waves and rarefactions

- Fisher-KPP equation
$$u_t = u_{xx} + u(1-u)$$
traveling waves

- Allen-Cahn equation
$$u_t = u_{xx} + u - u^3$$
Structure formation in materials

- Kuramoto-Sivashinsky equation
$$u_t + uu_x = -u_{xx} - u_{xxxx}$$
Flames, turbulence, chaos

- Ginzburg-Landau equation
$$u_t = (1+i\nu)u_{xx} + u - (1+i\mu)u|u|^2$$
nonlinear evolution and amplitude modulation of disturbances

- Perona-Malik equation
$$u_t = \nabla \cdot (g(|\nabla u|)\nabla u)$$
Sharpening of images

- Navier-Stokes equations
Fluid mechanics with viscosity

- Euler equations
Fluid mechanics without viscosity, gas dynamics

- Maxwell's equations
Electromagnetic radiation, light and radio waves

- Einstein equations
General relativity, black holes

- Hodgkin-Huxley equations
  Propagation of signals in neurons

- Black-Scholes equations
  Valuation of options in finance

- Cauchy-Riemann equations
  Analytic function theory, complex analysis

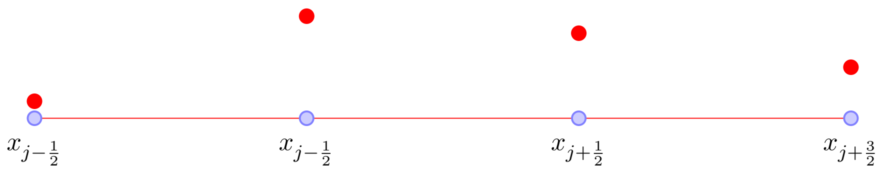
1.2 Numerical schemes

For PDEs, usually there are no exact solution formulae, and even if there is one, the formula can be demanding or dfficult to compute. One practial goal of numerical methods for PDEs is of course to provide a computationally *tractable* way for generating some kind of accurate approximations of the solution. Be aware that not all computational methods are *tractable* with given computational resources.

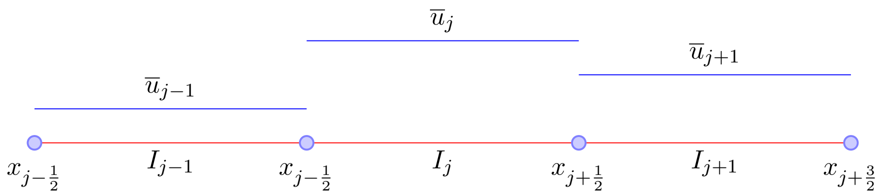
There are many popular numerical methods, which one may not have used but likely have heard of, such as *finite difference*, *finite element*, *finite volume* and *spectral methods*. As shown in Figure 1.1, approximations are obviously quite different in different numerical methods, which is however only a superficial way of understanding numerical schemes for PDEs. As a matter of fact, many of these different numerical methods can sometimes be regarded equivalent, especially for solving a one-dimensional problem.

The key is not the difference in the choice of approximation methods, but rather the PDEs that one needs to solve. For certain types of PDEs such as wave equations $u_{tt} = \Delta u$, almost all kinds of numerical methods can be used to obtain a useful numerical scheme. For many other types of PDEs, it can be hard to use even a very popular numerical method. Even though the popular finite element methods are equipped with various software packages and the most complete and beautiful mathematical theory, there are equations and problems that they cannot handle. There is no single numerical method to serve as a silver bullet, unless one is content with solving only particular kinds of PDEs.

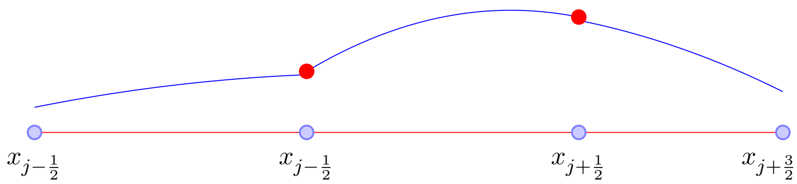
For example, finite volume schemes are successful for solving hyperbolic conservation laws and they are derived by discretizing the integral form of the conservation laws, and it is a perfectly natural thing to do because those PDEs are derived from the integral equations in the first place. On the other hand, it is very challenging to construct a scheme for hyperbolic conservation laws using spectral methods and continuous finite element methods.



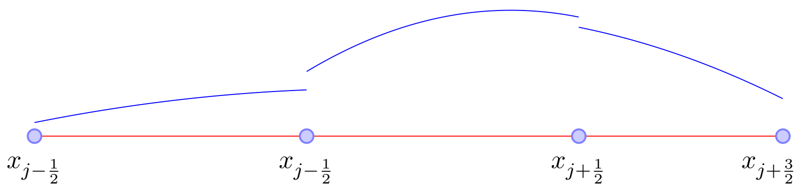
(a) Finite Difference



(b) Finite Volumne



(c) Continuous Finite Element



(d) Discontinuous Galerkin

Figure 1.1: An illustration of a few popular methods.

1.3 Computational tools

One particular emphasis of this lecture is the breadth of the scope. We will discuss a few different types of equations. The variety of different equations and different methods might seem overwhelming thus pose challenges, which however can become opportunities later because various methods provide ample inspiring perspectives of computational philosophy.

Many numerical methods go beyond solving PDEs. The simplest centered difference for Poisson's equation naturally extends to graph Laplacian on a graph. Numerical schemes for differential equations and numerical optimization algorithms are closely related. Many classical algorithms find roots in both territories. To name a few, the proximal point method for solving convex optimization, is nothing but backward Euler time discretization for numerical ODE.

Put simply, methods in numerical PDEs are also useful tools for other modern computational tasks.