Effective Control of Complex Turbulent Dynamical Systems through Statistical Functionals

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This manuscript was compiled on March 8, 2017

Turbulent dynamical systems characterized by both a high dimensional phase space and a large number of instabilities, are ubiquitous among complex systems in science and engineering including climate, material, and neural science. Control of these complex systems is a grand challenge, for example, in mitigating the effects of climate change or safe design of technology with fully developed shear turbulence. Control of flows in the transition to turbulence where there is a small dimension of instabilities about a basic mean state is an important and successful discipline. In complex turbulent dynamical systems, it is impossible to track and control the large dimension of instabilities which strongly interact and exchange energy, and new control strategies are needed. The goal of this paper is to propose an effective statistical control strategy for complex turbulent dynamical systems based on a recent statistical energy principle and statistical linear response theory. We illustrate the potential practical efficiency and verify this effective statistical control strategy on the forty dimensional Lorenz '96 model in forcing regimes with various types of fully turbulent dynamics with nearly half the phase space unstable.

statistical energy principle | response theory | statistical control

Turbulent dynamical systems characterized by both a high dimensional phase space and a large number of instabilities, are ubiquitous among complex systems in science and engineering (1–4) including climate, material, and neural science. Control of these complex systems is a grand challenge, for example, in mitigating the effects of climate change (5, 6) or safe design of technology with fully developed shear turbulence. Control of flows in the transition to turbulence where there is a small dimension of instabilities about a basic mean state is an important and successful discipline (7, 8). In complex turbulent dynamical systems, it is impossible to track and control the large dimension of instabilities which strongly interact and exchange energy (9), and new control strategies are needed.

The goal here is to propose an effective statistical control strategy for complex turbulent dynamical systems based on a recent statistical energy principle (10, 11) and statistical linear response theory (12–14). We illustrate the potential practical efficiency and verify this effective statistical control strategy on the forty dimensional Lorenz '96 (L-96) model in forcing regimes with various types of fully turbulent dynamics with nearly half the phase space unstable.

I) The statistical control theory proposed here has the goal and theoretical steps in its design:

A) Goal and statistical energy: The statistical energy, E, is the sum of the energy of the statistical mean and the trace of the statistical covariance (10, 11). A turbulent dynamical system is subjected to poorly known external forcing and the goal of the statistical control strategy is to find an effective deterministic feedback control to drive the statistical energy measured at some time back to a small neighborhood of a prescribed statistical steady state with energy, E_∞, in a finite time with a given cost.

B) Statistical energy as a Lyapunov functional: According to general recent theory (10, 11) the time rate of change of the statistical energy has a tendency to decay subject to forcing by the product of the current statistical mean, $\bar{u}\left(t\right)$, and the forcing control, $F\left(t\right)$.

C) Statistical linear response theory to define the control: With the perturbed forcing control, $\delta F\left(t\right)$, perturbed from the statistical steady state forcing $\bar{F}_\infty$, given the target statistical mean, $\bar{u}_\infty$, compute the linear statistical mean response (14),

$$\delta\bar{u}\left(t\right)=\int_0^t\mathcal{R}_{\bar{u}}\left(t-s\right)\delta F\left(s\right)ds,$$

by the fluctuation-dissipation theorem, perhaps with simple Gaussian approximation (12, 13, 15, 16). This procedure defines a memory dependent non-Markovian control $\delta F\left(t\right)$ for the statistical energy.

D) Explicit optimal local control: Transform the nonlocal control in C) to a local one and exactly solve the resulting

Significance Statement

Turbulent dynamical systems characterized by both a high dimensional phase space and a large number of instabilities, are ubiquitous among complex systems in science and engineering including climate, material, and neural science. Control of these complex systems is a grand challenge, for example, in mitigating the effects of climate change or safe design of technology with fully developed shear turbulence. In complex turbulent dynamical systems, it is impossible to track and control the large dimension of instabilities which strongly interact and exchange energy, and new control strategies are needed. The goal here is to propose an effective statistical control strategy for complex turbulent dynamical systems based on a recent statistical energy principle and statistical linear response theory.

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A.J.M. designed the research. A.J.M. and D.Q. performed the research. A.J.M. wrote the paper.

The authors declare no conflict of interest.

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quadratic linear regulator problem by Bellman's method (17, 18) to find an effective local feedback control, $\mathcal{C}^*(t)$.

E) Attribution of the local control $\mathcal{C}^*(t)$ to an effective forcing control, $\delta F^*(t)$: Explicitly invert step C) to determine $\delta F^*(t)$ from $\mathcal{C}^*(t)$.

II) A successful implementation and verification of the above strategy for control by statistical functionals has several very *attractive features*:

A) Only detailed statistical information in the target statistical steady state defined by $\bar{u}_\infty$ and E_∞ is needed. This can be determine by detailed observation or experiments.

B) Only an estimate of the statistical energy at the initial time of control and not any details of the forcing history are needed to set up the effective statistical control in *I)*.

C) Control of statistical energy by *I)* automatically gives control bounds on the mean and variance of the random state at spatial locations (10). For a climate mitigation scenario, this could be the mean and variance of the temperature at spatial locations; in general this is key information and provides important bounds for uncertainty quantification (19–21).

D) Various cost functions and specific forcing control strategies for using *I)* can be determined offline, without the need to run the actually complex turbulent system.

E) No explicit tracking or control of local instabilities is needed.

In the remainder of this paper, we sketch some background details of the statistical control strategy in *I)* and provide a detailed illustration, implementation, and verification on the L-96 model with various forcing and control scenarios, and explicitly demonstrate the attractive features in *II)*.

1. The Mathematical Structure of Turbulent Dynamical Systems

Consider the statistical behavior and control of quadratic systems with conservative nonlinear dynamics and unstable directions. In particular, consider the general turbulent dynamical system:

$$\frac{d\mathbf{u}}{dt} = (L + D)\,\mathbf{u} + B(\mathbf{u}, \mathbf{u}) + \mathbf{F}(t), \qquad [1]$$

acting on $\mathbf{u} \in \mathbb{R}^N$.

In the above equation we have:

- L, being a skew-symmetric linear operator representing the β-effect of Earth's curvature, topography, etc., and satisfying,

$$L^* = -L. \qquad [2a]$$

- D, being a negative definite symmetric operator,

$$D^* = D, \qquad [2b]$$

representing dissipative processes such as surface drag, radiative damping, viscosity, etc.

- The quadratic operator $B(\mathbf{u}, \mathbf{u})$ conserves the energy by itself so that it satisfies

$$\mathbf{u} \cdot B(\mathbf{u}, \mathbf{u}) = 0. \qquad [2c]$$

Such turbulent dynamical systems have a general statistical energy principle (10, 11) with many applications (19–21) and form the basis of the statistical control strategy. For simplicity in exposition here, assume that the damping above is constant multiple of the identity, $D = -dI$. Here is the statistical energy principle:

Under suitable general assumptions (10, 11), assume $D = -dI$, with $d > 0$, then the turbulent dynamical system [1] satisfies the closed statistical energy equation for $E = \frac{1}{2}\bar{\mathbf{u}} \cdot \bar{\mathbf{u}} + \frac{1}{2}\mathrm{tr}R$,

$$\frac{dE}{dt} = -2dE + \bar{\mathbf{u}} \cdot \mathbf{F}, \qquad [3]$$

where $\bar{\mathbf{u}}(t)$ is the statistical mean and R is the covariance matrix.

2. General control with linearized statistical energy functional equation

In a statistical equilibrium state, we have the relation

$$2dE_\infty = \bar{\mathbf{u}}_\infty \cdot \bar{\mathbf{F}}_\infty. \qquad [4]$$

Thus the equilibrium statistical energy E_∞ can be calculated through the equilibrium mean.

Focus on the small amplitude fluctuations about the equilibrium mean state $\mathbf{u}' = \mathbf{u} - \bar{\mathbf{u}}_\infty$, thus the statistical energy fluctuation functional becomes

$$E' = E - E_\infty = \bar{\mathbf{u}}_\infty \cdot \delta\bar{\mathbf{u}} + \frac{1}{2}\delta\mathrm{tr}R + \frac{1}{2}\left|\delta\bar{\mathbf{u}}\right|^2, \qquad [5]$$

where we define $\delta\bar{\mathbf{u}} = \bar{\mathbf{u}} - \bar{\mathbf{u}}_\infty$ as the fluctuation about the equilibrium mean, and $\delta\mathrm{tr}R = \mathrm{tr}R - \mathrm{tr}R_\infty$ as the fluctuation about the total variance (equivalently the single-point variance at each grid point). We want to control the statistical energy fluctuation E' back to zero (thus the system goes back to unperturbed equilibrium) via control on the mean state with only deterministic control forcing added. If we achieve the goal in controlling the total statistical energy to zero, automatically we succeed in controlling the mean state fluctuation, $\delta\bar{\mathbf{u}}$, and the single-point variance fluctuation, $\delta\mathrm{tr}R$, at the same time.

Linearized statistical energy identity about fluctuation. By subtracting the mean equilibrium statistics [4] from the original statistical energy equation [3], we have the *Perturbed Statistical Energy Equation*

$$\frac{dE'}{dt} = -2dE' + \bar{\mathbf{u}}_\infty \cdot \delta\mathbf{F} + \bar{\mathbf{F}}_\infty \cdot \delta\bar{\mathbf{u}} + O\left(\delta^2\right), \qquad [6]$$

where $O\left(\delta^2\right) = \delta\bar{\mathbf{F}} \cdot \delta\bar{\mathbf{u}}$ is for the higher order terms. Here we assume the external forcing perturbation is kept in small amplitude, thus the perturbed response in the mean is also small. Then we only need to focus on the leading order responses in $O(\delta)$. The task here is to find proper control to drive the perturbed energy E' back to zero efficiently with minimum cost.

Statistical response for the mean state from statistical linear response theory. In the above linearized equation, we only consider the linearized first-order terms on the right hand side of the equation [6]. Use the fluctuation-dissipation theorem

(FDT) (12, 13, 15, 22) to replace the response in the mean, $\delta\bar{\mathbf{u}}$, using the mean response operator

$$\delta\bar{\mathbf{u}} = \int_0^t \mathcal{R}_{\bar{u}}\left(t-s\right)\delta\mathbf{F}\left(s\right)ds + O\left(\delta^2\right). \quad [7]$$

Above $\mathcal{R}_{\bar{u}}$ is called the *linear response operator* about the statistical mean state, thus it only requires information from the equilibrium distribution p_∞ with the unperturbed system

$$\mathcal{R}_{\bar{u}}\left(t\right) = \left\langle \mathbf{u}\left(t\right)B\left[\mathbf{u}\left(0\right)\right]\right\rangle_\infty, \quad B\left(\mathbf{u}\right) = -\frac{\mathrm{div}_{\mathbf{u}}\left(\mathbf{w}p_\infty\right)}{p_\infty}, \quad [8]$$

with the forcing perturbation in the form $\delta\mathbf{F} = \mathbf{w}\left(\mathbf{u}\right)\delta f\left(t\right)$. In the present application with changes in external forcing, $\mathbf{w}\left(\mathbf{u}\right)$ is simply a constant vector. The linear response from FDT forms a non-Markovian delayed control. Especially if we make the quasi-Gaussian approximation for [8], that is, set $p_\infty \propto \exp\left(-\frac{1}{2}\mathbf{u}'^T R_\infty^{-1}\mathbf{u}'\right)$, the linear response operator for the mean becomes

$$\mathcal{R}_{\bar{u},ij}\left(t\right) = \left\langle \left(u_i\left(t+s\right) - \bar{u}_{i,\infty}\right)\mathbf{e}_j \cdot R_\infty^{-1}\left(\mathbf{u}\left(s\right) - \bar{\mathbf{u}}_\infty\right)^T\right\rangle. \quad [9]$$

Note that when we use linear Gaussian models (12, 16) to approximate the system, this above formula in [9] becomes exact for the linear response operator. There is high skill in approximating the mean both theoretically (13, 14) and for many complex turbulent dynamical systems (12, 19, 23, 24).

The L-96 model as a turbulent dynamical system. The simplest prototype example of a turbulent dynamical system to illustrate and verify the statistical control strategy is due to Lorenz and is called the L-96 model (25). It is widely used as a test model for algorithms for prediction, filtering, and low frequency climate response (13), as well as algorithms for UQ (19, 26). The L-96 model is a discrete periodic model given by the following system

$$\frac{du_j}{dt} = \left(u_{j+1} - u_{j-2}\right)u_{j-1} - u_j + F, \quad j = 0, \cdots, J-1, \quad [10]$$

with $J = 40$ and with F the forcing parameter. The model is designed to mimic baroclinic turbulence in the midlatitude atmosphere with the effects of energy conserving nonlinear advection and dissipation represented by the first two terms in [10]. In order to quantify and compare the different types of turbulent chaotic dynamics in the L-96 model as F is varied, the transformation $u_j = \bar{u} + E_p^{1/2}\tilde{u}_j$, $t = \tilde{t}E_p^{-1/2}$ is utilized where E_p is the energy fluctuations (13). After this normalization, the dynamical equation in terms of the new variables, $\tilde{u}_j$, becomes

$$\frac{d\tilde{u}_j}{d\tilde{t}} = \left(\tilde{u}_{j+1} - \tilde{u}_{j-2}\right)\tilde{u}_{j-1} + E_p^{-1/2}\left(\left(\tilde{u}_{j+1} - \tilde{u}_{j-2}\right)\bar{u} - \tilde{u}_j\right) + E_p^{-1}\left(F - \bar{u}\right). \quad [11]$$

Table S1 in *SI* lists in the non-dimensional coordinates, the stability analysis and statistical data in the L-96 model as F is varied through $F = 6, 8, 16$. Snapshots of the time series for [10], as depicted in Figure S1 in *SI*, qualitatively confirm the quantitative intuition with weakly turbulent patterns for $F = 6$, strongly chaotic wave turbulence for $F = 8$, and fully developed wave turbulence for $F = 16$.

Deterministic control of the unstable modes. It is worthwhile to briefly comment on a standard deterministic control strategy (7, 8) for the L-96 model and its limitations. To control the instabilities about the mean state, it is natural to use the formulation in [11]. The linear operator in [11] has sixteen unstable modes for $F = 6$ and eighteen for $F = 8, 16$ (13). Thus nearly half of the modes of the forty dimensional system need to be controlled.

3. Effective statistical control of the L-96 Model

The L-96 model is invariant to spatial translation and has homogeneous statistics (19), so the statistical mean is a time dependent scalar in response to homogeneous forcing, $F\left(t\right) = F + \delta F\left(t\right)$ in [10], which is assumed here. We follow the above general strategy for statistical control of perturbed energy [5] through statistical linear response in [7]-[9].

Thus, we consider the statistical energy fluctuation defined as

$$E' = E - E_\infty, \quad E_\infty = \left(2d\right)^{-1}\bar{u}_\infty\bar{F}_\infty,$$

according to a scalar (deterministic) control $\kappa\left(t\right)$ about the mean state. Considering all these simplifications, the *homogeneous linear statistical control equation* for the L-96 model can be rewritten as

$$\frac{dE}{dt} = -2dE + \bar{u}_\infty\kappa\left(t\right) + \bar{F}_\infty\int_0^t \mathcal{R}_{\bar{u}}\left(t-s\right)\kappa\left(s\right)ds, \; E\left(0\right) = E_0. \quad [12]$$

Above the primes are dropped in the statistical energy fluctuation E', and $\mathcal{R}_{\bar{u}}$ is the linear response operator for the scalar mean state defined in [9]. According to the statistical energy control equation, we can introduce a *local control* $\mathcal{C}\left(t\right)$ for the statistical energy identity as a functional of the control forcing $\kappa\left(t\right)$

$$\mathcal{C}\left(t\right) = \bar{u}_\infty\kappa\left(t\right) + \bar{F}_\infty\int_0^t \mathcal{R}_{\bar{u}}\left(t-s\right)\kappa\left(s\right)ds. \quad [13]$$

Then the general control problem becomes: i) find the optimal control strategy for $\mathcal{C}$; and then ii) invert the (nonlocal) functional $\mathcal{C}$ to get the explicit forcing control strategy for κ. In this way, we can first only focus on the general control functional $\mathcal{C}\left(t\right)$ for the statistical energy identity, and then consider the inversion problem.

The linear statistical control problem can be solved directly following *dynamic programming* (17, 18). Next we construct the linear statistical control problem by proposing a cost function to optimize. The control system is defined as

$$\frac{dE}{ds} = -2dE\left(s\right) + \mathcal{C}\left(s\right), \quad E\left(t\right) = x, \; t \le s \le T, \quad [14]$$

with $\mathcal{C}\left(t\right)$ a general control functional. To find the optimal control $\mathcal{C}^*\left(t\right)$ the cost function to minimize is proposed in the following form

$$\mathcal{F}_\alpha\left[\mathcal{C}\left(\cdot\right)\right] \equiv \int_t^T E^2\left(s\right) + \alpha\mathcal{C}^2\left(s\right)ds,$$
$$\mathcal{C}^* = \arg\min_{\mathcal{C}}\mathcal{F}_\alpha\left[\mathcal{C}\left(\cdot\right)\right]. \quad [15]$$

The cost function is defined in the simplest form as a combination about the energy and control. The parameter $\alpha > 0$ is introduced to add a balance between the two components in energy E and control $\mathcal{C}$. The larger value of α adds more weight on the control parameter in the process.

Remark 1 *The statistical control problem in [14] is quite universal representing a large group of systems with homogeneous damping. The true turbulent system could be nonlinear and complicated, as long as it has the energy conserving property and the symmetries that guarantee the statistical energy identity as the abstract form in [1]. Later we can see that the control parameter $\mathcal{C}$ can even include the random forcing control in the system. Furthermore by introducing the local control $\mathcal{C}$, no specific forcing and mean statistics are required in explicit form. Thus in the first step, we only need to concentrate on the general control equation [14] according to the cost [15].*

Optimal control from scalar Riccati equation. Now we derive the robust optimal control $C^*(t)$ for time interval $[0,T]$ with varying cost depending on α. It is well known (17, 18) that the scalar control problem in [14] and [15] is solved by a scalar Riccati equation, that is,

$$\begin{aligned}\frac{dK}{dt} &= \alpha^{-1}K^2 + 4dK - 1, \quad 0 \le t < T,\\ K(T) &= k_T.\end{aligned} \qquad [16]$$

Above it is a backward equation in time about $K(t)$. Therefore the optimal feedback control $\mathcal{C}^*(t)$ together with the optimal control statistical equation for E^* becomes

$$\begin{aligned}\mathcal{C}^*(t) &= -\alpha^{-1}K(t)E^*(t), \quad 0 \le t < T,\\ \frac{dE^*}{dt} &= -2dE^*(t) - \alpha^{-1}K(t)E^*(t),\end{aligned} \qquad [17]$$

with the initial fluctuation energy condition $E^*(0) = E_0$. Above $-\alpha^{-1}K(t)E^*(t)$ defines the feedback control due to the minimum cost constraint.

Suppose we have the optimal control $K(t)$ by solving [16], then the exact solution of [17] can be calculated directly, that is,

$$E^*(t) = E_0 \exp\left(-2dt - \alpha^{-1}\int_0^t K(s)\,ds\right). \qquad [18]$$

Note that $E(t)$ is actually the energy fluctuation, thus it can be either positive or negative depending on its initial value. Further notice that the above optimal solution has one additional degree of freedom about the final endpoint value k_T. Therefore *the statistical energy control auxiliary problem* can be formulated as

$$\max_{k_T}\left(2\alpha dT + \int_0^T K(t;k_T)\,dt\right) \Leftrightarrow \max_{k_T}\int_0^T K(t;k_T)\,dt. \qquad [19]$$

We calculate the explicit solution for the scalar Riccati equation in [16]

$$\left|\frac{K(t)-K_-}{K(t)-K_+}\right| = C(k_T)\exp\left[2\left(4d^2+\alpha^{-1}\right)^{1/2}(T-t)\right], \quad [20]$$

where

$$K_\pm = -2\alpha d \pm \left(4\alpha^2 d^2 + \alpha\right)^{1/2}$$

are the two roots (fixed points) of the quadratic polynomial on the right hand side of [16], and $C(k_T) = \left|\frac{k_T-K_-}{k_T-K_+}\right|$ is the coefficient due to the endpoint condition. As a special fixed-in-time solution, if we take $K(t) \equiv K_+$, the optimal solution in [18] becomes

$$E^*(t) = \exp\left[-\left(4d^2+\alpha^{-1}\right)^{1/2} t\right] E_0. \qquad [21]$$

See *SI* for the derivation and properties of the exact solution.

Attribution of the optimal local control to a forcing control. In the final step of the statistical control strategy, given the local optimal control $\mathcal{C}^*(t)$, one needs to invert the nonlocal operator in [13] to determine the forcing control strategy, $\kappa^*(t)$. For the L-96 model, $\mathcal{R}_{\bar{u}}(t)$ is a real scalar operator and with the Gaussian approximation in [9], we build a linear regression model to approximate the autocorrelation, $\mathcal{R}_{\bar{u}}(t)$, by $\mathcal{R}_{\bar{u}}^M(t) = \exp(-\gamma_M t)$ (12, 27). Therefore we get the dynamical equations for the autocorrelation $\mathcal{R}_{\bar{u}}^M$

$$\frac{d\mathcal{R}_{\bar{u}}^M}{dt} = -\gamma_M \mathcal{R}_{\bar{u}}^M(t), \quad \mathcal{R}_k^M(0) = 1. \qquad [22]$$

and the corresponding linear response $\mathcal{L}_{\bar{u}} = \int_0^t \mathcal{R}_{\bar{u}}(t-s)\kappa(s)\,ds$ is exact for the response in the mean state

$$\begin{aligned}\frac{d\mathcal{L}_{\bar{u}}^M}{dt} &= -\gamma_M\mathcal{L}_{\bar{u}}^M(t) + \kappa(t), \quad \mathcal{L}_{\bar{u}}(0) = 0,\\ \mathcal{L}_{\bar{u}}^M(t) &= \int_0^t \mathcal{R}_{\bar{u}}^M(t-s)\kappa(s)\,ds.\end{aligned} \qquad [23]$$

The optimal parameter for γ_M is chosen by a spectral information criterion (27). The fit of the true autocorrelation by their approximation is shown in Figure S2 in *SI* for $F = 6, 8, 16$ with good results.

With the above regression model, the problem is to find the optimal forcing control $\kappa^*(t)$ through the inversion about [13]

$$\mathcal{C}(t) = \bar{u}_\infty\kappa(t) + \bar{F}_\infty\mathcal{L}(t), \quad \mathcal{L}(t) = \int_0^t \mathcal{R}_{\bar{u}}^M(t-s)\kappa(s)\,ds.$$

From the optimal solution by statistical control of the energy equation

$$\begin{aligned}\mathcal{C}^*(t) &= -\alpha^{-1}K(t)E^*(t),\\ E^*(t) &= \exp\left(-2dt - \alpha^{-1}\int_0^t K(s)\,ds\right)E_0,\end{aligned}$$

we calculate using again the scalar Riccati equation [16]

$$\begin{aligned}\frac{d\mathcal{C}^*}{dt} &= -\alpha^{-1}\dot{K}E^* + \mathcal{C}^*\left(-2d - \alpha^{-1}K(t)\right)\\ &= 2d\mathcal{C}^* + \alpha^{-1}E^*.\end{aligned}$$

Above we explicitly use the scalar Riccati equation and the explicit form of the optimal solution E^*. On the other hand, the derivative about the right hand side of [13] gives

$$\begin{aligned}\frac{d\mathcal{C}^*}{dt} &= \bar{u}_\infty\frac{d\kappa}{dt} + \bar{F}_\infty\frac{d\mathcal{L}}{dt}\\ &= \bar{u}_\infty\frac{d\kappa}{dt} + \bar{F}_\infty(-\gamma\mathcal{L}) + \bar{F}_\infty\kappa\\ &= \bar{u}_\infty\dot{\kappa} - \gamma\left(\mathcal{C}^* - \bar{u}_\infty\kappa\right) + \bar{F}_\infty\kappa.\end{aligned}$$

Above the second equality uses [23] and the third equality uses [13] to replace $\mathcal{L}$ again. Combining the above two equations, we find the dynamical equation to solve for κ, that is,

$$\begin{aligned}\frac{d\kappa}{dt} + \left(\gamma_M + {}^{\bar{F}_\infty}/_{\bar{u}_\infty}\right)\kappa(t) + \frac{(\gamma_M + 2d)K(t) - 1}{\alpha\bar{u}_\infty}E^*(t) = 0,\\ \kappa(0) = \mathcal{C}^*(0)/\bar{u}_\infty = -\alpha^{-1}K(0)E_0/\bar{u}_\infty.\end{aligned} \qquad [24]$$

with initial value $\kappa(0)$ from $\mathcal{C}^*(0)$. Actually once we get the smooth solutions for $\mathcal{C}^*(t), E^*(t)$, the above equation [24] is just a first-order ODE with constant coefficients. Thus it can be solved efficiently.

As a special example, if we use the approximated solution of E^* in [21]

$$E^*(t) = \exp(-\lambda_1 t) E_0, \quad \lambda_1 = \left(4d^2 + \alpha^{-1}\right)^{1/2},$$

the explicit solution can be written as

$$\kappa^*(t) = E_0 e^{-\lambda_2 t}\left[-\frac{K_+}{\alpha \bar{u}_\infty} + G\frac{1-e^{(\lambda_2-\lambda_1)t}}{\lambda_2-\lambda_1}\right], \qquad [25]$$

where

$$\lambda_2 = \gamma_M + {}^{\bar{F}_\infty}/_{\bar{u}_\infty}, \quad G = \frac{(\gamma_M + 2d) K_+ - 1}{\alpha \bar{u}_\infty}.$$

Notice that if we assume E_0 as a random variable in the initial time, the proceeding optimal control forcing $\kappa^*(t)$ can also be random dependent on the randomness in E_0. The randomness from the initial value will be linearly related with the later state of the control forcing $\kappa(t)$.

4. Numerical verification for the optimal statistical control

Setup of the statistical control problem for L96 system. We consider the statistical control for the homogeneous 40-dimensional L-96 system [10] with state variables such that

$$\frac{du_j}{dt} = (u_{j+1} - u_{j-2}) u_{j-1} - u_j + \bar{F}_\infty + \delta F(t) + dH(t).$$

The equilibrium forcing varies as $\bar{F}_\infty = 6, 8, 16$ where the system is changing from strongly non-Gaussian statistics to a near Gaussian regime with full turbulence. The deterministic forcing perturbation is taken as a ramp-type forcing

$$\delta F(t) = f_0 \frac{\tanh a(t-t_c) + \tanh a t_c}{1 + \tanh a t_c},$$

with upward forcing for $\bar{F}_\infty = 8, 16$, and downward forcing for $F = 6$ with a 10% ramp amplitude compared with $\bar{F}_\infty$. In the test we add random perturbation to f_0 by a small amplitude (homogeneous) random forcing as white noise

$$dH = \sigma_0 dW_t.$$

As a result the final energy spectrum will be changed even with small perturbation σ_0. We use this additional random forcing here to test *the method's robustness* due to small random perturbations.

In this homogeneous setup, the mean state is uniform in each grid point and the covariance matrix is diagonal in the spectral domain. The statistical energy functional can be defined as

$$E(t) = \frac{1}{2}\bar{u}^2(t) + \frac{1}{2J}\sum_{j=1}^{J} r_j(t), \qquad [26]$$

The model parameters in two test cases with and without random forcing and unperturbed model statistics are listed in Table S2 in *SI*. Furthermore, we show the equilibrium energy spectra in these test cases in Figure S3 in *SI*. As illustrated, adding even small random forcing in the system can greatly increase the variability in the zero mode, and thus vastly change the entire energy spectrum to a more active state.

The dynamical equation for the statistical energy in [26] in this homogeneous case can be derived as

$$\frac{dE}{dt} = -2dE + \bar{u}F + \sigma^2.$$

Above $\sigma^2 = \frac{1}{J}\sum_{j=1}^{N}\sigma_j^2 = \sigma_0^2$ is the total effect from random forcing in the system. In statistical equilibrium state we have the relation

$$2dE_\infty = \bar{u}_\infty \bar{F}_\infty + \sigma^2. \qquad [27]$$

Verification of the optimal statistical control. In this final section, we verify the optimal control achieved from the previous optimal statistical control strategy and test it on the L-96 model to check the control performance. We run the true L-96 system using Monte-Carlo simulations with an ensemble size $N = 10000$ to get accurate statistics. To check the control skill with different perturbed initial data, we first apply the ramp-type perturbation in the system, and then replace the forcing with the control at a later time. The ramp amplitude is taken as $f_0 = -0.4, 1, 1.5$ for $F = 6, 8, 16$ cases respectively. Note that in the weakly chaotic case $F = 6$, we choose the downward ramp so that the statistics of the system will change drastically. Besides, we use the parameter value $\alpha = 0.1$ for all the tests.

Still we need to decide the time instant to add the control and the initial value (with perturbation) for the control to begin with. In general the verification can be carried out according to the following steps:

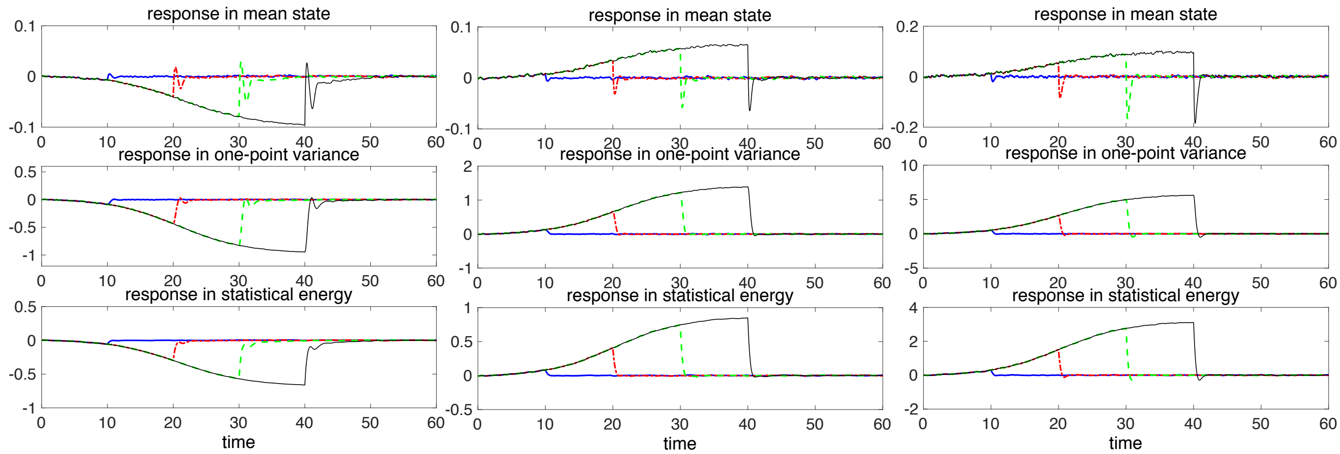
1) Choose the time T_{ctrl} as the start time to apply control. Then run the original model with original forcing perturbations δF up to the control time T_{ctrl};

2) Use the statistics at time T_{ctrl} as the initial value of the control, and switch the original forcing perturbation δF to the optimal control forcing κ from this time on as the forcing perturbation;

3) Run the model up to the final time T, and check the model responses in the statistics going back to the unperturbed state as the control κ is applied.

In the case with small random forcing perturbation, we can also consider additional Gaussian perturbations in the model with small amplitude. This setup is used to test the robustness of the control strategy. In the randomly perturbed case, we still use the same set of optimal control parameters as the case without random perturbation and check whether the control parameter can maintain the performance with the random noise.

Control verification on full Monte-Carlo simulation. We consider four different control cases by adding the control at $T_{\mathrm{ctrl}} = 10, 20, 30, 40$, while the total run time is $T = 60$. Through the ramp-type forcing the system is gradually shifted to another state, and the control added at different time T_{ctrl} can be used to test the skill of control with various out of equilibrium perturbed states. In Figure 1, time-series of the responses in the mean $\delta\bar{u} = \bar{u} - \bar{u}_\infty$, in the one-point variance $\delta \mathrm{tr} R/J = r_{1\mathrm{pt}} - r_{1\mathrm{pt},\infty}$, and in the total statistical energy $\delta E = E - E_\infty$ are compared in the three test regimes $F = 6$ (weakly chaotic), $F = 8$ (strongly chaotic), and $F = 16$ (fully turbulent) without random perturbation.



(a) Weakly chaotic regime $F = 6$ **(b)** Strongly chaotic regime $F = 8$ **(c)** Fully turbulent regime $F = 16$

Fig. 1. Statistical control of L-96 system applied at four different states $T_{\mathrm{ctrl}} = 10, 20, 30, 40$ without random forcing perturbation. Controlled responses (subtracting the equilibrium states) in mean state, one-point variance, and total statistical energy through true MC model using the optimal control forcing $\kappa(t)$ are shown.

First in the cases without random forcing, the optimal control κ efficiently drives the system back to the unperturbed state. Notice the initial mean overshoot error for $\bar{F}_\infty = 6$, but the error in the mean is much smaller than the response error in the variance, while the variance can always converge in a fast rate with no further oscillation under the control. The case with random forcing $\sigma_0 \dot{W}$ is shown in Figure S4 in *SI*. The control $\kappa(t)$ may no longer be optimal. Still we can use the achieved control to test the robustness of this method due to small random perturbations. As a result, larger fluctuations appear, nevertheless the optimal control displays significant skill in driving the system back to equilibrium with the original optimal control.

5. Concluding discussion

An efficient general statistical control strategy for complex turbulent dynamical systems as outlined in *I)* with all the attractive features in *II)* has been introduced here. The method has been illustrated and developed in detail for the L-96 model. The proposed statistical control strategy has been verified with significant robust skill through extensive numerical experiments. The statistical control strategies here are potentially very useful for extremely complex turbulent systems (20, 21), but this requires further detailed investigation.

ACKNOWLEDGMENTS. This research of the A.J.M. is partially supported by the Office of Naval Research through MURI N00014-16-1-2161 and DARPA through W911NF-15-1-0636. D.Q. is supported as a graduate research assistant on these grant.

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Supporting Information (SI): Effective Control of Complex Turbulent Dynamical Systems through Statistical Functionals

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March 6, 2017

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1 Illustration about the statistics in L-96 system

We illustrate the stability and statistical properties of the 40-dimensional L-96 model used as the test model for control strategies in the main text

$$\frac{du_j}{dt} = (u_{j+1} - u_{j-2})\, u_{j-1} - u_j + F, \quad j = 0, \cdots, J-1, \tag{S1}$$

in various dynamical regimes with numerical simulations. The model is designed to mimic baroclinic turbulence in the midlatitude atmosphere with the effects of energy conserving nonlinear advection and dissipation represented by the first two terms in (S1). For sufficiently strong forcing values such as $F = 6, 8, 16$, the L-96 is a prototype turbulent dynamical system which exhibits features of weakly chaotic turbulence ($F = 6$), strong chaotic turbulence ($F = 8$), and strong turbulence ($F = 16$) [1] as the strength of forcing, F, is increased. Table S1 lists in the non-dimensional coordinates, the leading Lyapunov exponent, λ_1, the dimension of the unstable manifold, N^+, the sum of the positive Lyapunov exponents (the KS entropy), and the correlation time, T_{corr}, of any $\tilde{u}_j$ variable with itself as F is varied through $F = 6, 8, 16$. Note that λ_1, N^+ and KS increase significantly as F increases while T_{corr} decreases in these non-dimensional units; furthermore, the weakly turbulent case with $F = 6$ already has a twelve dimensional unstable manifold in the forty dimensional phase space. Snapshots of the time series for (S1) with $F = 6, 8, 16$, as depicted in Figure S1, qualitatively confirm the above quantitative intuition with weakly turbulent patterns for $F = 6$, strongly chaotic wave turbulence for $F = 8$, and fully developed wave turbulence for $F = 16$. It is worth remarking here that smaller values of F around $F = 4$ exhibit the more familiar low-dimensional weakly chaotic behavior associated with the transition to turbulence.

In the attribution of the optimal local control for the L-96 model, we use the Gaussian approximation for the autocorrelation $\mathcal{R}_{\bar{u}}(t)$ which is a real scalar operator. A linear regression model is built to approximate the autocorrelation, $\mathcal{R}_{\bar{u}}(t)$, by $\mathcal{R}_{\bar{u}}^M(t) = \exp(-\gamma_M t)$. The optimal parameter for γ_M is chosen by a spectral information criterion [2]. The fit of the true autocorrelation by their approximation is shown in Figure S2 for $F = 6, 8, 16$ with excellent results.

	F	λ_1	N^+	KS	T_{corr}
weakly chaotic	6	1.02	12	5.547	8.23
strongly chaotic	8	1.74	13	10.94	6.704
fully turbulent	16	3.945	16	27.94	5.594

Table S1: Dynamical properties of L-96 model for regimes with $F = 6, 8, 16$. λ_1 denotes the largest Lyapunov exponent, N^+ denotes the dimension of the expanding subspace of the attractor, KS denotes the Kolmogorov-Sinai entropy, and T_{corr} denotes the decorrelation time of energy-rescaled time correlation function.

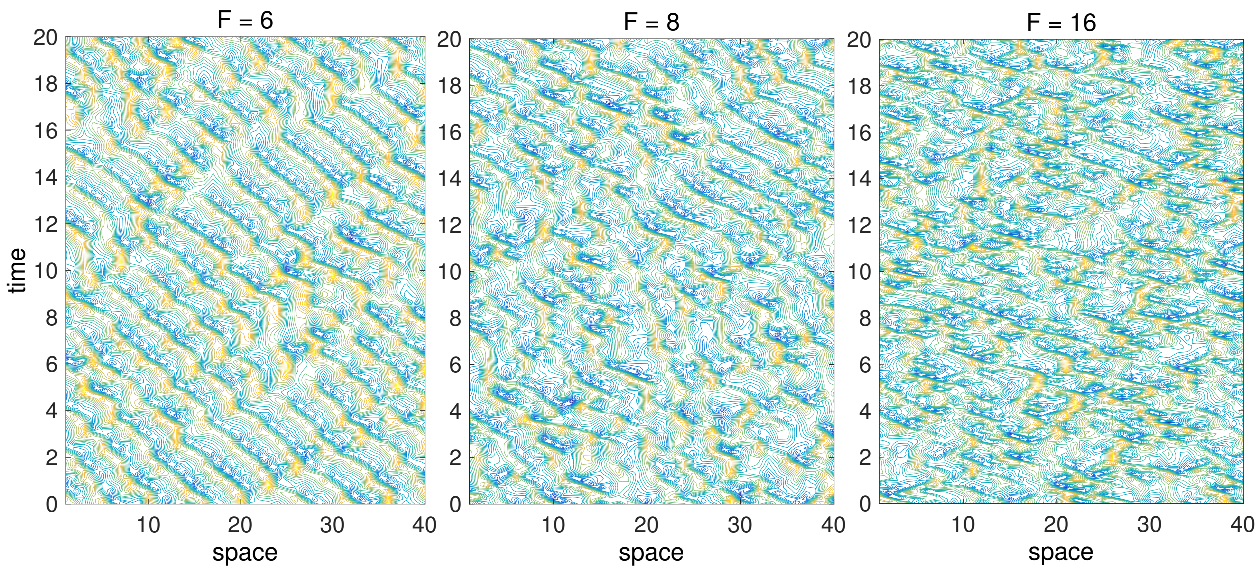


Figure S1: Space-time of numerical solutions of L-96 model for weakly chaotic ($F = 6$), strongly chaotic ($F = 8$), and fully turbulent ($F = 16$) regime.

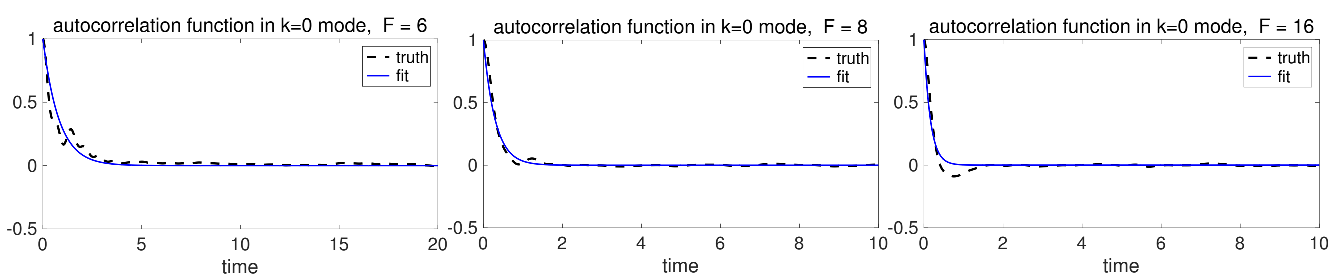


Figure S2: Autocorrelation (AC) functions for three regimes $F = 6, 8, 16$ in L-96 model. The truth from MC simulation is compared with approximation $\mathcal{R}_{\bar{u}}^{M}(t) = \exp(-\gamma_M t)$. The fits for ACs in the zero mode $\langle \hat{u}_0(t)\, \hat{u}_0(0) \rangle$ are shown for predicting responses in the mean state.

2 Explicit solution of scalar Riccati equation

We display the calculation of the explicit solution for the scalar Riccati equation as in the main text

$$\frac{dK}{dt}=\alpha^{-1}K^{2}+4dK-1=P_{2}\left(K\right),\ K\left(T\right)=k_{T},\quad 0\leq t<T.$$

There exist two roots (fixed points) of the quadratic polynomial P_2, that is,

$$K_{\pm}=-2\alpha d\pm\left(4\alpha^{2}d^{2}+\alpha\right)^{1/2}. \tag{S2}$$

Note that $K_+>0$ and $K_-<0$. And the stability of the two steady state can be implied by

- $P_2'\left(K_+\right)<0$, the equation is backward stable at K_+;
- $P_2'\left(K_-\right)>0$, the equation is backward unstable at K_-.

Thus we can solve the equation by integrating the above Riccati equation directly

$$\frac{1}{K_{+}-K_{-}}d\left(\ln\left|K-K_{+}\right|-\ln\left|K-K_{-}\right|\right)=\alpha^{-1}dt,$$

from t to the final time T and using the final end-point value. Therefore we have the explicit solution

$$\left|1+\frac{2\left(4\alpha^{2}d^{2}+\alpha\right)^{1/2}}{K\left(t\right)-K_{+}}\right|=\left|\frac{K\left(t\right)-K_{-}}{K\left(t\right)-K_{+}}\right|=C\left(k_{T}\right)\exp\left[2\left(4d^{2}+\alpha^{-1}\right)^{1/2}\left(T-t\right)\right], \tag{S3}$$

where $C\left(k_T\right)=\left|\frac{k_T-K_-}{k_T-K_+}\right|$ is the coefficient due to the endpoint condition. We have the non-negative constraint for $K\left(t\right)\geq 0$ always guaranteed through the explicit solution in (S3).

Special fixed-in-time solution

The simplest strategy is to use the fixed point K_+ as the steady state solution of $K\left(t\right)$. Thus the optimal solution $E^*\left(t\right)$ for the statistical energy

$$E^{*}\left(t\right)=E_{0}\exp\left(-2dt-\alpha^{-1}\int_{0}^{t}K\left(s\right)ds\right).$$

becomes the simpler form

$$E^{*}\left(t\right)=\exp\left[-\left(4d^{2}+\alpha^{-1}\right)^{1/2}t\right]E_{0}. \tag{S4}$$

We can observe the solutions in the asymptotic limit

- low cost limit: $\alpha\to 0$, $E\sim\exp\left(-\alpha^{-1/2}t\right)E_0$, energy decays in rate $1/\sqrt{\alpha}$ to achieve fast statistical energy decay;
- high cost regime: $\alpha^{-1}\to 0$, $E\sim\exp\left(-2dt-\left(4d\alpha\right)^{-1}t\right)E_0$, no control is needed in the leading order.

From the asymptotic behavior in (S4), large values of $\alpha\gg 1$ is equivalent to the case with no control added at all; while small values of $\alpha\ll 1$ means stronger forcing from the control $\mathcal{C}$ but increasing the cost in control $\int_0^T\mathcal{C}^2\left(s\right)ds$ at the same time.

		σ_0	$\bar{u}_\infty$	$r_{1\text{pt},\infty}$	$r_{u_0,\infty}$	E_∞
$F=6$	without random forcing	0	2.0123	8.0244	2.5959	6.0370
	with Gaussian random forcing	1.5	1.9366	8.9941	20.6986	6.3723
$F=8$	without random forcing	0	2.3416	13.2503	5.4544	9.3668
	with Gaussian random forcing	2	2.2400	14.9025	32.1200	9.9601
$F=16$	without random forcing	0	3.0863	39.8572	23.2976	24.6913
	with Gaussian random forcing	4	2.8923	45.9138	94.8999	27.1396

Table S2: Model parameters for the test cases with and without random forcing. The unperturbed model statistics in equilibrium, $\delta F=0$, are listed in the following columns. r_{u_0} is the variance in the base mode $k=0$. The random forcing can effective increase the energy in the zero mode.

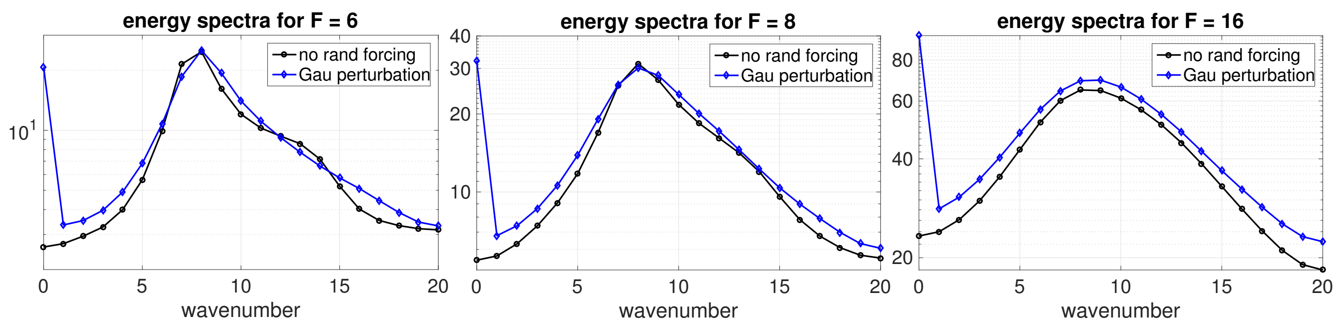


Figure S3: Equilibrium energy spectra in the three test regimes $F=6,8,16$ in L-96 model. The differences between the case with no random forcing and small white noise perturbation are compared.

3 Optimal statistical control for L-96 system with random noise perturbation

We show the statistical control for the homogeneous 40-dimensional L-96 system

$$\frac{du_j}{dt}=\left(u_{j+1}-u_{j-2}\right)u_{j-1}-u_j+\bar{F}_\infty+\delta F\left(t\right)+dH\left(t\right),\quad j=1,\cdots,J=40,$$

perturbed by random forcing as white noise

$$dH=\sigma_0 dW_t.$$

The equilibrium forcing varies as $\bar{F}_\infty=6,8,16$ where the system is changing from strongly non-Gaussian statistics to a near Gaussian regime with full turbulence, and the deterministic forcing perturbation is taken as a ramp-type forcing

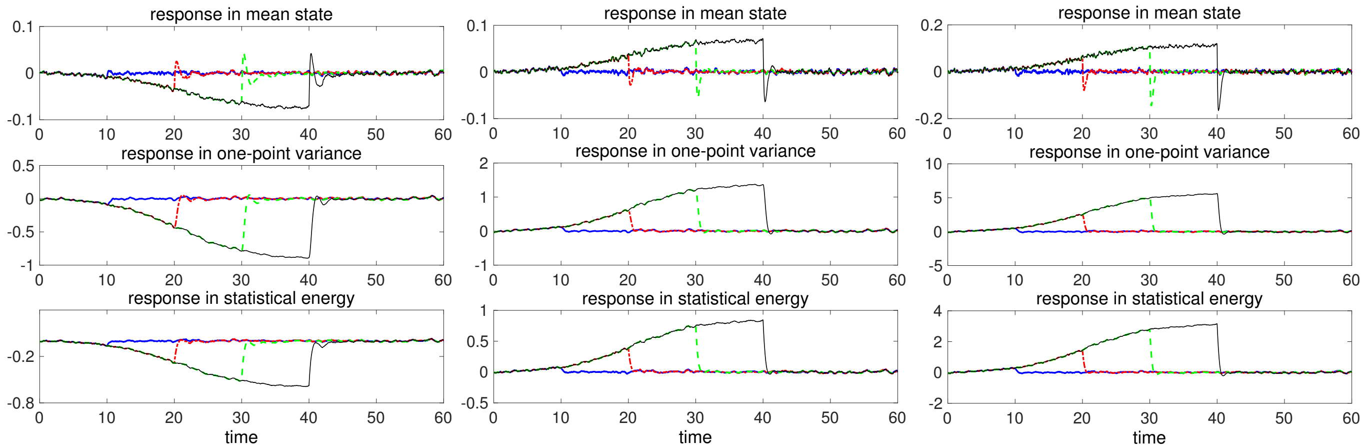
$$\delta F\left(t\right)=f_0\frac{\tanh a\left(t-t_c\right)+\tanh at_c}{1+\tanh at_c},$$

the same as in the main text. This additional random forcing case is tested here to check *the method's robustness* due to small random perturbations.

The model parameters in two test cases with and without random forcing and unperturbed model statistics are listed in Table S2. Furthermore, we show the equilibrium energy spectra in these test cases in Figure S3. As illustrated, adding even small random forcing in the system can greatly increase the variability in the zero mode $\hat{u}_0$, and thus vastly change the entire energy spectrum to a more active state through the nonlinear interactions.

Control verification with random forcing perturbation

In Figure S4, time-series of the responses in the mean $\delta\bar{u}=\bar{u}-\bar{u}_\infty$, in the one-point variance $\delta\text{tr}R/J=r_{1\text{pt}}-r_{1\text{pt},\infty}$, and in the total statistical energy $\delta E=E-E_\infty$ are compared in the three test regimes $F=6$ (weakly chaotic),



(a) Weakly chaotic regime $F = 6$ (b) Strongly chaotic regime $F = 8$ (c) Fully turbulent regime $F = 16$

Figure S4: Statistical control of L-96 system applied at four different states $T_{\mathrm{ctrl}} = 10, 20, 30, 40$ with random perturbation $\sigma_0 \dot{W}$. Controlled responses (subtracting the equilibrium states) in mean state, one-point variance, and total statistical energy through true MC model using the optimal control forcing $\kappa(t)$ are shown.

$F = 8$ (strongly chaotic), and $F = 16$ (fully turbulent) with random forcing perturbation. We use the parameter value $\alpha = 0.1$ for all the tests. The other set-ups stay the same as the unperturbed case shown in the main text. In the case with random forcing $\sigma_0 \dot{W}$, the control $\kappa(t)$ may not be optimal. Still we can use the achieved control to test the robustness of this method due to small random perturbations. As a result, larger fluctuations appear, nevertheless the optimal control displays significant skill in driving the system back to equilibrium with the original optimal control.

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