A Brief Introduction to Operator Spaces

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Abstract

These lecture notes are meant as a tool to go over the very first principles and structure of operator spaces. In particular we give an overview of Ruan’s Axioms and Ruan’s Representation theorem. We adopt the presentation as provided by Effros-Ruan [ER00], and Pisier [Pis03].

1 Introduction

Beginning with inception of Z.J. Ruan’s dissertation, the field of operator spaces has been greatly developed and used as a tool in many areas of operator algebras. Concretely, operator spaces are simply closed subspaces of \( \mathcal{B}(\mathcal{H}) \), where \( \mathcal{H} \) is a Hilbert space, but abstractly, operator spaces are completely determined by their norm structure not only on the level 1, but on amplifications to spaces of matrices. Our first goal is to present Ruan’s Axioms which characterize operator spaces, and following that we will state Ruan’s Representation theorem. This fundamental result is our tool in not having to distinguish between “concrete” and “abstract” operator spaces. This is our analogue to the Gelfand-Naimark-Segal construction for \( C^* \)-algebras, and the Choi-Effros characterization for operator systems [CE77].

Suppose we begin with a normed space \( \mathcal{E} \). A function space is defined as a linear subspace of \( \ell_\infty(s) = \{ (x_s)_{s \in s} : \sup \| x_s \| < \infty \} \), and by the Hahn-Banach theorem we always have an isometric identification of a normed space and a function space. In particular, if \( s = B_{E^*} \) denotes the closed unit ball of \( \mathcal{E}^* \), then we may define the isometry

\[
\Phi : \mathcal{E} \rightarrow \ell_\infty(s), \Phi(e)(f) := f(e), f \in s.
\]

Thus, we may consider any normed linear space as an “abstract” function space, and any function space may be regarded as a realization of a normed linear space. This is useful to us because given a normed space \( \mathcal{E} \) and a closed subspace \( \mathcal{N} \subset \mathcal{E} \), it is not obvious that the Banach dual \( \mathcal{E}^* \) and the quotient normed space \( \mathcal{E}/\mathcal{N} \) are function spaces, but since these are both normed spaces then we have a realization of both as function spaces.

We let \( \ell_\infty^n(\mathcal{E}) \) denote the Banach space of all \( n \)-tuples of elements of \( \mathcal{E} \), \( (x_j)_1^n \), such that

\[
\| x \|_\infty = \max_j \| x_j \| < \infty.
\]  \hspace{1cm} (1)

Representing \( \mathcal{E} \subset \ell_\infty(s) \), this norm is also induced by the inclusion

\[
\mathcal{E}^n \subset \ell_\infty(s \times n),
\]  \hspace{1cm} (2)

where \( n = \{1, ..., n\} \) and \( s \times n \) denotes the disjoint union of \( n \) copies of \( s \).

We wish to replace functions by operators and \( \ell_\infty(s) \) by \( \mathcal{B}(\mathcal{H}) \).

Definition 1.1. A concrete operator space is a closed linear subspace \( \mathcal{B}(\mathcal{H}) \).
Though Effros-Ruan define an operator space as simply a linear subspace of $\mathcal{B}(H)$, we will also assume the space be closed here since we will be needing completeness when one speaks on the tensor theory of operator spaces. Note that in [Pis03] Pisier also defines a concrete operator space as a closed linear subspace of $\mathcal{B}(H)$. In this setting we have the analogue of (2). To determine a “matrix norm” on the space $M_n(V)$ ($M_n(V)$ denotes the linear space of $n \times n$ matrices over $V$ with no other assumed structure) we first look at the natural inclusions

$$M_n(V) \subset M_n(\mathcal{B}(H)) = \mathcal{B}(H^n).$$

This determines a norm $\|\cdot\|_n$ on $M_n(V)$ and we will denote this normed space by $M_n(V)$. An important observation is that we have no real analogue of (1) since many distinct operator spaces may have the same underlying normed space. Thus, we can not hope to relate the norm of a matrix with the norm of its individual entries. Ruan’s axioms for operator spaces are motivated by the concrete observation.

**Theorem 1.1.** Suppose $V \subset \mathcal{B}(H)$ is a concrete operator space. The following properties are satisfied:

1. $\mathcal{A}_1$: For $v \in M_m(V), w \in M_n(V)$ then
   $$\|v \otimes w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}.$$

2. $\mathcal{A}_2$: For $\alpha \in M_{n,m}, \beta \in M_{m,n}, v \in M_m(V)$, we have
   $$\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_m \|\beta\|.$$

*Proof.* $\mathcal{A}_1$ is a special case on direct sums of operators on Hilbert spaces and thus is immediate. Realizing $M_m(V)$ as acting on $\mathbb{C}^m \otimes H$, we decompose $\alpha v \beta = (\alpha \otimes I)v(\beta \otimes I)$, and using the fact that the Hilbertian tensor norm is multiplicative we have

$$\|\alpha v \beta\|_n = \|((\alpha \otimes I)v(\beta \otimes I))\| \leq \|\alpha\| \|v\|_m \|\beta\|.$$

\[\square\]

## 2 Ruan’s Axioms

**Definition 2.1.** Given a linear space $V$, a *matrix norm* $\|\cdot\| = \{\|\cdot\|_n\}_n$ is an assignment of a norm $\|\cdot\|_n$ on each linear space $M_n(V)$ for $n \in \mathbb{N}$. An *abstract operator space* is the pair $(V, \{\|\cdot\|_n\}_n)$ where $V$ is a linear space and $\{\|\cdot\|_n\}_n$ is a matrix norm that satisfies the following properties:

1. $\mathcal{A}_1$: $\|v \otimes w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$;

2. $\mathcal{A}_2$: $\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_m \|\beta\|$

for all $v \in M_m(V), w \in M_n(V), \alpha \in M_{n,m},$ and $\beta \in M_{m,n}$. In this case we call $\|\cdot\| = \{\|\cdot\|_n\}_n$ an operator space matrix norm.

We refer to $\mathcal{A}_1, \mathcal{A}_2$ as *Ruan’s axioms*. Though we will not prove it in these notes there is an alternate version to Ruan’s first axiom where instead of equality we only need that the norm of the direct sum is less than or equal to the maximum of the norms. See (4.2).

Suppose that $\mathcal{A}$ is a $C^*$-algebra, and let $\pi : \mathcal{A} \hookrightarrow \mathcal{B}(H)$ be a faithful representation of $\mathcal{A}$ gotten via the Gelfand-Naimark-Segal construction. Then we may regard $M_n(\mathcal{A})$ as a subalgebra of operators acting on $H^n$. The $*$-algebraic structure is uniquely determined by that of $\mathcal{A}$ and thus the norm on $M_n(\mathcal{A})$ is independent of our faithful embedding. Thus, we will refer to this as the *canonical operator space structure on* $\mathcal{A}$.

It follows that given an abstract operator space $V$ and $v \in M_n(V)$, then for any unitary $\mu \in M_n$, $\|\mu v\| \leq \|v\| = \|\mu^{-1} \mu v\| \leq \|\mu v\|$,
and thus we see that the action of a unitary does not affect the norm of an element. Furthermore, we may thus permute rows and columns of \( v \) without affecting the norm since the operation corresponds to multiplication on the left or right by a permutation matrix.

It follows that rectangular matrices also carry distinguished norms that satisfy Ruan’s axioms. We may view the linear space \( M_{m,n}(V) \) as a subspace of \( M_p(V) \) where \( p = \max\{m, n\} \), and denote the inherited normed linear space as \( M_{m,n}(V) \). It is important that we point some properties that do constrain the properties of our matrix norms. Given \( v \in M_n(V) \) we see that

\[
\|v_{ij}\| = \|E_i v E_j^*\| \leq \|v\|,
\]

and

\[
\|v\| = \left\| \sum E_i^* v_{ij} E_j \right\| = \left\| \sum v_{ij} \right\|.
\]

Here we have let \( E_j \in M_{1,n} \) denote the row matrix with 1 in the \( j \)th entry and 0’s elsewhere. What precisely do these two equations tell us? We see that a sequence \( \{v_n\} \) converges if and only if \( v_{ij}(k) \) converges for all \( i, j \). Furthermore, any two such norms on \( M_n(V) \) must be equivalent. Finally, we see that \( V \) will be complete if and only if \( M_n(V) \) is complete for all \( n \in \mathbb{N} \).

A final remark is that a map \( F = [F_{ij}] : V \to M_n \) is continuous if and only if each \( F_{ij} \) is continuous and thus we define the pairing

\[
\langle \cdot, \cdot \rangle : M_n(V) \times M_n(V^*) \to \mathbb{C} : \langle v, f \rangle := \sum_{i,j} f_{ij}(v_{ij}),
\]

to identify the linear space \( M_n(V^*) \) with the Banach dual \( M_n(V)^* \).

**Theorem 2.1** (Polar Decomposition). Given \( \alpha \in M_{m,n} \) there exists a partial isometry \( \nu : \mathbb{C}^n \to \mathbb{C}^m \) mapping \( (\ker \alpha)^\perp \) onto \( \text{range} \alpha, \alpha = \nu |\alpha| \), and \( \ker \alpha = \ker \nu \). For \( \alpha \in M_{m,n} \) (resp. \( \alpha \in \mathcal{HS}_{m,n}, \alpha \in T_{m,n} \)) then \( |\alpha| \in M_n \) (resp. \( |\alpha| \in \mathcal{HS}_n, |\alpha| \in T_n \)) has the same norm as \( \alpha \). If \( m = n \) then there exists a unitary \( \mu \) with \( \alpha = \mu |\alpha| \).

**Proof.** First note that given \( \eta \in \mathbb{C}^n \),

\[
|||\alpha| \eta||^2 = (|\alpha| \eta, |\alpha| \eta) = (|\alpha|^2 \eta, \eta) = \|\alpha \eta\|^2,
\]

implying that if we define the map \( \nu : \text{range} |\alpha| \to \text{range} \alpha \subset \mathbb{C}^m, |\alpha| \eta \mapsto \alpha \eta \), then \( \nu \) is an isometry. Extend \( \nu \) to a linear map on all of \( \mathbb{C}^n \), still denoted \( \nu \), setting \( \nu \eta = 0 \) for all \( \eta \in (\text{range} |\alpha|)^\perp \), giving us the partial isometry \( \nu : \mathbb{C}^n \to \mathbb{C}^m \). Of course

\[
(\ker \alpha)^\perp = (\ker |\alpha|)^\perp \cong (\text{range} |\alpha|),
\]

which is mapped onto \( \text{range} \alpha \) by construction. It is clear that \( \alpha = \nu |\alpha| \). Furthermore we have that \( \nu^* \nu \) is the orthogonal projection onto \( \text{range} \alpha \), giving us that \( \nu^* \alpha = |\alpha| \).

Suppose now that \( m = n \). Define the canonical isometry \( \nu' : \ker \alpha \cong (\text{range} \alpha)^\perp \). We then define the unitary \( \mu : \mathbb{C}^n \to \mathbb{C}^n \), by

\[
\mu(\xi + \zeta) = \nu \xi + \nu' \zeta.
\]

It then is checked that \( \alpha = \mu |\alpha| \).

If \( \alpha \in T_{m,n} \), then we see

\[
|||\alpha||_1 = \text{tr}(|\alpha|^2) = \text{tr} |\alpha| = \|\alpha\|_1, \quad \text{and if } \alpha \in \mathcal{HS}_{m,n}, \text{ then } |||\alpha||_2^2 = (\text{tr} |\alpha|^2)^{\frac{2}{2}} = (\text{tr} \alpha^* \alpha)^{\frac{2}{2}} = \|\alpha\|_2.
\]

\( \square \)
3 Some Remarks on Completely Bounded Maps

As we have seen in the operator system case, we need to discuss the morphisms in this category. Rather than looking at simply bounded maps as in Banach space theory, we will be looking at completely bounded maps. Given a linear map \( \varphi : V \to W \) between abstract operator spaces, we once again denote \( \varphi^{(n)} : M_n(V) \to M_n(W) \) as the \( n \)th amplification defined by \( \varphi^{(n)}(v) = [\varphi(v_{ij})]_{ij} \). Following standard terminology, if the induced linear map \( \tilde{\varphi} : V / \ker \varphi \to W \) is an isometry then we will say that \( \varphi \) is a quotient map. If \( \varphi \) maps \( B_{M_n(V)} \) onto \( B_{M_n(W)} \) then we will say that \( \varphi \) is an exact quotient map.

**Definition 3.1.** Let \( V, W \) be abstract operator spaces and let \( \varphi : V \to W \) be a linear map.

1. \( \varphi \) will be called **completely bounded** if \( \| \varphi\|_{cb} = \sup_n \| \varphi^{(n)} \| < \infty \).
2. \( \varphi \) will be called a **complete isometry** if \( \varphi^{(n)} \) is an isometry for all \( n \in \mathbb{N} \).
3. \( \varphi \) will be called a **complete isomorphism** if \( \varphi^{(n)} \) is a linear isomorphism for all \( n \in \mathbb{N} \) with \( \| \varphi\|_{cb} \cdot \| \varphi^{-1}\|_{cb} < \infty \).
4. \( \varphi \) will be called a **complete quotient mapping** if \( \varphi^{(n)} \) is a quotient map for all \( n \in \mathbb{N} \).
5. \( \varphi \) will be called an **exact quotient mapping** if \( \varphi \) is an exact quotient map for all \( n \in \mathbb{N} \).

If \( V \) is an abstract operator space with \( \mathcal{H} \) a Hilbert space, then if \( \varphi : V \to \mathcal{B}(\mathcal{H}) \) is a completely isometric injection, we will say that \( \varphi \) is a realization of \( V \). A completely isometric injection between abstract operator spaces will be denoted by \( \hookrightarrow \) and a complete quotient surjection will be denoted by \( \twoheadrightarrow \).

There is only one operator space of dimension 1. To see this we first show the following lemma;

**Lemma 3.1.** Given an abstract operator space \( V \) with \( v \in M_n(V) \) and \( \alpha \in M_p \) we always have

\[
\| \alpha \otimes v \| = \| v \otimes \alpha \| = \| \alpha \| \| v \| .
\]

*Proof.* Using polar decompositon (2.1) we write \( \alpha = \mu |\alpha| \) for some unitary \( \mu \in M_p \). Using the finite-dimensional spectral theorem we have a unitary \( \lambda \in M_p \) and scalars \( c_1 \geq \ldots \geq c_p \geq 0 \) with \( \| \alpha \| = c_1 \), which give us the decomposition

\[
|\alpha| = \lambda^* (c_1 \oplus \ldots \oplus c_p) \lambda.
\]

Let \( c = \oplus_i c_i \) and \( \hat{v} = c \otimes v \in M_{np}(V) \). We then see the following

\[
\| \alpha \otimes v \| = \| \mu |\alpha| \otimes v \| = \| \mu (\lambda^* c \lambda) \otimes v \| = \| (\lambda^* \otimes I_p) \hat{v} (\lambda \otimes I_p) \| \leq \| \hat{v} \| = \| c \otimes v \| = \| \alpha \| \| v \| .
\]

(5)

Since the algebraic tensor product is symmetric we have our result. \( \square \)

Using this result we then have the there is only one operator space of dimension 1. To see this define the map

\[
\theta_v : \mathbb{C} \to V, \alpha \mapsto \alpha v
\]

where \( v \in S_{M_n(V)} \) (this latter space denoting the unit sphere of \( M_n(V) \)). We then have that \( \theta_v \) is a complete isometry since

\[
\| \theta_v^{(n)}(\alpha) \| = \| \alpha \otimes v \| = \| \alpha \| , \forall \alpha \in M_n.
\]

(6)

As is proven in the operator system case for completely positive maps, we have some analogous properties regarding completely bounded maps.

**Lemma 3.2.** Given \( \eta \in \mathbb{C}^m \otimes \mathbb{C}^n \) with \( m \geq n \) there exists an isometry \( \beta : \mathbb{C}^n \to \mathbb{C}^m \) and a vector \( \tilde{\eta} \in \mathbb{C}^n \otimes \mathbb{C}^n \) such that \( \beta \otimes I_n(\tilde{\eta}) = \eta \).
Proof. Begin by taking unique vectors $\eta_j \in \mathbb{C}^m$, $j = 1, \ldots, n$, such that

$$\eta = \sum_j \eta_j \otimes e_j^{(n)}.$$ 

Here we have let $e_j^{(n)}$ denote the basis vector in $\mathbb{C}^n$ with 1 in the $j$th position. Letting $F = \text{span}_j \eta_j \subset \mathbb{C}^m$ we have $\dim F \leq n \leq m$ and moreover we have an isometry $\beta : \mathbb{C}^n \to \mathbb{C}^m$ whose image contains $F$. Thus, there exist unique vectors $\tilde{\eta}_j \in \mathbb{C}^n$ such that

$$\beta(\tilde{\eta}_j) = \eta_j.$$ 

Letting

$$\tilde{\eta} = \sum \tilde{\eta}_j \otimes e_j^{(n)},$$

we have $\beta \otimes I_n(\tilde{\eta}) = \eta$.

\[ \text{Theorem 3.1 (Smith’s Lemma). Let } \varphi : V \to M_n \text{ denote a linear map from an abstract operator space to a matrix algebra. Then} \]

$$\|\varphi\|_{cb} = \|\varphi^{(n)}\|.$$ 

Proof. Given $m \geq n$ we need only show that $\|\varphi^{(m)}\| \leq \|\varphi^{(n)}\|$. Begin by letting $\epsilon > 0$ and then choosing $v \in B_{M_n(V)}$ such that $\|\varphi^{(m)}(v) - \epsilon < \|\varphi^{(m)}(v)\|$. Also choose unit vectors $\eta, \xi \in \mathbb{C}^m \otimes \mathbb{C}^n$ such that

$$\|\varphi^{(m)}(v)\eta\xi\| - \epsilon < \|\varphi^{(m)}(v)\eta\xi\|.$$ 

Using (3.2) we have isometries $\alpha, \beta : \mathbb{C}^n \to \mathbb{C}^m$ and vectors $\tilde{\eta}, \tilde{\xi} \in \mathbb{C}^n \otimes \mathbb{C}^n$ such that

$$\alpha \otimes I_n(\tilde{\xi}) = \xi, \beta \otimes I_n(\tilde{\eta}) = \eta.$$ 

We now have the following

$$\|\varphi^{(m)}(v)\eta\xi\| - \epsilon < \|\varphi^{(m)}(v)\eta\xi\|.$$ 

Corollary 3.1. Given an abstract operator space $V$ and a linear functional $f : V \to \mathbb{C}$ then

$$\|f\|_{cb} = \|f\|.$$ 

Thus, any bounded functional is automatically completely bounded. (Recall that in the operator system case positive functionals are completely positive). Finally we point out that if the codomain of the linear map $\varphi : V \to A$ is a commutative $C^*$-algebra then

$$\|\varphi\|_{cb} = \|\varphi\|$$

which also agrees with the operator system case.

4 Ruan’s Representation Theorem

As is true for $C^*$-algebras (GNS construction and the theorem of Gelfand-Naimark) and operator systems (Choi-Effros), we need not distinguish between ‘abstract’ and “concrete” operator spaces. We will reserve the symbol $\simeq$ to denote completely isometric objects.
Theorem 4.1 (Ruan). Given an abstract operator space $\mathcal{V}$ there exists a Hilbert space $\mathcal{H}$ and a concrete operator space $\mathcal{W} \subset \mathcal{B}(\mathcal{H})$ such that

$$\mathcal{V} \simeq \mathcal{W}.$$  

Conversely, any concrete operator space is an abstract operator space. If $\mathcal{H}$ is separable we may assume it is $\ell_2$.

As we had mentioned when first presenting Ruan’s axioms, we have an alternate version which becomes easier to use in practice.

Theorem 4.2 (Ruan’s Axioms 2.0). Let $\mathcal{V}$ be an operator space and suppose we have a sequence of maps

$$\|\cdot\|_n : M_n(\mathcal{V}) \rightarrow [0, \infty), n \in \mathbb{N},$$

which satisfy the following properties:

1. $R_1'$: Given $v \in M_m(\mathcal{V}), w \in M_n(\mathcal{V})$ we have

$$\|v \oplus w\|_{m+n} \leq \max \{\|v\|_m, \|w\|_n\};$$

2. $R_2$: Given $v \in M_m(\mathcal{V}), \alpha \in M_{n,m}, \beta \in M_{m,n},$ we have

$$\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_m \|\beta\|.$$  

Then these maps are seminorms which satisfy $R_1$ and $R_2$. If $\|\cdot\|_1$ is a norm then $\|\cdot\|_n$ is a norm for all $n \in \mathbb{N}$ and this yields an operator space structure on $\mathcal{V}$.

References

