A Survey of Lance’s Weak Expectation Property, The Double Commutant Expectation Property, and The Operator System Local Lifting Property

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Abstract

This brief survey is meant to put together the key results involving Lance’s Weak Expectation Property, the Double Commutant Expectation Property, and the Operator System Local Lifting Property. They are used as my personal notes for reference and the presentation and many results are due to Kavruk et al. [Kav+13], though we introduce some different terminology, notation, and alter some of the proofs. For a more detailed take on the topics then we refer the reader to the original paper.

1 Introduction

We assume that if $A$ is a C*-algebra, then $A$ is necessarily unital. Given two operator systems $S, T$, we let $S \otimes T$ denote their algebraic tensor product, and thus, if $\tau$ is any operator system structure (oss) on the algebraic tensor product, we let $S \otimes_\tau T$ denote the operator system gotten from $\tau$. Given operator systems $S, T, U, V$, and two maps $\phi : S \rightarrow T, \psi : U \rightarrow V$, we denote the induced linear map on the algebraic tensors

$$S \otimes T \rightarrow U \otimes V$$

by $\phi \otimes \psi$, and if $\tau$ is an oss on $S \otimes T$, and $\kappa$ is an oss on $U \otimes V$, then we denote the map from

$$S \otimes_\tau T \rightarrow U \otimes_\kappa V,$$

by $\phi \otimes_{\tau,\kappa} \psi$. If we have the same oss $\tau$ in both the domain and codomain, we will denote the map from $S \otimes_\tau T \rightarrow U \otimes_\tau V$ by $\phi \otimes_{\tau} \psi$.

In operator system theory we are concerned with six different operator system structures on the algebraic tensor product. These tensor product structures are the minimal (min), enveloping (e), enveloping left (el), enveloping right (er), commuting (c), and the maximal (max), tensor products. For properties of these tensor structures we refer the reader to [Kav+11] which gives a detailed account on all these structures and their basic properties. Though, it is important to mention the ordering of these oss’ which is

$$\text{min} \leq e \leq \text{el, er} \leq c \leq \text{max}.$$

Recall that this is telling us that the cones given by the minimal structure are the largest, and the cones given by the maximal structure are the smallest.

Given an operator system $S$ and two operator system tensor products $\tau, \kappa$, we say that $S$ is $(\tau, \kappa)$-nuclear if for any operator system $T$, we have

$$S \otimes_\tau T = S \otimes_\kappa T,$$
i.e., the identity map from $S \otimes \tau T$ to $S \otimes \kappa T$ is a complete order isomorphism. We will say that $S$ is weak$(\tau, \kappa)$-nuclear if
\[ S \otimes \tau T = S \otimes \kappa S. \]
Finally if $S$ satisfies
\[ S \otimes \tau T_o = S \otimes \kappa T_o \]
for only the operator system $T_o$, then we will say that $S$ is weak$_{T_o}$(\tau, \kappa)$-nuclear.

Finally, we state the remarkable theorem of Choi and Effros stating that we may always realize any operator system concretely as a self-adjoint unital subspace of $B(H)$.

**Theorem 1.1** (The Choi-Effros Characterization [CE77]). Given an operator system $S$, there exists a Hilbert space $H$ and a self-adjoint unital subspace $V \subset B(H)$ such that $S \equiv_{coi} V$. Here we let c.o.i. denote complete order injection. Conversely, every self-adjoint unital subspace of $B(H)$ is an abstract operator system.

## 2 Lance’s Weak Expectation Property

The weak expectation property had been formulated by Lance in [Lan73] in terms of C*-algebras. We say that a C*-algebra $A \subset B(H)$ has Lance’s weak expectation property (WEP) if the canonical embedding $\iota : A \rightarrow A^{**}$ into the double dual of $A$, extends to a unital completely positive map (ucp) $\hat{\iota} : B(H) \rightarrow A^{**}$ such that $\hat{\iota}(a) = a$ for all $a \in A$. For completeness sake we point out that in his original paper [Lan73], Lance proved that a C*-algebra having the weak expectation property was equivalent to $\otimes_{C^* - \text{max}}$ being left injective. To be more specific, a C*-algebra $A$ has Lance’s weak expectation property if and only if for any C*-algebra $C \supset A$, and C*-algebra $B$, we have the following
\[ A \otimes_{\text{max}} B \subset_{coi} C \otimes_{\text{max}} B. \]

It follows that these two definitions coincide.

Lance’s weak expectation property has been studied extensively and we refer the reader to [Pis03] for an excellent write up of this property. We wish to extend this notion to operator systems, and it will follow from the construction. An object we refer the reader to either [Pau02] or [Ham79], though we will mention some key points of the construction.

Letting $S \subset_{coi} B(H)$ be the Choi-Effros characterization of $S$, we look at the maps $\phi : B(H) \rightarrow B(H)$ such that $\phi$ is c.p., and $\phi$ fixes $S$, that is to say that $\phi(s) = s$ for all $s \in S$ (thus making any such $\phi$ unital). We will call such maps $S$-maps. If $\phi$ is an $S$-map that satisfies $\phi \circ \phi = \phi$, then such a $\phi$ is called an $S$-projection. Define the partial ordering $\preceq$ by declaring $\psi \preceq \phi$ for $S$-maps $\psi, \phi$ if $\phi \circ \psi = \psi \circ \phi = \psi$. Throughout the survey we will let $\mathcal{O}$ denote the category whose objects are operator systems and whose morphisms are unital completely positive maps.

**Theorem 2.1.** Let $S \in \mathcal{O}$ and let $S \subset_{coi} B(H)$ be the Choi-Effros characterization of $S$. Then $S$ is injective if and only if there exists a completely positive projection $\phi : B(S) \rightarrow S$.

**Proof.** If $S$ is injective then it follows since id : $S \rightarrow S$ is u.c.p. there exists a u.c.p. extension $\phi : B(H) \rightarrow S$ which is necessarily a projection since it extends the identity.

Conversely, if $E, F \in \mathcal{O}$ with $E \subset F$, and $\phi : B(H) \rightarrow S$ is a u.c.p. projection, then using injectivity of $B(H)$ we have if $\gamma : E \rightarrow S \subset B(H)$ is u.c.p. then letting $\tilde{\gamma} : F \rightarrow B(H)$ denote its u.c.p. extension, we have the desired u.c.p. map $\phi \circ \tilde{\gamma} : F \rightarrow S$ extending $\gamma$. \[ \square \]

Given any $S$-map, we can look at an $S$-seminorm $\rho_\phi : B(S) \rightarrow \mathbb{R}$ defined by $\rho_\phi(a) = \|\phi(a)\|$ for $a \in B(H)$.

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Finally we may define partial orderings on the set of $S$-seminorms by saying that $\sigma \leq \rho$ if $\sigma(x) \leq \rho(x)$ for all $x \in \mathcal{B}(\mathcal{H})$. At this point one proves that there exists a minimal $S$-seminorm, and necessarily the $S$-map that gives the minimal $S$-seminorm will give us our injective envelope, i.e., if $\rho_o$ denotes the minimal $S$-seminorm, then $\phi(\mathcal{B}(\mathcal{H}))$ is an injective envelope for $S$ and it is unique up to complete order isomorphism.

**Proposition 2.1.** Let $S, T$ and $U$ be operator systems and suppose that $\phi : S \rightarrow T$ and $\psi : T \rightarrow U$ are unital completely positive maps. Then if the composition map $\psi \circ \phi$ is a complete order injection, then the map $\phi$ is necessarily a complete order injection.

**Proof.** Let $t = [t_{ij}] \in M_n(\phi(S))^+$. Letting $s = [s_{ij}] \in M_n(S)$ be such that $\phi^n(s) = t$, we know that $\psi^n(t) = \psi^n \circ \phi^n(s) = (\psi \circ \phi^n)(s) \in M_n(U)^+ \iff s \in M_n(S)^+$.

Thus, $\phi^{-1} : \phi(S) \rightarrow S$ is u.c.p. as desired which implies $\phi : S \rightarrow T$ is a complete order injection. \hfill \Box

**Proposition 2.2.** Let $S$ be an operator system. Then $S$ is $(el, \max)$-nuclear if and only if for all operator systems $S_o \supseteq S$ and $T$,

$$S \otimes_{\max} T \subset_{\text{coi}} S_o \otimes_{\max} T,$$

where we let $\subset_{\text{coi}}$ denote a complete order injection. We will call this property weak$_S$-left injectivity of $\otimes_{\max}$.

**Proof.** Suppose that $S$ is $(el, \max)$-nuclear. Let $\iota : S \rightarrow S_o$ denote the inclusion mapping, $id : T \rightarrow T$ the identity mapping on $T$, and let $j : S \rightarrow I(S)$ denote the complete order injection into the injective envelope. We then have a ucp extension $j : S_o \rightarrow I(S)$ of $j$. We now look at the sequence of maps

$$(j \otimes_{\max} id) \circ (\iota \otimes_{\max} id) : S \otimes_{\max} T = S \otimes_{\max} T \rightarrow S_o \otimes_{\max} T \rightarrow I(S) \otimes_{\max} T.$$ 

But by definition we have that

$$S \otimes_{\text{el}} T \subset_{\text{coi}} I(S) \otimes_{\max} T,$$

and therefore by applying Proposition 2.1 we have that $\iota \otimes_{\max} id$ is a complete order injection.

Now suppose that for every operator system $S_o \supseteq S$, we have the stated completely order inclusion with respect to $\otimes_{\max}$. It then follows that

$$S \otimes_{\text{el}} T \subset_{\text{coi}} I(S) \otimes_{\max} T \supset_{\text{coi}} S \otimes_{\max} T.$$

Thus we have that both $S \otimes_{\text{el}} T$, and $S \otimes_{\max} T$ are operator subsystems of $I(S) \otimes_{\max} T$, and therefore they must be equal. \hfill \Box

If $S^{**}$ denotes the double dual of the operator system $S$, then it is easily verified that the canonical inclusion $\iota : S \rightarrow S^{**}$ is a complete order injection. Futhermore, $S^{**}$ is an operator system with archimedean order unit $1 = \iota(1)$, where 1 denotes the archimedean order unit of $S$. Recall that the dual $S^*$ is given an order structure by saying that $(f_{ij})_{ij} \in M_n(S^*)^+$ if the map $S \rightarrow M_n$ defined by $s \mapsto (f_{ij}(s))_{ij}$ is completely positive. Thus, the order structure on $S^{**}$ is gotten via the dual structure on $S^*$.

Suppose $A$ and $B$ are C*-algebras and we look at a map $f \in (A \otimes B)^*$. It then follows that $f$ induces a map $T_f \in \mathcal{B}(A, B^*)$ by $T_f(a)(b) := f(a \otimes b)$. The following lemma was shown in [Lan73].

**Lemma 2.1 (Lance).** Let $A$ and $B$ be C*-algebras with $f \in (A \otimes B)^*$. Then $f$ is positive if and only if the induced map $T_f \in \mathcal{B}(A, B^*)$ is completely positive.
Proof. We have the following string of implications;

\[ f \geq 0 \iff f \left( \left( \sum_i a_i \otimes b_i \right)^* \left( \sum_j a_j \otimes b_j \right) \right) \geq 0 \]
\[ \iff \sum_{i,j} f ((a_i^* a_j) \otimes (b_i^* b_j)) \geq 0 \]
\[ \iff \sum_{i,j} T f (a_i^* a_j)(b_i^* b_j) \geq 0 \]
\[ \iff T \otimes id_{M_n} \left( \sum_{i,j} a_i^* a_j \otimes e_{ij} \right) \left( \sum_{i,j} b_i^* b_j \otimes e_{ij} \right) \geq 0 \]
\[ \iff T \in \text{cp}(\mathcal{A}, \mathcal{B}^*) . \]

Here we used the notation \( \text{cp}(\mathcal{A}, \mathcal{B}^*) \) to denote all completely positive maps from \( \mathcal{A} \) to \( \mathcal{B}^* \).

In the above lemma we directly used the fact that if \( \mathcal{A} \) is a C*-algebra, and \( a \in M_n(\mathcal{A})^+ \) then we may write \( a = (a_i^* a_j) \) for some \( \{a_1, ..., a_n\} \subset \mathcal{A} \).

Lemma 2.2. Given an operator system \( \mathcal{S} \), then for any operator system \( \mathcal{T} \) it follows

\[ \mathcal{S} \otimes_{\text{max}} \mathcal{T} \subset \coten \mathcal{S}^{**} \otimes_{\text{max}} \mathcal{T} . \]

Proof. Let \( \iota_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}^{**} \) denote the canonical injection into the double dual of \( \mathcal{T} \). Since \( \iota_{\mathcal{T}} \) is completely positive implies that

\[ \iota_{\mathcal{T}}^* : \mathcal{T}^{****} \rightarrow \mathcal{T}^{**} \]

is completely positive. Let \( f \in \mathcal{S} (\mathcal{S} \otimes_{\text{max}} \mathcal{T}) \), where \( \mathcal{S} (\mathcal{S} \otimes_{\text{max}} \mathcal{T}) \) denotes the state space of \( \mathcal{S} \otimes_{\text{max}} \mathcal{T} \). By applying Lemma 2.1 we have the induced map \( T_f \in \text{cp}(\mathcal{S}, \mathcal{T}^*) \). Once again we have that

\[ T_f^{**} : S^{**} \rightarrow \mathcal{T}^{****} \]

is completely positive, and therefore the composition map

\[ \iota_{\mathcal{T}}^* \circ T_f^{**} : \mathcal{S}^{**} \rightarrow \mathcal{T}^{**} \]

is completely positive. Let \( g \in \mathcal{S}^{**} (\mathcal{S} \otimes_{\text{max}} \mathcal{T}) \) be the state induced by the composition map. It then can be checked that \( g|_{\mathcal{S} \otimes_{\text{max}} \mathcal{T}} = f \), and therefore we have shown that every state on \( \mathcal{S} \otimes_{\text{max}} \mathcal{T} \) extends to a state on \( \mathcal{S}^{**} \otimes_{\text{max}} \mathcal{T} \). Thus, it follows that

\[ (\mathcal{S} \otimes_{\text{max}} \mathcal{T})^+ = (\mathcal{S}^{**} \otimes_{\text{max}} \mathcal{T})^+ \cap (\mathcal{S} \otimes \mathcal{T}) , \]

since the positive elements are characterized by remaining positive when acted on by a state.

Now, for \( n \in \mathbb{N} \), we have that

\[ M_n(\mathcal{S} \otimes_{\text{max}} \mathcal{T}) = M_n(\mathcal{S} \otimes_{\text{max}} \mathcal{T}) = S \otimes_{\text{max}} M_n(\mathcal{T}) , \]

which is a result of the fact that \( \otimes_{\text{max}} \) is associative and symmetric. Thus, we have our desired inclusion. \( \square \)

Corollary 2.1. Given operator systems \( \mathcal{S} \) and \( \mathcal{T} \), we have that

\[ \mathcal{S} \otimes_c \mathcal{T} \subset \coten \mathcal{S}^{**} \otimes_c \mathcal{T} . \]
Proof. This proof is a direct result of Lemma 2.2, the fact that
\[ S \otimes_c T \subset_{\text{coi}} S \otimes_{\text{max}} C_u^*(T) \]
where \( C_u^*(T) \) denotes the universal C*-algebra of \( T \), and that if \( B \) is a C*-algebra, then
\[ S \otimes_{\text{max}} B = S \otimes_c B. \]

For a proof of the last claim see Theorem 3.1. □

Definition 2.1. Let \( S \) be an operator system. We will say that \( S \) has Lance’s weak expectation property if the complete order injection \( \iota : S \rightarrow S^{**} \) lifts to a unital completely positive map \( \tilde{i} : \mathcal{I}(S) \rightarrow S^{**} \).

Theorem 2.2. Let \( S \) be an operator system. Then the following are equivalent.

1. \( S \) has Lance’s WEP.
2. \( S \) is \((el, max)\)-nuclear.
3. \( \otimes_{\text{max}} \) is weakly left injective.
4. There exists a complete order inclusion \( S \subset \mathcal{B}(H) \) such that the complete order injection \( \iota : S \rightarrow S^{**} \) lifts to a completely positive map \( \tilde{i} : \mathcal{B}(H) \rightarrow S^{**} \).
5. The complete order injection \( \iota : S \rightarrow S^{**} \) factors through an injective operator system via unital completely positive maps.

Proof. The equivalence of Lance’s WEP and \((el, max)\)-nuclearity is due to Han [Han11] but we will prove the forward implication. Suppose that the operator system \( S \) has Lance’s WEP. Let \( j : S \rightarrow \mathcal{I}(S) \) denote the c.o.i. into the injective envelope, and let \( \tilde{i} : \mathcal{I}(S) \rightarrow S^{**} \) denote the u.c.p. lift of the c.o.i. \( \iota : S \rightarrow S^{**} \). We then have
\[ (\tilde{i} \otimes_{\text{max}} id_T) \circ (j \otimes_{\text{max}} id_T) : S \otimes_{\text{max}} T \rightarrow \mathcal{I}(S) \otimes_{\text{max}} T \rightarrow S^{**} \otimes_{\text{max}} T. \]

But by applying Lemma 2.2 we have that the composition is a c.o.i. and therefore
\[ S \otimes_{\text{max}} T \subset_{\text{coi}} \mathcal{I}(S) \otimes_{\text{max}} T. \]

By definition we also have that
\[ S \otimes_{el} T \subset_{\text{coi}} \mathcal{I}(S) \otimes_{\text{max}} T, \]
therefore giving us that Lance’s WEP implies \((el, max)\)-nuclearity since for every \( n \in \mathbb{N} \),
\[ M_n(S \otimes_{\text{max}} T)^+ = M_n(\mathcal{I}(S) \otimes_{\text{max}} T)^+ \cap M_n(S \otimes T) = M_n(S \otimes_{el} T)^+. \]

The proof of the converse can be found in [Han11]. Though, currently we are working to reprove this result via different methods. Therefore we have \((1) \iff (2)\).
\((2) \iff (3)\) was proven in Proposition 2.2.

We show that \((1) \iff (5) \implies (3)\).

Now if the operator system \( S \) has Lance’s WEP then by definition the c.o.i. \( \iota : S \rightarrow S^{**} \) factors through an injective operator system, namely the injective envelope \( \mathcal{I}(S) \), via u.c.p. maps. Thus, \((1) \implies (5)\). Let \( \mathcal{R} \) be an injective operator system such that \( S \) factors through \( \mathcal{R} \) via the u.c.p. maps
\[ \phi : S \rightarrow \mathcal{R}, \psi : \mathcal{R} \rightarrow S^{**}. \]
Let \( S_o \supset S \) be an operator system containing \( S \). We then have for any operator system \( T \), if \( \iota : S \rightarrow S_o \) denotes the inclusion, and \( \hat{\phi} : S_o \rightarrow R \) the u.c.p. lift of \( \phi \), we have

\[
S \otimes_{\text{max}} T \rightarrow S_o \otimes_{\text{max}} T \rightarrow R \otimes_{\text{max}} T \rightarrow S^{**} \otimes_{\text{max}} T,
\]
gotten via the maps

\[
\left( (\psi \otimes_{\text{max}} \text{id}_T) \circ (\hat{\phi} \otimes_{\text{max}} \text{id}_T) \right) \circ (\iota \otimes_{\text{max}} \text{id}_T).
\]

By applying Lemma 2.2, we know that the composition is a c.o.i. and therefore the first map must be a c.o.i., i.e.,

\[
S \otimes_{\text{max}} T \subset \text{coi} S_o \otimes_{\text{max}} T,
\]
for all operator systems \( S_o \) containing \( S \) and every operator system \( T \). Thus, (5) \( \Rightarrow \) (3).

Finally we show (1) \( \iff \) (4).

If \( S \) has Lance’s WEP then we denote the u.c.p. lift of \( \iota \) by \( \tilde{\iota} : \mathcal{I}(S) \rightarrow S^{**} \). Let \( S \subset \mathcal{B}(\mathcal{H}) \) be the Choi-Effros characterization of \( S \). Letting \( j : S \rightarrow \mathcal{I}(S) \) denote the complete order injection into the injective envelope, we lift to

\[
\tilde{j} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{I}(S),
\]
and therefore our desired lift of \( \iota \) to \( \mathcal{B}(\mathcal{H}) \) is

\[
\tilde{\iota} \circ \tilde{j} : \mathcal{B}(\mathcal{H}) \rightarrow S^{**}.
\]

The proof of the converse is similar, using the facts that \( \mathcal{B}(\mathcal{H}) \) is injective, and realizing \( S \) as an operator subsystem of its injective envelope \( \mathcal{I}(S) \).

\[\square\]

**Theorem 2.3.** An bidual operator system is injective if and only if it has Lance’s WEP.

**Proof.** Let \( R \) denote our bidual operator system. If \( R \) is injective then we have \( R = \mathcal{I}(R) \) and therefore \( R \) clearly has Lance’s WEP by looking at the inclusion \( \iota : R \rightarrow R^{**} \).

Conversely, suppose that \( R \) has Lance’s WEP. Write \( R = S^{**} \), and let

\[
\iota : S^* \rightarrow S^{***}
\]
denote the inclusion mapping. Therefore its adjoint

\[
\iota^* : S^{****} \rightarrow S^{**}
\]
is a c.p. projection of \( R^{**} \) onto \( R \). Let

\[
\psi_1 : R \rightarrow \mathcal{I}(R), \psi_2 : \mathcal{I}(R) \rightarrow R^{**}
\]
denote the u.c.p. maps gotten from Lance’s WEP.

Let \( T_1 \subset T_2 \) be operator systems and suppose that \( \phi : T_1 \rightarrow R \) is a c.p. map, and let \( \phi' : T_2 \rightarrow \mathcal{I}(R) \) be a c.p. extension of this. We then have

\[
\iota^* \circ \psi_2 \circ \phi' : T_2 \rightarrow R,
\]
is a c.p. lift of \( \phi \) implying that \( R \) is indeed injective as desired.

\[\square\]

**Theorem 2.4.** Let \( S \) be a finite-dimensional operator system. Then the following are equivalent:

1. \( S \) has Lance’s WEP.
2. \( S \) is injective.
3. \( S \) is (el,max)-nuclear.
4. \( S \) is \((\text{min}, \text{max})\)-nuclear.

5. \( S \) is completely order isomorphic to a \( C^* \)-algebra.

6. \( S \) is weak\(_S\) \((el, \text{max})\)-nuclear.

Proof. By our previous results we know 
\( (2) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1) \Rightarrow (2) \).

Note that the first implication is a result of Choi-Effros. By Theorem 2.2 we know that \( (1) \Leftrightarrow (3) \).
Trivially \( (3) \Rightarrow (6) \), and therefore we show that \( (6) \Rightarrow (1) \).
Thus, suppose that for an operator system \( S \), \( S \) is weak\(_S\) \((el, \text{max})\)-nuclear. Since \( S \) is finite-dimensional we have that \( S = S'' \). Let \( \text{id} : S \rightarrow S'' \) denote the identity on \( S \) and since this is c.p., by applying Lemma 2.1 we know that we have an induced map \( f : S \odot_{\text{max}} S^* \rightarrow C \). By applying our assumption we have 
\( S \odot_{\text{max}} S^* = S \odot_{el} S^* \subset_{\text{coi}} I(S) \odot_{\text{max}} S^* \).
In particular we see that \( f \) has an extension \( \tilde{f} : I(S) \odot_{\text{max}} S^* \rightarrow C \), and therefore we let \( \phi : I(S) \rightarrow S'' \) denote the induced c.p. map, \( \phi \) is necessarily an extension of the identity on \( S \) implying that \( S \) has Lance’s WEP.

\[ \square \]

3 The Double Commutant Expectation Property

We now wish to turn our attention to the double commutant expectation property (DCEP). Using tools we have already shown, it will follow that a \( C^* \)-algebra has Lance’s WEP if and only if it has DCEP.

Definition 3.1. Let \( S \) be an operator system. We say that \( S \) has the double commutant expectation property (DCEP) if for every completely order isomorphic inclusion \( S \subset B(H) \) there exists a completely positive map 
\( \phi : B(H) \rightarrow S'' \)
such that \( \phi(s) = s \) for all \( s \in S \), thus implying that any such \( \phi \) is unital.

If a map \( \phi \) satisfies \( \phi(s) = s \) for all \( s \in S \) then we will say that \( \phi \) fixes \( S \).

As we did for Lance’s WEP we wish to prove equivalences to when an operator system has DCEP. First though it is necessary to state and prove a theorem that we will use throughout the rest of the survey. It is the fact that \( \otimes_{c} = \otimes_{\text{max}} \) as long as one of the objects being tensored is a \( C^* \)-algebra. The proof will rely on results found in [Pau02].

Theorem 3.1. Given any unital \( C^* \)-algebra \( A \), and operator system \( S \), then 
\[ A \otimes_{c} S = A \otimes_{\text{max}} S. \]

Proof. We first begin by showing that \( A \otimes_{\text{max}} S \) is an operator \( A \)-system, that is to say that \( A \otimes_{\text{max}} S \) is an \( A \)-bimodule and furthermore that positive elements remain positive under conjugation by elements of \( A \), i.e., given \( U \in M_n(A \otimes_{\text{max}} S)^+ \) then for any \( B \in M_n(A) \) we have \( B^* UB \in M_n(A \otimes_{\text{max}} S)^+ \). Declare the bimodule multiplication on elementary tensors by 
\[ a_1(a \otimes s)a_2 := a_1aa_2 \otimes s. \]

First suppose that \( U \in M_n(A \otimes_{\text{max}} S)^+ \) and therefore we assume 
\[ U = \alpha(P \otimes Q)\alpha^*, P \in M_n(A)^+, Q \in M_n(S)^+, \alpha \in M_{n,pq}. \]
Given any matrix $B$ it then follows
\[ B^* (\alpha (P \otimes Q) \alpha^*) B = (\alpha^* B)^* (P \otimes Q) (\alpha^* B). \]

Thus, in showing that $A \otimes_{\text{max}} S$ is an operator $A$-system, we may assume $U = P \otimes Q$, where $P \in M_n(A)^+$, $Q = (s_{ij}) \in M_q(S)^+$, $pq = n$.

Let $B = (B_1 \cdots B_q)^T \in M_{p,k}$, it then follows that
\[ B^* U B = \sum_{i,j} (B_i^* P B_j) \otimes (s_{ij}). \]

Denote $\Lambda = (B_i^* P B_j)_{i,j} \in M_{kq}(A)^+$. Finally, let
\[ X = (e_1 \otimes I_k \cdots e_q \otimes I_k)^T, e_i \otimes I_k \in M_{kq}. \]

We then see that
\[ B^* U B = X^*(\Lambda \otimes Q) X \in M_n(A \otimes_{\text{max}} S)^+, \tag{1} \]
therefore giving us that $A \otimes_{\text{max}} S$ is an operator $A$-system.

Define the following maps
\[ \pi : A \rightarrow \mathcal{I}(A \otimes_{\text{max}} S), \rho : S \rightarrow \mathcal{I}(A \otimes_{\text{max}} S) \]
by
\[ a \mapsto a \otimes 1_S, \ s \mapsto 1_A \otimes s, \]
respectively. Using the Choi-Effros result proving that injective operator systems are necessarily completely order isomorphic to C*-algebras, and letting $\cdot$ denote the bimodule action, one shows that $\pi$ defined above is a $*$-homomorphism. Furthermore, $\rho$ is a c.o.i. Thus, since
\[ \pi(A) \rho(S) = \rho(S) \pi(A), \]
and both maps are c.p., we have that
\[ \pi \times \rho : A \otimes_c S \rightarrow \mathcal{I}(A \otimes_{\text{max}} S), \]
is c.p. with range $A \otimes_{\text{max}} S$. Thus, since $\pi \times \rho(a \otimes s) = a \otimes s$, we have that
\[ A \otimes_c S = A \otimes_{\text{max}} S. \]

\[ \square \]

**Theorem 3.2.** Let $S$ be an operator system. Then the following are equivalent.

1. $S$ is (el,c)-nuclear.
2. Given any operator system $S_o \supset S$, then for every operator system $T$,
\[ S \otimes_c T \subset_{\text{coi}} S_o \otimes_c T. \]
3. Given any operator system $S_o \supset S$, then for every C*-algebra $B$, we have
\[ S \otimes_c B \subset_{\text{coi}} S_o \otimes_c B. \]
4. There exists an inclusion $S \subset \mathcal{B}(\mathcal{H})$ such that for every C*-algebra $B$ we have
\[ S \otimes_c B \subset_{\text{coi}} B(\mathcal{H}) \otimes_c B. \]
5. There exists an injective operator system \( R \supset S \) such that for every operator system \( T \) we have

\[
S \otimes_c T \subseteq_{coi} R \otimes_c T.
\]

**Proof.** It follows that (1) \( \iff \) (2) \( \iff \) (3) \( \iff \) (4) \( \iff \) (1). All implications are trivial except the first and last. Therefore, suppose that \( S \) is \((el, c)\)-nuclear, with \( S_o \supset S \), and \( T \) both operator systems. Realizing \( S \subseteq_{coi} \mathcal{B}(H) \) via the Choi-Effros characterization, we let \( j : S_o \rightarrow \mathcal{B}(H) \) be the u.c.p. extension of the inclusion map of \( S \) into \( \mathcal{B}(H) \), gotten from injectivity of \( \mathcal{B}(H) \). Furthermore, let \( \iota : S \rightarrow S_o \) denote the inclusion mapping. We then have the following sequence of u.c.p. maps

\[
(j \otimes id_T) \circ (\iota \otimes id_T) : S \otimes_{el} T = S \otimes_c T \rightarrow S_o \otimes_c T \rightarrow \mathcal{B}(H) \otimes_{el=c=\text{max}} T.
\]

The composition is a c.o.i. since by left injectivity of \( \otimes_{el} \),

\[
S \otimes_{el} T \subseteq_{coi} \mathcal{B}(H) \otimes_{el=c=\text{max}} T.
\]

Thus, \( j \otimes id_T \) is a complete order injection.

Now suppose that there exists an inclusion \( S \subset \mathcal{B}(H) \) such that

\[
S \otimes_c B \subseteq_{coi} \mathcal{B}(H) \otimes_c B
\]

for all C*-algebras \( B \). It then follows that if \( T \) is an operator system

\[
S \otimes_{el} T \subseteq_{coi} \mathcal{B}(H) \otimes_{el} T = \mathcal{B}(H) \otimes_{c=\text{max}} T \subseteq_{coi} \mathcal{B}(H) \otimes_{c=\text{max}} C_u^*(T),
\]

\[
\mathcal{B}(H) \otimes_{c=\text{max}} C_u^*(T) \subseteq_{coi} S \otimes_{e=\text{max}} C_u^*(T) \subseteq_{coi} S \otimes_c T.
\]

This implies that

\[
S \otimes_{el} T = S \otimes_c T.
\]

We now show that (4) \( \iff \) (5). Suppose that (4) holds, and let \( T \) be an arbitrary operator system. Notice

\[
S \otimes_c T \subseteq_{coi} S \otimes_{e=\text{max}} C_u^*(T) \subseteq_{coi} \mathcal{B}(H) \otimes_{c=\text{max}} C_u^*(T),
\]

and

\[
\mathcal{B}(H) \otimes_{c=\text{max}} T \subseteq_{coi} \mathcal{B}(H) \otimes_{c=\text{max}} C_u^*(T).
\]

Thus,

\[
S \otimes_c T \subseteq_{coi} \mathcal{B}(T) \otimes_c T,
\]

since

\[
M_n(S \otimes_c T)^+ = M_n(\mathcal{B}(H) \otimes_{c=\text{max}} C_u^*(T))^+ \cap (S \otimes T)
\]

\[
\subseteq M_n(\mathcal{B}(H) \otimes_{c=\text{max}} C_u^*(T))^+ \cap (\mathcal{B}(H) \otimes C_u^*(T)) = M_n(\mathcal{B}(H) \otimes_{c=\text{max}} T)^+.
\]

Conversely, if we let \( R \supset S \) be an injective operator system such that (5) holds. The result then follows by representing \( S \subset \mathcal{B}(H) \) and then applying Proposition 2.1, using once again the fact that \( \mathcal{B}(H) \) is injective.

\[\square\]

**Theorem 3.3.** Given an operator system \( S \), the following are equivalent.

1. \( S \) has DCEP.
2. for all inclusions \( S \subset \mathcal{B}(H) \) there exists completely positive map \( \phi : \mathcal{I}(S) \rightarrow S'' \) fixing \( S \).
3. There exists an injective C*-algebra \( R \supset S \) such that for all inclusions \( S \subset \mathcal{B}(H) \) there exists a completely positive map \( \phi : R \rightarrow S'' \) fixing \( S \).
4. For all injective C*-algebras \( R \supset S \) there exists a completely positive map \( \phi : R \rightarrow S'' \) fixing \( S \).

5. \( S \) is \((el,c)\)-nuclear.

Proof. It will follow

\[
(1) \implies (2) \implies (5) \implies (4) \implies (1), (2) \iff (3).
\]

First suppose that \( S \) has DCEP. Let \( S \subset B(H) \) with \( \iota : S \hookrightarrow B(H) \) denoting the inclusion map. By injectivity we have a u.c.p. extension \( \tilde{\iota} : I(S) \rightarrow B(H) \). This map fixes \( S \), and therefore if \( \phi : B(H) \rightarrow S'' \) denotes the c.p. map fixing \( S \) gotten by assumption then clearly \( \phi \circ \tilde{\iota} : I(S) \rightarrow S'' \), is our desired map. Thus, \( (1) \implies (2) \).

Now, assume \( (2) \) and let \( T \) be an arbitrary operator system. We wish to show that \( S \) is \((el,c)\)-nuclear. We know that

\[
S \otimes_c T \subset \coi C_u(S) \otimes_{max} C_u(T),
\]

and now we wish to represent

\[
C_u(S) \otimes_{max} C_u(T) \subset B(H),
\]

faithfully. Thus, by invoking the universal property of the maximal C*-tensor product, we realize \( S \) and \( T \) as commuting operator subsystems of \( B(H) \), and furthermore we have that the map

\[
f : S \otimes_c T \rightarrow B(H), x \otimes y \mapsto xy
\]

is a complete order injection. Let \( j : S \hookrightarrow I(S) \) denote the complete order injection into the injective envelope. Finally, let \( \phi : B(H) \rightarrow S'' \) be the c.p. map fixing \( S \) gotten by assumption. By construction, \( S'' \subset T' \), and therefore the map

\[
\tilde{\phi} : I(S) \otimes_{max} T \rightarrow B(H), x \otimes y \mapsto \phi(x)y
\]

is completely positive. Looking at the composition

\[
\tilde{\phi} \circ (j \otimes_{el,max} id_T) : S \otimes_{el} T \rightarrow I(S) \otimes_{max} T \rightarrow B(H) \supset \coi S \otimes_c T,
\]

this is a c.p. map whose image is precisely \( S \otimes_c T \). This, along with the fact that

\[
el \leq c
\]

implies that

\[
S \otimes_{el} T = S \otimes_c T.
\]

Thus we have \( (2) \implies (5) \).

Now, suppose that the operator system \( S \) is \((el,c)\)-nuclear, and let \( R \) be an injective C*-algebra containing \( S \). We have that \( S' \) is a C*-algebra, and therefore we look at the map

\[
\phi : S \otimes_{el=c=\max} S' \rightarrow R, x \otimes y \mapsto xy.
\]

This map is unital and by looking at elementary tensors, if

\[
P \otimes Q \in M_n(S \otimes_c S')^+,
\]

then \( PQ = QP \in M_n(R)^+ \). Using injectivity of \( R \) and left-injectivity of \( \otimes_{el} \), we lift \( \phi \) to a unital completely positive map

\[
\tilde{\phi} : R \otimes_{el=c=\max} S' \rightarrow R.
\]

We notice that \( \tilde{\phi}|_{C_1 \otimes S'} \) is a *-homomorphism and by using Choi’s results on multiplicative domains (see [Pau02]) we know that \( \tilde{\phi} \) is an \( S' \)-bimodule map. We define the map \( \psi : R \rightarrow R \) by \( \psi(a) := \tilde{\phi}(a \otimes 1) \). Then we see that \( \psi \) is u.c.p., and since \( \tilde{\phi} \) is an \( S' \) bimodule map, this gives us that \( \psi(R) \subset S'' \). Furthermore it is
simple to see that \( \psi \) fixes \( S \). Thus, (5) \( \implies \) (4).

(4) \( \implies \) (1) is trivial, so it remains to show that (2) \( \iff \) (3). Now, (2) \( \implies \) (3) is trivial therefore suppose that there exists an injective \( C^* \)-algebra \( \mathcal{R} \) such that the statement holds. Letting \( \iota : S \rightarrow \mathcal{R} \) denote the inclusion map, we have by injectivity a u.c.p. lift of \( \iota \), \( \tilde{\iota} : \mathcal{I}(S) \rightarrow \mathcal{R} \). Thus by looking at the composition

\[
\phi \circ \tilde{\phi} : \mathcal{I}(S) \rightarrow S''
\]

we have our desired map. \( \square \)

We have now shown that an operator system having DCEP is equivalent to it being \((el,c)\)-nuclear, and using the tools we have developed we want to start looking at how the full group \( C^* \)-algebra of the free group behaves under various oss’.

**Theorem 3.4** (Kirchberg [Kir94]). \( C^*(F) \) is weak\(_{\mathcal{B}(\mathcal{H})}(\text{min, max})\)-nuclear for all free groups \( F \).

**Theorem 3.5.** Let \( S \) be an operator system. Then the following are equivalent:

1. \( S \) is weak\(_{C^*(F)}(\text{min, max})\)-nuclear for all free groups \( F \).
2. There exists a complete order inclusion \( S \subset \mathcal{B}(\mathcal{H}) \) such that
   \[
   S \otimes_{\text{max}} C^*(F) \subset_{\text{coi}} \mathcal{B}(\mathcal{H}) \otimes_{\text{max}} C^*(F).
   \]
3. \( S \) is weak\(_{C^*(F_{\infty})}(\text{min, max})\)-nuclear.
4. There exists a complete order inclusion \( S \subset \mathcal{B}(\mathcal{H}) \) such that
   \[
   S \otimes_{\text{max}} C^*(F_{\infty}) \subset_{\text{coi}} \mathcal{B}(\mathcal{H}) \otimes_{\text{max}} C^*(F_{\infty}).
   \]

**Proof.** If \( S \) is weak\(_{C^*(F)}(\text{min, max})\)-nuclear, then realize \( S \subset_{\text{coi}} \mathcal{B}(\mathcal{H}) \) via the Choi-Effros characterization. Then by using injectivity of \( \otimes_{\text{min}} \) and Kirchberg’s theorem we have,

\[
S \otimes_{\text{max}} C^*(F) = S \otimes_{\text{min}} C^*(F) \subset_{\text{coi}} \mathcal{B}(\mathcal{H}) \otimes_{\text{min}} C^*(F) = \mathcal{B}(\mathcal{H}) \otimes_{\text{max}} C^*(F).
\]

Now suppose that (2) is satisfied. We then have by applying Kirchberg’s theorem that both \( S \otimes_{\text{min}} C^*(F) \) and \( S \otimes_{\text{max}} C^*(F) \) are operator subsystems of \( \mathcal{B}(\mathcal{H}) \otimes_{\text{max}} C^*(F) \), implying that for all \( n \in \mathbb{N} \),

\[
M_n(S \otimes_{\text{min}} C^*(F))^+ = M_n(\mathcal{B}(\mathcal{H}) \otimes_{\text{max}} C^*(F))^+ \cap (S \otimes C^*(F)) = M_n(S \otimes_{\text{max}} C^*(F))^+.
\]

Thus we have (1) \( \iff \) (2) which implies (3) \( \iff \) (4) therefore we need only show that (3) \( \implies \) (1). To this end, let there be a complete order inclusion \( S \subset_{\text{coi}} \mathcal{B}(\mathcal{H}) \) such that

\[
S \otimes_{\text{max}} C^*(F_{\infty}) \subset_{\text{coi}} \mathcal{B}(\mathcal{H}) \otimes_{\text{max}} C^*(F_{\infty}).
\]

Suppose \( X \subset Y \) are sets which implies \( C^*(F_X) \subset C^*(F_Y) \). Let \( \text{id}_S : S \rightarrow S \) denote the identity on \( S \) and let \( \iota : C^*(F_X) \hookrightarrow C^*(F_Y) \) denote the inclusion mapping. We then have a c.p. projection \( p_X : C^*(F_Y) \rightarrow C^*(F_X) \) which is the left inverse to \( \iota \). Therefore

\[
(id_S \otimes_{\text{max}} p_X) \circ (id_S \otimes_{\text{max}} \iota) : S \otimes_{\text{max}} C^*(F_X) \rightarrow S \otimes_{\text{max}} C^*(F_Y) \rightarrow S \otimes_{\text{max}} C^*(F_X).
\]
Once again by invoking Proposition 2.1, since the composition is a c.o.i. implies that the first map \( \text{id}_S \otimes_{\text{max}} t \) is a c.o.i. and thus
\[
S \otimes_{\text{max}} C^*(F_X) \subseteq_{\text{coi}} S \otimes_{\text{max}} C^*(F_Y).
\]
Thus, if \( X \) is indeed countable we have
\[
S \otimes_{\text{max}} C^*(F_X) \subseteq_{\text{coi}} S \otimes_{\text{max}} C^*(F_\infty).
\]
Using Kirchberg’s theorem along with injectivity of \( \otimes_{\text{min}} \) we have
\[
S \otimes_{\text{max}} C^*(F_X) \subseteq_{\text{coi}} S \otimes_{\text{max}} C^*(F_Y) \subseteq_{\text{coi}} \mathcal{B}(H) \otimes_{\text{max}} C^*(F_\infty) = \mathcal{B}(H) \otimes_{\text{min}} C^*(F_\infty) \subseteq_{\text{coi}} S \otimes_{\text{min}} C^*(F_X).
\]
Now, suppose that \( X \) is uncountable and
\[
S \otimes_{\text{min}} C^*(F_X) \neq S \otimes_{\text{max}} C^*(F_X).
\]
Let \( u \in S \otimes C^*(F_X) \) be such that \( \|u\|_{\text{min}} \neq \|u\|_{\text{max}} \), with \( u = \sum_{\ell \in L} a_\ell \otimes \delta_\ell \), where \( \delta_\ell \) corresponds to the generator \( \ell \in X, L \subset F_X \). But \( L \) must be countable and this would imply
\[
S \otimes_{\text{min}} C^*(F_L) \neq S \otimes_{\text{max}} C^*(F_L),
\]
which contradicts the first case over a countable set. Thus, no such \( u \) can exist and we have \( S \) is weak\(_{C^*(F)}(\text{min, max})\)-nuclear for any free group \( F \).

\[\Box\]

**Theorem 3.6.** Let \( S \) be an operator system. Then \( S \) is \((el, c)\)-nuclear if and only if there exists a completely order isomorphic inclusion \( S \subset \mathcal{B}(H) \) such that
\[
S \otimes_{\text{max}} C^*(F) \subseteq_{\text{coi}} \mathcal{B}(H) \otimes_{\text{max}} C^*(F),
\]
for all free groups \( F \).

**Proof.** If \( S \) is \((el, c)\)-nuclear then by using the equivalence (4) in Theorem 3.2 we have our result letting the C*-algebra that we are tensoring against be \( C^*(F) \). Conversely, let
\[
S \otimes_{\text{max}} C^*(F) \subseteq_{\text{coi}} \mathcal{B}(H) \otimes_{\text{max}} C^*(F),
\]
for some c.o.i. \( S \subset \mathcal{B}(H) \), and any free group \( F \). Let \( A \) be a C*-algebra and \( F \) a free group such that
\[
A = C^*(F)/\mathcal{I}
\]
for some ideal \( \mathcal{I} \subset C^*(F) \). We will let \( \hat{\otimes} \), denote the completed operator system tensor product arising from the respective inclusion, and \( \hat{\otimes} \) denotes the closure of the algebraic tensor product. We must use a result from [Kav+13] which we will not prove. Suppose that \( S \subset A \), where \( S \) is an operator system, and \( A \) a C*-algebra. Then
\[
\frac{S \hat{\otimes}_{\text{min}} B}{S \hat{\otimes} \mathcal{I}} \subset \frac{A \hat{\otimes}_{\text{max}} B}{A \hat{\otimes} \mathcal{I}},
\]
completely isometrically for any C*-algebra \( B \). Furthermore the operator space and operator system quotients coincide.

We then apply exactness with respect to \( \hat{\otimes}_{\text{max}} \), the equivalence of (1) and (2) in Theorem 3.5, our remark above, and Kirchberg’s theorem to get
\[
S \hat{\otimes}_{\text{max}} A = S \hat{\otimes}_{\text{max}} C^*(F)/\mathcal{I} = \frac{S \hat{\otimes}_{\text{max}} C^*(F)}{S \hat{\otimes} \mathcal{I}} = \frac{S \hat{\otimes}_{\text{min}} C^*(F)}{S \hat{\otimes} \mathcal{I}} = \mathcal{B}(H) \hat{\otimes}_{\text{max}} A.
\]
\[\Box\]
4 The Operator System Local Lifting Property

Thus far we have related Lance’s WEP to \((el, max)\)-nuclearity, and DCEP to \((el, c)\)-nuclearity. In this final section we will relate the operator system local lifting property with \((min, er)\)-nuclearity.

**Definition 4.1.** Let \(S\) be an operator system, and \(A\) a C*-algebra with ideal \(I \subset A\). Given a unital completely positive map \(\phi : S \to A/I\), we say that \(\phi\) lifts locally if given any finite-dimensional operator subsystem \(S_o \subset S\) there exists a unital completely positive map \(\psi : S_o \to A\) such that \(q \circ \psi = \phi|_{S_o}\), where here \(q : A \to A/I\) denotes the canonical quotient map. An operator system \(S\) has the operator system local lifting property if for every C*-algebra \(A\) and every ideal \(I \subset A\), every unital completely positive map \(\phi : S \to A/I\) lifts locally.

**Theorem 4.1.** Given an operator system \(S\), the following are equivalent;

1. \(S\) is \((min, er)\)-nuclear.
2. \(S\) is \(weak_{\mathcal{B}(H)}(min, max)\)-nuclear for every Hilbert space \(H\).
3. \(S \otimes_{\min} T = S \otimes_{\max} T\) for all operator systems \(T\) having Lance’s WEP.

Proof. Supposing that \(S\) is \((min, er)\)-nuclear, then we have by injectivity of \(\mathcal{B}(H)\),

\[
S \otimes_{\min} \mathcal{B}(H) = \mathcal{S} \otimes_{er} \mathcal{B}(H) = \mathcal{S} \otimes_{\max} \mathcal{B}(H).
\]

Conversely, suppose that \(S\) is \(weak_{\mathcal{B}(H)}(min, max)\)-nuclear. Let \(T\) be an arbitrary operator system and let \(T \subset_{\text{coi}} \mathcal{B}(H)\) be the Choi-Effros characterization of \(T\). We then have by right injectivity of \(\otimes_{er}\), and injectivity of \(\otimes_{\min}\), that

\[
S \otimes_{er} T \subset_{\text{coi}} S \otimes_{\text{er=max}} \mathcal{B}(H) = S \otimes_{\min} \mathcal{B}(H) \supset_{\text{coi}} S \otimes_{\min} T.
\]

Thus, we have \(S\) is \((min, er)\)-nuclear.

By Kirchberg’s theorem we know that \(\mathcal{B}(H)\) has Lance’s WEP and thus \((3) \implies (2)\) is direct. Conversely, suppose that \(S\) is \(weak_{\mathcal{B}(H)}(min, max)\)-nuclear. Let \(T\) be an operator system having Lance’s WEP. We then have by injectivity of \(\otimes_{\min}\), symmetry of \(\otimes_{\max}\) and property \((3)\) of Theorem 2.2, that

\[
S \otimes_{\min} T \subset_{\text{coi}} S \otimes_{\min} \mathcal{B}(H) = S \otimes_{\max} \mathcal{B}(H) \supset_{\text{coi}} S \otimes_{\max} T,
\]

where of course we realize \(T \subset_{\text{coi}} \mathcal{B}(H)\) via the Choi-Effros characterization.

\[\Box\]

**Theorem 4.2.** An operator system \(S\) has the OSLLP if and only if \(S\) is \(weak_{\mathcal{B}(H)}(min, max)\)-nuclear. In particular, \(S\) has the OSLLP if and only if \(S\) is \((min, er)\)-nuclear.

Proof. Suppose that \(S\) has the OSLLP. Let \(\phi : S \to C_u(S)\) denote the u.c.p. map into the universal C*-algebra of \(S\). Let \(F\) be a free group such that

\[
C^*(F)/I = C_u(S),
\]

for some ideal \(I \subset C^*(F)\), and let \(q : C^*(F) \to C^*(F)/I\) denote the quotient map. Let \(a \in (S \otimes_{\min} \mathcal{B}(H))^+\), and let \(S_o \subset S\) be a finite-dimensional operator subsystem such that \(a \in (S_o \otimes_{\min} \mathcal{B}(H))^+\). Since \(S\) has the OSLLP we know that there exists a c.p. map \(\psi : S_o \to C^*(F)\) such that \(q \circ \psi = \phi|_{S_o}\).

We then have the following,

\[
S_o \otimes_{\min} \mathcal{B}(H) \to C^*(F) \otimes_{\min} \mathcal{B}(H) = C^*(F) \otimes_{\max} \mathcal{B}(H) \to C^*(F)/I \otimes_{\max} \mathcal{B}(H) = C_u(S) \otimes_{\max} \mathcal{B}(H) \supset_{\text{coi}} S \otimes_{\text{c=max}} \mathcal{B}(H).
\]

Thus we see that \(t \in (S \otimes_{\max} \mathcal{B}(H))^+\) and applying this method for all \(n \in \mathbb{N}\) will yield \(S\) is \(weak_{\mathcal{B}(H)}(min, max)\)-nuclear.

Though we will not write it here, we are currently writing the converse using Sinclair’s result [Sin17] that the local lifting property is equivalent to CP-stability.

\[\Box\]
References


