1 Introduction

The goal of this lecture series is to characterize properties of operator systems by looking at their nuclearity properties. In recent papers [Kav+11] and [Kav+13], the tensor theory of operator systems was developed, and further it was shown that one can view the famous Kirchberg’s conjecture as an operator system theoretic question.

Given any C*-algebra \( A \), we always assume that \( A \) is unital. It is first necessary to explain the objects we will be working with. Let \( S \) denote a vector space with involution, with a positive cone \( P_1 \subset S_h \) satisfying \( P_1 \cap -P_1 = \{0\} \). We will call the pair \((S, P_1)\) an ordered \( \ast \)-vector space. Here we define a partial order \( \leq \) on \( S_h \) such that \( s \geq t \) if and only if \( s - t \in P_1 \). Furthermore, let \( e \in P_1 \) be such that for every \( s \in S_h \) there exists \( r > 0 \) such that \( re \geq s \). We will say that \( e \) is an order unit for \( S \). If \( e \) satisfies the property that for all \( s \in S \), if

\[
re + s \in P_1 \quad \forall r > 0 \implies s \in P_1,
\]

then we will say that \( e \) is an archimedean order unit. We will call the triple \((S, P_1, e)\) an Archimedean order unit space (AU).

In [PT09], the theory of archimedean ordered vector spaces was developed (Kadison’s Function Systems). We let the sequence \( \{P_n\}_n \) denote a matrix ordering on \( S \) which implies for all \( n \in \mathbb{N} \)

1. \( P_n \subset M_n(S)_h \) is a cone.
2. \( P_n \cap -P_n = \{0\} \).
3. \( X^*P_nX \subset P_m \) for all \( X \in M_{n,m} \). (Compatibility of the family \( \{P_n\}_n \).)

We call the pair \((S, \{P_n\}_n)\) a matrix ordered \( \ast \)-vector space. Letting \( e_n := \text{diag} \{e, ..., e\} \in M_n(S) \), if \( e_n \) is an Archimedean order unit for the ordered \( \ast \)-vector space \((M_n(S), P_n)\) and this holds for all \( n \in \mathbb{N} \) then we will call \( e \) an Archimedean matrix order unit, and furthermore we will say that an abstract operator system is a triple \((S, \{P_n\}_n, e)\) where \( S \) is a \( \ast \)-vector space, \( \{P_n\}_n \) is a matrix ordering on \( S \), and \( e \) is an Archimedean matrix order unit.

Some examples of operator systems are

1. \( M_n \), the complex \( n \times n \) matrices.
2. \( \mathcal{B}(H) \), in particular, any unital C*-algebra.
3. Given \( A \in \mathcal{B}(H)^{(d)} \), \( A = (A_1, ..., A_d) \) we may look at the operator system generated by the span of the components. Thus, let \( S_A = \text{span} \{A_i; -d \leq i \leq d\} \), such that \( A_0 = 1 \), and \( A_{-i} = A_i^\ast \).
4. Let $\mathcal{M}$ be an operator space, think of a closed subspace of $\mathcal{B}(\mathcal{H})$. Then we have the induced operator system

$$S_{\mathcal{M}} = \left\{ \begin{pmatrix} \lambda & x \\ y & \mu \end{pmatrix} : x, y \in \mathcal{M}, \lambda, \mu \in \mathbb{C} \right\}$$

Given two operator systems $\mathcal{S}, \mathcal{T}$, we say that $\phi : \mathcal{S} \rightarrow \mathcal{T}$ is unital completely positive (ucp) if $\phi$ is unital, and for every $n \in \mathbb{N}$, $\phi(M_n(\mathcal{S})^+) \subset M_n(\mathcal{T})^+$. For us, the morphisms will always be u.c.p. maps. We will call a linear map $\phi : \mathcal{S} \rightarrow \mathcal{T}$ a complete order isomorphism (c.o.i.) if $\phi$ is a c.p. bijection, and $\phi^{-1}$ is also c.p.. Operator systems were characterized in a fundamental paper [CE77] by Choi and Effros.

**Theorem 1.1 (The Choi-Effros Characterization).** Given an operator system $\mathcal{S}$, there exists a Hilbert space $\mathcal{H}$ and a self-adjoint unital subspace $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ such that $\mathcal{S} \cong_{\text{coi}} \mathcal{V}$. Conversely, every self-adjoint unital subspace of $\mathcal{B}(\mathcal{H})$ is an abstract operator system.

This powerful theorem thus tells us that we do not need to distinguish between (abstract) and (concrete) operator systems, which becomes an indispensable tool in certain proofs. It is worth mentioning that this was also done by Ruan for operator spaces. We refer the reader to [Pis03] for an excellent write up of these topics.

Letting $(\mathcal{S}, \{P_n\}, e)$ and $(\mathcal{T}, \{Q_n\}, f)$ be two operator systems, denote the algebraic tensor product of these two operator systems by $\mathcal{S} \otimes \mathcal{T}$. By an operator system structure (oss) on $\mathcal{S} \otimes \mathcal{T}$ we mean a sequence of cones $\tau = \{C_n\}$ satisfying

1. $(\mathcal{S} \otimes \mathcal{T}, \{C_n\})$ is an operator system which we denote $\mathcal{S} \otimes_{\tau} \mathcal{T}$;
2. $P_n \otimes Q_n \subset C_{mn}$ for all $m, n \in \mathbb{N}$;
3. given u.c.p. maps $\phi : \mathcal{S} \rightarrow M_p$ and $\psi : \mathcal{T} \rightarrow M_q$, the induced map on the tensor product

$$\phi \otimes_{\tau} \psi : \mathcal{S} \otimes_{\tau} \mathcal{T} \rightarrow M_{pq}$$

is u.c.p. (matrix functoriality of $\tau$.) Thus, when we say an operator system tensor product we mean a map $\tau : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$ such that $\tau(\mathcal{S}, \mathcal{T})$ is an oss on $\mathcal{S} \otimes \mathcal{T}$ for every $\mathcal{S}, \mathcal{T} \in \mathcal{O}$. Here $\mathcal{O}$ denotes the category whose objects are operator systems and whose morphisms are u.c.p. maps. We will say that an oss $\tau$ is functorial if given any operator systems $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}$, and u.c.p. maps $\phi : \mathcal{S} \rightarrow \mathcal{U}$, $\psi : \mathcal{T} \rightarrow \mathcal{V}$, the induced linear map

$$\phi \otimes_{\tau} \psi : \mathcal{S} \otimes_{\tau} \mathcal{T} \rightarrow \mathcal{U} \otimes_{\tau} \mathcal{V}$$

is u.c.p.. Suppose that $\tau, \kappa$ are two oss’, then we denote the map from

$$\mathcal{S} \otimes_{\tau} \mathcal{T} \rightarrow \mathcal{U} \otimes_{\kappa} \mathcal{V}$$

by $\phi \otimes_{\tau,\kappa} \psi$. If the oss’ in the domain and codomain are the same ($\tau = \kappa$) then we will simply write $\phi \otimes_{\tau} \psi$. We will reserve the notation $\phi \otimes \psi$ to denote the induced linear map on the algebraic tensor product, not assuming any operator system structure. We may put a partial ordering on the set of oss $\leq$ by saying $\tau \geq \kappa$ if the cone structure induced by $\tau$ are contained in those induced by $\kappa$, i.e. for $\mathcal{S}, \mathcal{T} \in \mathcal{O}$ and for every $n \in \mathbb{N}$ we have

$$M_n(\mathcal{S} \otimes_{\tau} \mathcal{T})^+ \subset M_n(\mathcal{S} \otimes_{\kappa} \mathcal{T})^+.$$ 

Given an operator system $\mathcal{S}$ we will say that $\mathcal{S}$ is $(\tau, \kappa)$-nuclear if for any $\mathcal{T} \in \mathcal{O}$, we have

$$\text{id} : \mathcal{S} \otimes_{\tau} \mathcal{T} \rightarrow \mathcal{S} \otimes_{\kappa} \mathcal{T}$$

is a complete order isomorphism. In other words

$$\mathcal{S} \otimes_{\tau} \mathcal{T} = \mathcal{S} \otimes_{\kappa} \mathcal{T}.$$
We now want to talk about oss' on $S \otimes T$ induced by inclusions into larger operator systems. Suppose that $S_o \supset S$ is an operator system containing $S$ as an operator subsystem. Let $\tau_o$ be an operator system structure on $S_o \otimes T$. Then we say that $\tau$ is induced by $\tau_o$ if for all $n \in \mathbb{N}$,

$$M_n(\tau \otimes, T)^+ = M_n(S_o \otimes \tau_o T)^+ \cap M_n(S \otimes T).$$

This means that the cone structure given by $\tau$ is the same as the cone structure on $S \otimes T$ gotten via the embedding $S \otimes T \subset S \otimes \tau_o T$. We will denote this property by

$$S \otimes, T \approx S \otimes T \subset S \otimes \tau_o T.$$

In operator system theory we are concerned with six different operator system structures on our algebraic tensor product, the minimal (min), enveloping (e), enveloping left (el), enveloping right (er), commuting (c), and maximal (max). It was shown in [Kav+11] that these oss' have the following ordering

$$min \leq e \leq el, er \leq c \leq max.$$

Thus, we see that $\otimes_{max}$ induces the smallest cone structure on the algebraic tensor product, and $\otimes_{min}$ produces the largest. We refer the reader to [Kav+11] for various properties concerning how these structures are induced but we will provide some of the main points here.

2 Six Operator System Structures on $\otimes$

Let $S, T \in \mathcal{O}$. For all $n \in \mathbb{N}$ we define the following cones

$$\mathcal{C}^{min}_n = \left\{ A \in M_n(S \otimes T) : (\phi \otimes \psi)^{(n)}(A) \in M_{qn^2}, \forall \phi \in \mathcal{J}_p(S), \forall \psi \in \mathcal{J}_q(T), \forall p, q \in \mathbb{N} \right\},$$

where here we let $\mathcal{J}_p(S)$ denote the p-state space of $S$, define $\mathcal{J}_q(T)$ analogously, and $(\phi \otimes \psi)^{(n)}$ denotes the nth-amplification of the induced map $\phi \otimes \psi$. It can be checked that $\otimes_{min} = \{\mathcal{C}^{min}_n\}_n$ is an oss on $S \otimes T$. We summarize the main properties of $\otimes_{min}$, which were proved in [Kav+11], in a single theorem:

**Theorem 2.1** (Kavruk et al.). Given $S, T \in \mathcal{O}$, it follows that $\otimes_{min}$ is a symmetric, associative, injective, functorial operator system tensor product on $S \otimes T$. Furthermore

$$S \otimes_{min} T \approx S \otimes T \subset \mathcal{B}(\mathcal{H} \otimes_2 K),$$

where $S \subset \mathcal{B}(\mathcal{H})$ and $T \subset \mathcal{B}(K)$ are the Choi-Effros characterizations of the operator systems $S$ and $T$, and $\otimes_2$ denotes the Hilbertian tensor product. Finally, $\otimes_{min}$ generates the smallest operator system structure on $S \otimes T$, i.e., given any other oss $\tau$ on the algebraic tensor product, then for every $n \in \mathbb{N}$ we have

$$M_n(S \otimes \tau T)^+ \subset M_n(S \otimes_{min} T)^+.$$

As we have seen in the tensor theory for C*-algebras, $\otimes_{min}$ is the "best" behaved oss we look at, except when we speak of exactness. Denoting the units of $S$ and $T$ both by 1, we let

$$S \otimes_{min} T = (S \otimes T, \{\mathcal{C}^{min}_n\}_n, 1 \otimes 1)$$

denote the minimal tensor product of $S$ and $T$.

Next we want to define the maximal cone structure, but a few remarks are in order. By definition, the order unit of an operator system must necessarily be archimedean, but this is not always the case when looking at a general matrix ordering. But thanks to the work of Paulsen and Tomforde in [PT09] it was shown that any order unit space can be "archimedeanized." We will not prove this fact here but the construction itself is not very difficult. Once again for $S, T \in \mathcal{O}$ and for all $n \in \mathbb{N}$ we define the matrix ordering on the elementary tensors of $S \otimes T$ by

$$\mathcal{D}^{max}_n = \{\alpha(P \otimes Q)\alpha^* : P \in M_p(S)^+, Q \in M_q(T)^+, \alpha \in M_{n,pq}\}.$$
We now define
\[ C_n^{\max} = \{ A \in M_n(S \otimes T) : \text{re}_n + P \in D_n^{\max} \forall r > 0 \} . \]

It follows that \( \{ C_n^{\max} \}_n \) is the archimedeanization of the matrix ordering \( \{ D_n^{\max} \}_n \), though we will not prove that here. We define the operator system
\[ (S \otimes T, \{ C_n^{\max} \}_n, 1 \otimes 1) = S \otimes_{\max} T \]
to be the maximal operator system tensor product.

As we did for the minimal tensor product we summarize the results of [Kav+11] with regard to \( \otimes_{\max} \) in one theorem.

**Theorem 2.2** (Kavruk et al.). Given \( S, T \in \mathcal{O} \) then it follows \( \otimes_{\max} \) is a symmetric, associative, and functorial operator system tensor product on \( S \otimes T \). Furthermore it follows that \( \otimes_{\max} \) is the largest operator system structure on the algebraic tensor product, i.e., given any other oss \( \tau \) on \( S \otimes T \), then for every \( n \in \mathbb{N} \)
\[ M_n(S \otimes T)^+ \subset M_n(S \otimes_{\tau} T)^+ . \]

Given a Hilbert space \( \mathcal{H} \), we define the completely positive maps from \( S \) to \( \mathcal{B}(\mathcal{H}) \) by \( CP(S, \mathcal{B}(\mathcal{H})) \). Thus, given \( S, T \in \mathcal{O} \) we define the following set
\[ cp(S, T) = \{ (\phi, \psi) : \mathcal{H} \text{ is a Hilbert space, } \phi \in CP(S, \mathcal{B}(\mathcal{H})), \psi \in CP(T, \mathcal{B}(\mathcal{H})), \phi(S)\psi(T) = \psi(T)\phi(S) \} . \]

Thus, we define \( cp(S, T) \) to be the set of all pairs of c.p. maps into \( \mathcal{B}(\mathcal{H}) \) such that they have commuting ranges. Thus, given any \( (\phi, \psi) \in cp(S, T) \) we define the product map
\[ \phi \times \psi : S \otimes T \rightarrow \mathcal{B}(\mathcal{H}) \]
by \( (\phi \cdot \psi)(s \otimes t) = \phi(s)\psi(t) \).

Using pairs from \( cp(S, T) \) we may then define the cone structure given by
\[ C_n^c = \left\{ A \in M_n(S \otimes T) : (\phi \times \psi)^{(n)}(A) \geq 0, \forall (\phi, \psi) \in cp(S, T) \right\} . \]

One then shows that \( \{ C_n^c \}_n \) is a matrix ordering on \( S \otimes T \) and we then define the commuting tensor product as
\[ (S \otimes T, \{ C_n^c \}_n, 1 \otimes 1) := S \otimes_c T . \]

Recall that given an operator system there always exists its universal C*-algebra. Thus, the universal C*-algebra \( C_u^*(S) \) of the operator system \( S \) is the unique (up to *-homomorphism) C*-algebra such that there exists a u.c.p. map \( \iota : S \rightarrow C_u^*(S) \) with the property that \( \iota(S) \) generates \( C_u^*(S) \) as a C*-algebra, and \( C_u^*(S) \) satisfies the universal property that given any other C*-algebra \( A \) and a u.c.p. map \( \phi : S \rightarrow A \) there exists a *-homomorphism, \( \pi : C_u^*(S) \rightarrow A \) such that \( \phi = \pi \circ \iota \).

**Theorem 2.3** (Kavruk et al.). Let \( S, T \in \mathcal{O} \). It then follows that \( \otimes_c \) is a symmetric, and functorial operator system tensor product. Furthermore,
\[ S \otimes_c T \approx S \otimes T \subset C_u^*(S) \otimes_{\max} C_u^*(T) . \]

Thus, we see that the oss gotten from \( \otimes_c \) is the same as the one gotten via the embedding into the maximal tensor product of the respective universal C*-algebras of the operator systems. A very important fact is worth mentioning; many times in our proofs we are looking at an operator system tensor product where both of our objects are C*-algebras. Thus, in this last case, how does
\[ C_u^*(S) \otimes_{\max} C_u^*(T) \]
relate to
\[ C_u^*(S) \otimes_{C_{\max}} C_u^*(T) , \]
where $\otimes_{C^*,\max}$ denotes the maximal C*-algebra tensor product? Recall given C*-algebras $A, B$, we get the maximal C*-algebra tensor product by completing $A \otimes B$ with respect to the norm
\[ \|x\|_{C^*,\max} = \sup_{\pi} \{ \|\pi(x)\| : \pi : A \otimes B \to \mathcal{B}(\mathcal{H}) \}, \]
where the supremum is running over all representations of the algebraic tensor product into $\mathcal{B}(\mathcal{H})$. Well we have the following;

**Remark 2.1.** Let $A$ and $B$ be two C*-algebras. Then

1. $A \otimes_{\min} B \cong A \otimes B \subset A \otimes_{C^*,\min} B$.
2. $A \otimes_{\max} B \cong A \otimes B \subset A \otimes_{C^*,\max} B$.

One result that is very important in proving many nuclearity properties is the fact that $\otimes_c = \otimes_{\max}$ when one of the objects we are tensoring with is a C*-algebra.

**Theorem 2.4** (Kavruk et al.). *Given any unital C*-algebra $A$, and operator system $S$, then*

\[ A \otimes_c S = A \otimes_{\max} S. \]

**Proof.** We first begin by showing that $A \otimes_{\max} S$ is an operator $A$-system, that is to say that $A \otimes_{\max} S$ is an $A$-bimodule and furthermore that positive elements remain positive under conjugation by elements of $A$, i.e., given $U \in M_n(A \otimes_{\max} S)^+$ then for any $B \in M_n(A)$ we have $B^* UB \in M_n(A \otimes_{\max} S)^+$. Declare the bimodule multiplication on elementary tensors by
\[ a_1(a \otimes s)a_2 := a_1aa_2 \otimes s. \]

First suppose that $U \in M_n(A \otimes_{\max} S)^+$ and therefore we assume
\[ U = \alpha(P \otimes Q)^*, P \in M_n(A)^+, Q \in M_n(S)^+, \alpha \in M_{n,pq}. \]

Given any matrix $B$ it then follows
\[ B^* \alpha(P \otimes Q)^* B = (\alpha^* B)^*(P \otimes Q)(\alpha^* B). \]

Thus, in showing that $A \otimes_{\max} S$ is an operator $A$-system, we may assume $U = P \otimes Q$, where $P \in M_n(A)^+$, $Q = (s_{ij}) \in M_p(S)^+, pq = n$.

Let $B = (B_1 \cdots B_q)^T \in M_{p,k}$, it then follows that
\[ B^* UB = \sum_{i,j} (B_i^* PB_j) \otimes (s_{ij}). \]

Denote $\Lambda = (B_i^* PB_j)_{i,j} \in M_{kq}(A)^+$. Finally, let
\[ X = (e_1 \otimes I_k \cdots e_q \otimes I_k)^T, e_i \otimes I_k \in M_{kq}. \]

We then see that
\[ B^* UB = X^*(\Lambda \otimes Q)X \in M_n(A \otimes_{\max} S)^+, \]
therefore giving us that $A \otimes_{\max} S$ is an operator $A$-system.

Define the following maps
\[ \pi : A \to \mathcal{I}(A \otimes_{\max} S), \rho : S \to \mathcal{I}(A \otimes_{\max} S) \]

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by
\[ a \mapsto a \otimes 1_S, \quad s \mapsto 1_A \otimes s, \]
respectively. Using the Choi-Effros result proving that injective operator systems are necessarily completely
order isomorphic to C*-algebras, and letting \( \cdot \) denote the bimodule action, one shows that \( \pi \) defined above
is a *-homomorphism. Furthermore, \( \rho \) is a c.o.i.. Thus, since
\[ \pi(A)\rho(S) = \rho(S)\pi(A), \]
and both maps are c.p., we have that
\[ \pi \times \rho : A \otimes_c S \to I(A \otimes_{\text{max}} S), \]
is c.p. with range \( A \otimes_{\text{max}} S \). Thus, since \( \pi \times \rho(a \otimes s) = a \otimes s \), we have that
\[ A \otimes_c S = A \otimes_{\text{max}} S. \]

We now wish to look at operator system tensor products whose structure is purely defined by a given
embedding. Let \( I(S) \) denote the injective envelope of the operator system \( S \). Recall that this is the
minimal injective object in \( O \), and the construction of such an object will not be proven here, though we
will mention the main points. Letting \( S \subseteq S \subseteq B \)
\[ \phi \text{ and only if there exists a completely positive projection} \]
\[ \text{Theorem 2.5. Let } S \in O \text{ and let } S \subseteq B(\mathcal{H}) \text{ be the Choi-Effros characterization of } S \text{. Then } S \text{ is injective if and only if there exists a completely positive projection } \phi : B(S) \to S. \]

Proof. If \( S \) is injective then it follows since \( \text{id} : S \to S \) is u.c.p. there exists a u.c.p. extension
\[ \phi : B(\mathcal{H}) \to S \]
which is necessarily a projection since it extends the identity.

Conversely, if \( E, F \in O \) with \( E \subseteq F \), and \( \phi : B(\mathcal{H}) \to S \) is a u.c.p. projection, then using injectivity of
\( B(\mathcal{H}) \) we have if \( \gamma : E \to S \subseteq B(\mathcal{H}) \) is u.c.p. then letting \( \tilde{\gamma} : F \to B(\mathcal{H}) \) denote its u.c.p. extension, we
have the desired u.c.p. map \( \phi \circ \tilde{\gamma} : F \to S \) extending \( \gamma \).

Given any \( S \)-map, we can look at an \( S \)-seminorm \( \rho_\phi : B(\mathcal{H}) \to \mathbb{R} \) defined by \( \rho_\phi(a) = \| \phi(a) \| \) for
\( a \in B(\mathcal{H}) \).

Finally we may define partial orderings on the set of \( S \)-seminorms by saying that \( \sigma \leq \rho \) if \( \sigma(x) \leq \rho(x) \)
for all \( x \in B(\mathcal{H}) \). At this point one proves that there exists a minimal \( S \)-seminorm, and necessarily the
\( S \)-map that gives the minimal \( S \)-seminorm will give us our injective envelope, i.e., if \( \rho_\phi \) denotes the minimal
\( S \)-seminorm, then \( \phi(B(\mathcal{H})) \) is an injective envelope for \( S \) and it is unique up to complete order isomorphism.

At this point we are able to define our remaining three tensors. Letting \( S, T \in O \), we define the \textit{enveloping left} tensor product \( \otimes_{cl} \) by
\[ S \otimes_{cl} T \approx S \otimes T \subset I(S) \otimes_{\text{max}} T. \]
Thus, this tells us that the cone structure is given by
\[ M_n(S \otimes_{cl} T)^+ = M_n(I(S) \otimes_{\text{max}} T)^+ \cap M_n(S \otimes T). \]
Similarly, we define the \textit{enveloping right} tensor product \( \otimes_{cr} \) by
\[ S \otimes_{cr} T \approx S \otimes T \subset S \otimes_{\text{max}} I(T). \]
Finally, we define the \textit{enveloping} tensor product \( \otimes_e \) by
\[ S \otimes_e T \approx S \otimes T \subset I(S) \otimes_{\text{max}} I(T). \]

As we did with the other oss’ thus far, we state one theorem encompassing the main properties of these
three operator system tensor products.
Proposition 3.1. Let $S \in \mathcal{O}$. Then $\otimes_{el}, \otimes_{er}$, and $\otimes_{e}$ are all functorial operator system tensor products, and $\otimes_{el}$ is the largest left injective operator system tensor product, $\otimes_{er}$ is the largest right injective operator system tensor product, and $\otimes_{e}$ is the largest injective operator system tensor product.

3 Characterizations of Operator Systems in Relation to Tensor Theory Nuclearity

In [Lan73] Lance had presented what he called a weak expectation for a C*-algebra. In short, assuming that a C*-algebra was acting non-degenerately on an algebra of bounded operators on a Hilbert space, a weak expectation was a u.c.p. map from the algebra of bounded operators to the weak closure of the C*-algebra. A C*-algebra $A$ was then said to have the weak expectation property provided that every faithful representation of $A$, $\pi : A \rightarrow B(H)$, admitted a weak expectation. Lance went on to prove the following theorem which we will take as the definition of a C*-algebra having this property.

Theorem 3.1 (Lance). A C*-algebra $A$ has the weak expectation property if and only if $\otimes_{C^{*}\text{-max}}$ is weak$_A$-left injective. That is to say that for any C*-algebra $B \supseteq A$ and any C*-algebra $C$, we have the following

$$A \otimes_{C^{*}\text{-max}} C \subseteq \text{coi} B \otimes_{C^{*}\text{-max}} C.$$

Therefore, given a C*-algebra $A$ we say that $A$ has Lance’s weak expectation property (WEP) if $\otimes_{C^{*}\text{-max}}$ is left injective with respect to $A$. Equivalently, we may say that $A \subseteq B(H)$ has Lance’s WEP if the canonical complete order isomorphism $\iota : A \hookrightarrow A^{**}$ lifts to a u.c.p. map

$$\tilde{\iota} : B(H) \rightarrow A^{**},$$

fixing $A$.

Letting $S \in \mathcal{O}$ we say that $S$ has Lance’s WEP if the the complete order isomorphism $\iota : S \hookrightarrow S^{**}$ has u.c.p. lift

$$\tilde{\iota} : \mathcal{I}(S) \rightarrow S^{**}.$$

The next result which relates Lance’s WEP with tensor nuclearity is due to Han in [Han11].

Theorem 3.2 (Han). Let $S$ be an operator system. Then $S$ has Lance’s WEP if and only if $S$ is $(el, max)$-nuclear

Proving that Lance’s WEP implies $(el, max)$-nuclearity was already known and proven in [Kav+13], but the converse was not known until Han’s proof. Moreover, it follows that $S$ being $(el, max)$-nuclear is equivalent to $\otimes_{max}$ being weak$_S$-left injective.

Proposition 3.1. Let $S$ be an operator system. Then $S$ is $(el, max)$-nuclear if and only if for all operator systems $S_o \supseteq S$ and $T$,

$$S \otimes_{max} T \subseteq \text{coi} S_o \otimes_{max} T,$$

where we let $\text{coi}$ denote a complete order injection. We will call this property weak$_S$-left injectivity of $\otimes_{max}$.

We will need the following lemma before proving the proposition;

Lemma 3.1. Let $S, T$ and $U$ be operator systems and suppose that $\phi : S \rightarrow T$ and $\psi : T \rightarrow U$ are unital completely positive maps. Then if the composition map $\psi \circ \phi$ is a complete order injection, then the map $\phi$ is necessarily a complete order injection.

Proof. Let $t = [t_{ij}] \in M_n(\phi(S))^+$. Letting $s = [s_{ij}] \subseteq M_n(S)$ be such that $\phi^{(n)}(s) = t$, we know that

$$\psi^{(n)}(t) = \psi^{(n)}(\phi^{(n)}(s)) = (\psi \circ \phi)^{(n)}(s) \in M_n(U)^+ \iff s \in M_n(S)^+.$$

Thus, $\phi^{-1} : \phi(S) \rightarrow S$ is u.c.p. as desired which implies $\phi : S \rightarrow T$ is a complete order injection.

\[ \Box \]
Proof of Proposition 3.1. Suppose that $\mathcal{S}$ is $(el, max)$-nuclear. Let $\iota : \mathcal{S} \rightarrow \mathcal{S}_o$ denote the inclusion mapping, $id : \mathcal{T} \rightarrow \mathcal{T}$ the identity mapping on $\mathcal{T}$, and let $j : \mathcal{S} \hookrightarrow \mathcal{I}(\mathcal{S})$ denote the complete order injection into the injective envelope. We then have a ucp extension $j : \mathcal{S}_o \rightarrow \mathcal{I}(\mathcal{S})$ of $j$. We now look at the sequence of maps

$$((\tilde{j} \otimes id) \circ (\iota \otimes max id)) : \mathcal{S} \otimes_{el} \mathcal{T} = \mathcal{S} \otimes_{max} \mathcal{T} \rightarrow \mathcal{S}_o \otimes_{max} \mathcal{T} \rightarrow \mathcal{I}(\mathcal{S}) \otimes_{max} \mathcal{T}.$$ 

But by definition we have that

$$\mathcal{S} \otimes_{el} \mathcal{T} \subset_{coi} \mathcal{I}(\mathcal{S}) \otimes_{max} \mathcal{T},$$

and therefore by applying Lemma 3.1 we have that $\iota \otimes_{max} id$ is a complete order injection.

Now suppose that for every operator system $\mathcal{S}_o \supset \mathcal{S}$, we have the stated completely order inclusion with respect to $\otimes_{max}$. It then follows that

$$\mathcal{S} \otimes_{el} \mathcal{T} \subset_{coi} \mathcal{I}(\mathcal{S}) \otimes_{max} \mathcal{T} \supset_{coi} \mathcal{S} \otimes_{max} \mathcal{T}.$$ 

Thus we have that both $\mathcal{S} \otimes_{el} \mathcal{T}$, and $\mathcal{S} \otimes_{max} \mathcal{T}$ are operator subsystems of $\mathcal{I}(\mathcal{S}) \otimes_{max} \mathcal{T}$, and therefore they must be equal.

Thus, this notion of the weak expectation property to operator systems is a natural extension to that developed by Lance for C*-algebras.

We now relate an characterization of operator systems to $(el, c)$-nuclearity. If for all completely order isomorphic inclusions $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ we have a completely positive map $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{S}''$ that fixes $\mathcal{S}$ (that is to say that $\phi(s) = s$ for every $s \in \mathcal{S}$) then we will say that $\mathcal{S}$ has the double commutant expectation property (DCEP). Though we cannot prove all the equivalences here, it is worth making note of some important observations. If an operator system $\mathcal{S}$ is $(el, c)$-nuclear, then it follows that $\otimes_c$ is weak$_S$-left injective. Notice how this equivalence parallels that of Lance’s weak expectation property. Furthermore, it follows that an operator system will have DCEP if the for every inclusion $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ we have a c.p. map

$$\phi : \mathcal{I}(\mathcal{S}) \rightarrow \mathcal{S}''$$ fixing $\mathcal{S}$.

If $\mathcal{A}$ is a C*-algebra, then WEP and DCEP coincide. This is a direct consequence of Theorem 2.4, that $\otimes_c = \otimes_{max}$ when one of the objects we are tensoring against is a C*-algebra. Thus, knowing if a C*-algebra has Lance’s WEP is equivalent to knowing if it has DCEP. Thus, it should be seen which nuclearity property is equivalent to DCEP, and that is being $(el, c)$-nuclear.

Theorem 3.3. Let $\mathcal{S} \in \mathcal{O}$. Then $\mathcal{S}$ has DCEP if and only if $\mathcal{S}$ is $(el, c)$ nuclear.

We only present an outline of the main points behind the proof of this equivalence.

Outline. First assuming that $\mathcal{S} \in \mathcal{O}$ has DCEP, you show that we may instead take c.p. maps from the injective envelope of $\mathcal{S}$ rather than $\mathcal{B}(\mathcal{H})$. Afterwards you prove that this property then implies $(el, c)$-nuclearity. This equivalence relies on the universal property of $\otimes_{C^{*-max}}$. After this one shows that $(el, c)$-nuclearity implies that for all injective C*-algebras $\mathcal{R} \supset \mathcal{S}$ there exists c.p. maps $\phi : \mathcal{R} \rightarrow \mathcal{S}''$ fixing $\mathcal{S}$. This last property will then give you DCEP.

Theorem 3.4 (Universality of $\otimes_{C^{*-max}}$). Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. Then given a *-homomorphism $\pi : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ there exists a unique extension of $\pi$ to a *-homomorphism $\hat{\pi} : \mathcal{A} \otimes_{C^{*-max}} \mathcal{B} \rightarrow \mathcal{C}$. Conversely, given two *-homomorphisms

$$\pi_1 : \mathcal{A} \rightarrow \mathcal{C}, \pi_2 : \mathcal{B} \rightarrow \mathcal{C}, \pi_1(\mathcal{A})\pi_2(\mathcal{B}) = \pi_2(\mathcal{B})\pi_1(\mathcal{A})$$

they induce a unique *-homomorphism on $\otimes_{C^{*-max}}$ given by

$$\pi_1 \times \pi_2 : \mathcal{A} \otimes_{C^{*-max}} \mathcal{B} \rightarrow \mathcal{C}.$$
We present the remarkable theorem of Kirchberg shown in [Kir94], which we use to also give a characterization of DCEP.

**Theorem 3.5** (Kirchberg). Given any free group $F$, the full group $C^*$-algebra $C^*(F)$ is weak-$\mathcal{B}(\mathcal{H})$ (min, max)-nuclear. That is to say that

$$C^*(F) \otimes_{\text{min}} \mathcal{B}(\mathcal{H}) = C^*(F) \otimes_{\text{max}} \mathcal{B}(\mathcal{H}).$$

Using Kirchberg’s theorem we are able to prove the following equivalence.

**Theorem 3.6** (Kavruk et al.). Let $S \in \mathcal{O}$. Then $S$ is (el, c)-nuclear if and only if there exists a completely order isomorphic inclusion $S \subset \mathcal{B}(\mathcal{H})$ such that

$$S \otimes_{\text{max}} C^*(F) \subset_{\text{coi}} \mathcal{B}(\mathcal{H}) \otimes_{\text{max}} C^*(F),$$

for all free groups $F$.

**Proof.** If $S$ is (el, c)-nuclear then though we did not prove it, this is equivalent to the existence of a Hilbert space $H$ such that $S \subset_{\text{coi}} \mathcal{B}(H)$ and

$$S \otimes_c B \subset_{\text{coi}} \mathcal{B}(H) \otimes_c B,$$

for all $C^*$-algebras $B$. Thus, simply let $B = C^*(F)$.

Conversely, let

$$S \otimes_{\text{max}} C^*(F) \subset_{\text{coi}} \mathcal{B}(H) \otimes_{\text{max}} C^*(F),$$

for some c.o.i. $S \subset B(H)$, and any free group $F$. Let $A$ be a $C^*$-algebra and $F$ a free group such that

$$A = C^*(F)/I$$

for some ideal $I \subset C^*(F)$. We will let $\hat{\otimes}$ denote the completed operator system tensor product arising from the respective inclusion, and $\hat{\otimes}$ denotes the closure of the algebraic tensor product. We then apply exactness with respect to $\hat{\otimes}$, the equivalence of

$$S \otimes_{\text{min}} C^*(F) = S \otimes_{\text{max}} C^*(F) \iff S \subset_{\text{coi}} \mathcal{B}(H), S \otimes_{\text{max}} C^*(F) \subset_{\text{coi}} \mathcal{B}(H) \otimes_{\text{max}} C^*(F),$$

for all free groups $F$, injectivity of $\hat{\otimes}_{\text{min}}$, isometric inclusions of quotients with respect to $\hat{\otimes}_{\text{min}}$, and Kirchberg’s theorem to get

$$S \hat{\otimes}_{\text{max}} A = S \hat{\otimes}_{\text{max}} C^*(F)/I = \frac{S \hat{\otimes}_{\text{max}} C^*(F)}{S \otimes I} = \frac{S \hat{\otimes}_{\text{min}} C^*(F)}{S \otimes I},$$

$$\subset_{\text{coi}} \frac{\mathcal{B}(H) \hat{\otimes}_{\text{min}} C^*(F)}{\mathcal{B}(H) \otimes I} = \mathcal{B}(H) \hat{\otimes}_{\text{min}} C^*(F)/I = \mathcal{B}(H) \hat{\otimes}_{\text{max}} C^*(F)/I$$

$$= \mathcal{B}(H) \hat{\otimes}_{\text{max}} A.$$

The final property we wish to talk about is Kirchberg’s operator system local lifting property. Suppose we are looking at a $C^*$-algebra $A$. Then given a contractive completely positive map (c.c.p.), say $\phi : A \to B/I$, where $B$ is any $C^*$-algebra and $I \subset B$ an ideal (closed two-sided), we say that $\phi$ lifts locally if given any finite-dimensional operator subsystem $S_o \subset A$, there exists a c.c.p. map $\psi : S_o \to B$ such that

$$q \circ \psi = \phi|_{S_o},$$

where $q : B \to B/I$ is the canonical quotient map. Thus, we will say that $A$ has the local lifting property if given any $C^*$-algebra and any ideal of that $C^*$-algebra, and c.c.p. map from $A$ to the quotient $C^*$-algebra
lifts locally. This will be precisely our notion for local lifting of operator systems except we will make the corresponding change to our morphisms.

Let $S \in \mathcal{O}$, then given a C*-algebra $B$ and any ideal $J \subset B$, we say that $\phi : S \rightarrow B/J$ lifts locally if given any finite-dimensional operator subsystem $S_o \subset S$, there exists a u.c.p. map $\psi : S_o \rightarrow B$ such that $q \circ \psi = \phi |_{S_o}$.

Thus, $S$ will have the operator system local lifting property (OSLLP) if for any C*-algebra and any ideal of the C*-algebra, every u.c.p. map from the operator system to the quotient C*-algebra lifts locally.

**Theorem 3.7.** Let $S \in \mathcal{O}$. Then the following are equivalent.

1. $S$ is $(\min, \text{er})$-nuclear.
2. $S$ is weak$_{\mathcal{B}(H)}(\min, \max)$-nuclear for any Hilbert space $H$.
3. $S \otimes_{\min} T = S \otimes_{\max} T$ for any operator system having Lance’s WEP.

**Proof.** We will prove the first equivalence, the other can be found in the author’s survey on the three characterizations WEP, DCEP, and OSLLP.

If $S$ is $(\min, \text{er})$-nuclear then

$$S \otimes_{\min} \mathcal{B}(H) = S \otimes_{\text{er}} \mathcal{B}(H) = S \otimes_{\max} \mathcal{B}(H),$$

where the last equality follows since $\mathcal{B}(H)$ is injective. Conversely, suppose that $S$ is weak$_{\mathcal{B}(H)}(\min, \max)$-nuclear, and let $T \in \mathcal{O}$, with $T \subset \mathcal{B}(H)$ being the Choi-Effros characterization of $T$. It follows

$$S \otimes_{\text{er}} T \subset_{\text{coi}} S \otimes_{\text{er}} \mathcal{B}(H) = S \otimes_{\max} \mathcal{B}(H) = S \otimes_{\min} \mathcal{B}(H) \subset_{\text{coi}} S \otimes_{\min} T,$$

where we have used right-injectivity of $\otimes_{\text{er}}$ and injectivity of $\otimes_{\min}$.

Note that in [Pis03] Pisier proves this exact equivalence in the C*-algebra case. In particular given a C*-algebra $A$, then Pisier proved that $A$ has LLP if and only if $A$ is weak$_{\mathcal{B}(H)}(\min, \max)$-nuclear. Equivalently he shows that you may replace the Hilbert space $H$ with $\ell_2$. Finally you may also replace $\mathcal{B}(H)$ with any C*-algebra having Lance’s WEP.

**Theorem 3.8.** Given $S \in \mathcal{O}$, $S$ has the operator system local lifting property if $S$ is weak$_{\mathcal{B}(H)}(\min, \max)$-nuclear for every Hilbert space $H$. Thus, $S$ having the OSLLP is equivalent to it being $(\min, \text{er})$-nuclear.

Thus, we have seen that all of these properties of operator systems are natural extensions of those that had already been defined for C*-algebras. Though it was not presented in these lecture notes, there are many open questions in operator algebras which may be viewed as an operator system theoretic question. The biggest question that we must answer is whether every operator system that is $(\min, \text{er})$-nuclear is necessarily $(el, c)$-nuclear. In other words, we are asking whether every operator system with the local lifting property has the double commutant expectation property. The reader who has followed up to this point should immediately recognize the equivalent formulations of this problem. If we were restricting ourselves to C*-algebras, we are then asking whether the local lifting property implies Lance’s weak expectation property, i.e., does LLP imply WEP?

For completeness we conclude the theorem showing this equivalence;

**Theorem 3.9** (Kavruk et al.). Then the following are equivalent.

1. $C^*(F_\infty)$ has Lance’s WEP.
2. OSLLP implies DCEP

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3. Every \((\text{min, er})\)-nuclear operator system is weak\((\text{min, c})\)-nuclear, i.e., if \(S \in \mathcal{O}\) is \((\text{min, er})\)-nuclear, then \(S \otimes_{\text{min}} S = S \otimes_{\text{c}} S\).

**Proof.** Suppose that (2) holds and \(S\) is an operator system that is \((\text{min, er})\)-nuclear. It then follows

\[
S \otimes_{\text{min}} S = S \otimes_{\text{er}} S \subset_{\text{coi}} S \otimes_{\text{er}} \mathcal{I}(S) = \mathcal{I}(S) \otimes_{\text{el}} S \supset_{\text{coi}} S \otimes_{\text{el}} S = S \otimes_{\text{c}} S,
\]

giving us weak\((\text{min, c})\)-nuclearity. Thus, (2) \(\implies\) (3).

Recall that Kirchberg’s theorem states that \(C^{*}(F_{\infty})\) is weak\(_{\mathcal{B}(H)}\)(min, max)-nuclear. In our last survey we presented the proof that being weak\(_{\mathcal{B}(H)}\)(min, max)-nuclear is indeed to equivalent to \((\text{min, er})\)-nuclearity. Thus, if \((\text{min, er})\)-nuclearity implies weak\((\text{min, c})\)-nuclearity, then since \(C^{*}(F_{\infty})\) is \((\text{min, er})\)-nuclear, then

\[
C^{*}(F_{\infty}) \otimes_{\text{min}} C^{*}(F_{\infty}) = C^{*}(F_{\infty}) \otimes_{\text{c}} C^{*}(F_{\infty}) = C^{*}(F_{\infty}) \otimes_{\text{max}} C^{*}(F_{\infty}),
\]

where we have used the fact that \(\otimes_{\text{max}} = \otimes_{\text{c}}\) when one of the involved objects is a \(C^{*}\)-algebra. Therefore we have that \(C^{*}(F_{\infty})\) has Lance’s WEP.

Finally we only need show that (1) implies (2). Suppose that \(C^{*}(F_{\infty})\) has Lance’s WEP, and let \(S\) be a \((\text{min, er})\)-nuclear operator system. Let \(S \subset_{\text{coi}} \mathcal{B}(\mathcal{H})\) be the Choi-Effros characterization of \(S\). It then follows

\[
S \otimes_{\text{max}} C^{*}(F_{\infty}) = C^{*}(F_{\infty}) \otimes_{\text{c}} S = C^{*}(F_{\infty}) \otimes_{\text{el}} S = S \otimes_{\text{er}} C^{*}(F_{\infty})
\]

\[
= S \otimes_{\text{min}} C^{*}(F_{\infty}) \subset_{\text{coi}} \mathcal{B}(\mathcal{H}) \otimes_{\text{min}} C^{*}(F_{\infty}) = \mathcal{B}(\mathcal{H}) \otimes_{\text{max}} C^{*}(F_{\infty}).
\]

Note that above we used symmetry of \(\otimes_{\text{max}}\), equivalence of Lance’s WEP and \((\text{el, max})\)-nuclearity, injectivity of \(\otimes_{\text{min}}\), and Kirchberg’s theorem.

\[\square\]

**References**


