CONCEPTUALLY BASED TASK DESIGN: MEGAN'S PROGRESS TO THE ANTICIPATORY STAGE OF MULTIPLICATIVE DOUBLE COUNTING (mDC)

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This study examined the application of a conceptual framework for learning new conceptions to the design and use of tasks/prompts that can lead students to construct multiplicative double counting (mDC) – a scheme underlying the development of multiplicative reasoning. Within the context of a teaching experiment with fourteen 4th-5th graders, we analyze the teacher-researcher's work with one student, Megan, as she progressed from having no such conception to the participatory and then anticipatory stage of mDC. Our analysis demonstrates how tasks can (a) draw on available conceptions and (b) be designed to engender the intended learning via orientation of reflective processes.

Introduction

How might tasks that promote conceptual understanding of multiplicative operations be designed and implemented based on students' available conceptions? In this study we addressed this critical pedagogical problem, which is consistent with the growing interest of mathematics educators in the role tasks play in students' learning of mathematics (Watson & Mason, 1998, 2007; Watson & Sullivan, 2008; Zaslavsky, 2007). In particular, we examined the application of Tzur and Simon's (2004) stage distinction (see below) to the process of instructional task design. This application contributes to the recent focus on task use, because it is rooted in a framework that explicitly links learning of new (to the learner) conceptions with interventions that can promote such learning. Thus, we applied the stage distinction, reflexively, to both the analysis of students' available conceptions and the tasks used for transforming these conceptions into intended, more advanced mathematical ideas.

We chose the difficult-to-grasp domain of multiplicative reasoning because of the central role it plays in empowering students' mathematics (e.g., algebra preparedness, see Confrey & Harel, 1994). We believe that inadequate conceptualization in this domain is one key cause for the ever-growing gaps among students during the upper elementary, middle, and early high school years. Consequently, this study focused on the commencement of multiplicative double counting (mDC, see details below), a milestone mental operation that constitutes a child's transition from a unitary counting stage to a binary counting stage (Vergnaud, 1994). Our central thesis is that the stage distinction, and the reflection on activity-effect relationship (Simon, Tzur, Heinz, & Kinzel, 2004) framework in which it is rooted, provide useful tools for creating and adjusting tasks/prompts conducive for nurturing mDC at a level necessary for students to independently carry out cross-context problem solving processes proper to a situation at hand.

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Conceptual Framework

In this section, we first briefly describe the general and content-specific constructs that guided this study, and then delineate how they were used to design a set of tasks/prompts for teaching mDC. The general constructs constitute the reflection on activity-effect relationship (*Ref*AER*) framework (Simon et al., 2004), itself an elaboration of Piaget's (1985) and von Glasersfeld's (1995) scheme-based theories. *Ref*AER* is the postulated mechanism by which the human mind forms novel conceptions. It commences with the learner's assimilation of a problem situation into her available conceptions, which set her goal and trigger the activities (usually an activity sequence) that the mind and body carry out to accomplish the goal. The learner's goal then regulates, from within the mental system, the progress of her activity sequence and her noticing of effects that this activity brings forth. Through two types of reflection, in the form of brain-based comparisons, the learner first relates the newly noticed effects with the activity and later with the situation in which she should anticipate such an activity-effect relationship (*AER*). *Type-I reflection* consists of comparison between the learner's goal and the actual effect of her activity sequence; *Type-II reflection* consists of comparison across records of experience in which the learner invariantly uses *AER* compounds for solving what then become similar problem situations for the learner. A novel anticipation of *AER* is formed via two stages (Tzur & Simon, 2004). In the first, *participatory* stage, the learner forms a provisional anticipation of *AER* that she cannot access directly from her available schemes. Rather, this anticipation can only be retrieved if the learner is somehow *prompted* for the activity, which generates the effect and hence the *AER* compound. In the second, *anticipatory* stage, the learner forms a robust *AER* that she can independently and spontaneously call up, use, and transfer to new situations. The anticipation encapsulated in the *AER* of both stages is the same; they differ in the learner's access to that anticipation.

The content-specific constructs are rooted in the work of Steffe et al. on children's construction of number schemes, particularly of numerical composite units (CU) through mental activities of iterating the unit of one (1's, see Steffe & Cobb, 1988; Steffe & von Glasersfeld, 1985). Steffe (1994; Steffe & Cobb, 1998) proposed that a child who has constructed CU can operate on such units not only additively (e.g., counting-on to solve a missing-addend problem), but also multiplicatively, via simultaneously applying her counting scheme to CU and to the 1's that constitute the CU (e.g., a child may find 3x4 by counting from 1 to 12 in 'triplets' as in '1-2-3, 4-5-6, 7-8-9, 10-11-12', while keeping track of those triplets on the other hand's fingers, '1, 2, 3, 4'). Most importantly, in using mDC, the child creates a scheme of correspondence, where one CU is distributed across the other. In our example, each CU of 3 is distributed into the composite unit items that make up '4'. Thus, mDC enables a child to quantify, in the absence of objects (i.e., in anticipation), the total number of 1's that are embedded in a given number of same-size CU without having to count each and every singleton. It is important to clarify here that mDC refers to the mental quantification of the units—not to the manner in which it is executed (e.g., using fingers, or making tally marks to monitor each CU, or mentally counting the CU).

To complement the *Ref*AER* with a pedagogical approach, Tzur (2008) elaborated on Simon's (1995) and Simon & Tzur's (2004) teaching approaches by proposing a 7-step cycle. It proceeds from specifying students' available conceptions and the intended mathematical ideas, through identifying an activity sequence they can carry out, designing and implementing tasks that may engage them in such a sequence, to monitoring students' progress and orienting their reflection via intentional introduction of follow-up tasks/prompts. In this study, the tasks were designed to trigger the learner's setting of a global goal of finding the total number of 1's (Unifix cubes) embedded within a given number of CU ('towers' of cubes). Once such situations are recognized, the teacher can hide the cubes to encourage the learner's creation of a fundamental sub-goal – namely, to keep track of how of the number of CU – and introduce the activity of mDC as a means to accomplish that sub-goal. Numbers for tasks were chosen to require more than two hands, hence a transformation in the child's available activity of counting all 1's as a single sequence of numbers (e.g., 1-2-3, 4-5-6, etc.). A child's inability to keep track could be resolved by introducing another set of items on which to keep track of CU accrual, and orient her attention to the stopping point of mDC when she accounted for all of the CU.

Methodology

This study was part of a teaching experiment (Steffe, Thompson, & von Glasersfeld, 2000) with three 4th graders and eleven 5th graders, designed to develop multiplicative reasoning in elementary school students with (or at risk of) learning disabilities in mathematics[i]. Three teaching episodes with one student, Megan (a student at risk), were conducted over the course of 4 weeks by the sixth author. Megan was selected for this study because, prior to the work presented in this paper, she had constructed the anticipatory stage of using composite units numerically (e.g., for missing addend tasks).

The teaching episodes consisted of students playing a game called "Please Go Bring Me..." (PGBM). It involves one player sending another to a box containing individual Unifix cubes and instructing her to create a tower m cubes high. The 'bringer' returns the tower to the table and the process repeats until she brings N towers of m (henceforth notated NT_m). Three principal questions are then asked: (a) How many towers did you bring? (b) How many cubes are in each tower? And (c) How many cubes do you have in all? These questions prompt the child to identify, respectively, the number of composite units (CU), the unit rate (UR), and the total number of cubes (1's). A version of the game that we used frequently utilized "What if?" tasks, which require figuring out her answers in the absence of the cubes (e.g., by asking her to pretend she brought them, or by covering the towers).

Data from the episodes consist of field notes, videotapes, transcripts, and notes from ongoing analysis sessions. The research team initially analyzed episodes soon after conducting them, focusing on significant events and on necessary modifications to the plan for the next teaching session(s). A second round of analysis highlighted critical events in the transcripts of the sessions, where the team inferred Megan's thinking processes at the participatory or anticipatory stages via attending to her language and actions. Final, retrospective analysis involved a team discussion of the highlighted segments, which were integrated into a story line of her growth in multiplicative reasoning. The episodes included in the analysis begin after Megan had become familiar and comfortable with the basic form of PGBM (with cubes).

Analysis

In session 1, the teacher asked Megan: "Pretend I send you to get towers of 4… and I asked you to bring 7 towers of 4. Can you figure it out, using any other way except bringing those and counting the cubes, how many cubes you would have now?" Megan simply could not answer the question. Even after the teacher offered paper and pencil, and later said she could use her fingers, she threw up her hands and exclaimed, "I can't do it." Her inability to attempt this initial task of $7T_4$ indicated that at this point she had no access to mDC. Below, we present excerpts of critical events from three consecutive teaching sessions and suggest how they promoted Megan's transition to an anticipatory stage of mDC.

Promoting Construction of the Participatory Stage

In order to move Megan to the participatory level of mDC, the teacher suggested that Megan use her fingers to keep track of the number of towers while counting the number of cubes:

Excerpt 1 (Session 1, Introduction to Double Counting)

T: Let me suggest the following. I'll give you my fingers for every tower. Okay? Every time we have a tower that's (holds up one finger) one tower of four– how many do we now have? You can use your fingers for [counting] the four [cubes].

M: So 4.

T: What if I brought another tower (raises a 2nd finger)?

M: 8.

T: What if I brought another tower (raises a 3rd finger)?

M: 12? (Doesn't look confident.) No wait.

T: You can use your fingers to figure it out.

M: (Counts under her breath.) 16? No!

T: So we had 4 – now let's use your fingers (shows counting on from 4 on his other hand) 5-6-7-8. And then, 9-10-11-12.

M: Yeah.

T: So now we have 3 [towers], what if we added another one (raises a 4th finger)?

M: 16.

T: Ok another one.

M: (Counts on her fingers under the table) 17-18-19-20.

T: So with 5 we have 20. We still have to go [bring towers] two more times.

M: (Counts on her own fingers) 21-22-23-24; 28.

The exchange in Excerpt 1 enabled Megan to start developing an intentional method for keeping track of the number of towers and to anticipate when to stop counting cubes. The teacher's continual prompts of, "What if I brought another tower?" oriented Megan's Type-I reflections between the accruing effects of the double counting activity and the global goal of finding the total number of cubes. This was possible because she could assimilate indicating a CU by the teacher's finger into her available numerical composite unit scheme. Consequently, Excerpt 1 provides a window to two important facets of the work: Megan's early shift to the participatory stage of mDC and how the teacher used tasks/prompts to promote this shift. Following the prompts, Megan knew to operate on the proper unit (cubes) with her number sequence, including anticipating that 4 cubes comprised a single CU (tower). Thus, the task, which required her to retrieve only one tower at a time and to count after each new acquisition, seemed to promote Megan's *coordination* of the two number sequences as evidenced by her finishing of the last two towers without needing to actually go get them.

Megan's construction of the participatory stage for mDC became evident in the task that followed (Excerpt 2).

Excerpt 2 (Session 1, Participatory Double Counting)

T: So you have 6 towers of 3 over there, and you use your fingers or your brain or (jokingly) your hair, or your blinking, or whatever, figure out how many are there all together… Can you put your fingers [above the desk] so I can see what you did?

M: (attempts to skip-count, first by 3's , then by 6's) 3-6-12-19? No, wait. (double counts, nodding her head as though counting, e.g., 4, 5, 6, but only speaking the total after each tower out loud) 3-6-9-12-15-18?

T: So 18? That's very good … I saw you putting 3, almost immediately; then 6 almost immediately. Then you started and recounted… So, could it be you said in your head, 7-8-9, 10-11-12, 13-14-15, 16-17-18 (puts a finger for each triplet)?

M: (nods yes)

Excerpt 2 indicates that the work on the previous task and the availability of counting triplets enabled Megan to solve $6T_3$ while internalizing the differentiation of 1's (cubes) from CUs (towers), distributing her units across each of the re-presented units, and coordinating the addition of the cubes and the number sequence of the CUs (tower). Two interventions were key to the formation of this coordination. The teacher began the task by imposing a constraint on Megan: she was not allowed to use paper and pencil to draw the $6T_3$. This oriented her to move away from simply counting individual (drawn) cubes. The teacher's prompt, "Can you put your fingers so I can see what you did?" provoked a Type-I reflection as Megan revisited the use of mDC *for* finding the total, as evidenced in her immediate self-correction after the first attempt ("3, …, 19") toward using mDC *intentionally* in distributing the unit rate (3 cubes/tower) over the number of CU (towers).

Megan's work on those two tasks indicated that she was in the participatory stage of mDC. We did not expect she could yet spontaneously call up the activity sequence, but we did expect she would use mDC when prompted. To test our hypothesis, we began the next episode by testing if Megan was at the anticipatory stage of mDC by engaging her in a prompt-less situation.

Excerpt 3 (Session 2, Test Anticipatory/Participatory Double Counting)

T: Pretend you were going (to retrieve a tower of Unifix cubes), and I sent you to get a tower of 3; another tower of 3, and another (etc.). And you brought, think of $7T_3$. Can you figure out how many cubes are there?

M: (Thinks – uses her fingers to count 1-2-3, 4-5-6, (inaudible speech), gets up to the 6th finger and gets lost.) Ok. I just forgot.

T: Ok. Just take your time. If you need my fingers, you can use them.

M: 3, 6... (Starts using teacher's fingers, but then goes back to her own.) 3, 6... 21. I think.

T: How did you get 21?

M: I added three 7 times.

T: Did you do this? (Demonstrates double counting with one hand monitoring) You raised one finger and said 1-2-3. Then you raised another finger and said 4-5-6 [and so on]. Is that what you did?

M: Yeah.

T: Ok. Let's see if your answer is true.

M: (Builds 7 towers of 3 and counts cubes.) 3, 6, 9, 12; 13-14-15; 16-17-18; 19-20-21.

Excerpt 3 indicates that Megan was yet to construct the anticipatory stage of mDC, as she became lost prior to the teacher's prompt for using *his* fingers. Once prompted, however, she could immediately regenerate an anticipation of the *AER* for mDC. She momentarily used the teacher's fingers, but then internalized the activity as evidenced in her shift to her own fingers for successfully completing mDC to reach her global goal. It confirmed our hypothesis, and led to interventions for promoting transition to the anticipatory stage via tasks with larger numbers.

Promoting Construction of the Anticipatory Stage

Excerpt 4 (Session 4, In transition to anticipatory stage of Double Counting)

T: Pretend you have a tower of 6, another tower of 6, and another [etc.]. Seven towers of 6. [How many cubes] would you get?

M: Um, 6, 12, 24 (gets lost on her fingers) I don't know. That's hard.

T: You can use my fingers.

M: I can't. That's hard.

T: That's harder, because I gave you larger numbers. [See if you can use] my fingers for the number of towers and use yours to count how many in each. So you said the first one is going to be 6 (puts out one of his fingers), then you said 12. (Puts down another finger) Then you started struggling. Use your fingers to add from 12, 6 more.

M: (Counts-on with her fingers) 12; 13-14-15-16-17-18.

T: Ok that's another tower (puts out another finger). That's 3 towers.

M: (Counting-on with her fingers.) 19-20-21-22-23-24 (Pauses for teacher to put a finger); 25-26-27-28-29-30 (Pauses for teacher); 31-32-33-34-35-36; 37-38-39-40-41-42.

T: Should we stop now, or go on? I said 7 towers.

M: Yeah, that's it.

When finding the number of cubes in $7T_6$, Megan struggled with the size of the numbers because each CU was larger than 5 (her fingers). This brought about her Type-2 reflection, evidenced in her realization ("I can't. That's hard") that using mDC would be difficult for the current situation because, *unlike previous situations*, she would not be able to simultaneously hold the number of towers and count the number of cubes. This Type-2 reflection, however, enabled her to easily assimilate the teacher's suggestion to use his fingers to keep track of CU and she immediately completed the task. It was her spontaneous contribution, evidenced in the intentional pause until the teacher raised his next finger, which led us to conjecture Megan might solve a similar task in the next week's episode at an anticipatory level.

To test our conjecture, we introduced a problem situation at the beginning of the following week's episode that required Megan to use mDC in a different context, asking her to create a PGBM situation (cubes, towers) that would be equivalent to having 7 baskets with 8 chicks in each. Megan immediately built a tower with seven same-color cubes and one different color cube, counting under her breath, "1-2-3-4-5-6-7-8." She then continued building more T_8 of the same color pattern until she had $7T_8$, at which point the teacher asked if she could figure out the total number of cubes. Megan spontaneously asked, "Can I use your hands?" and proceeded to count individual cubes on her fingers while counting towers on his, successfully stopping at 56. That is, Megan no longer needed a prompt. Rather, she clearly anticipated and spontaneously carried out the entire mDC activity sequence. The way she built her towers to present chicks and baskets, and her initiative for using the teacher's fingers, indicated that she intentionally (a) distinguished between the CU (towers) and UR (cubes/tower) and (b) used mDC to determine the total. Megan assimilated the chicks and baskets task into her global goal of finding total cubes and independently called up the activity sequence needed for multiplicative coordination of CU: differentiate 1's (cubes) from CUs (towers), distribute her units of 8 across each of the re-presented seven CU, coordinate the addition of the cubes and the number sequence of the CUs (tower), and employ two sets of objects (fingers) to keep track of both counts.

Discussion

This study demonstrated a fundamental transition to multiplicative thinking. At the beginning of our analysis, we saw that Megan, a student at risk in mathematics, had not constructed mDC, putting her at a disadvantage with her peers (2-3 years behind). Through a *Ref*AER* designed intervention, Megan learned to spontaneously call up mDC for reaching her goal in various multiplicative situations. Megan's intentional translation of and solution to the mDC task in the last episode, including her request to use another set of fingers, indicated the commencement of her anticipatory stage of a units-coordinating scheme to which Steffe (1994) refers as an implicit concept of multiplication. Most importantly, this study demonstrated how Tzur and Simon's (2004) stage distinction for determining a learner's available conceptions could guide selection of tasks and prompts for transforming these conceptions. Such guidance included the introduction of double counting on one's and another person's fingers upon shifting to 'for pretend' tasks as a means to the sub-goal of simultaneously keeping track of clearly differentiated CU and 1's, progressing from small to large numbers, and continually orienting the learner's reflection onto the critical questions of when to stop counting (e.g., when Megan paused for the teacher's next finger).

Endnotes

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