CHILDREN'S DEVELOPMENT OF MULTIPLICATIVE REASONING: A SCHEMES AND TASKS FRAMEWORK

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We present a synthesis of findings from constructivist teaching experiments—a developmental framework of six schemes that children construct for reasoning multiplicatively and tasks to promote them. The framework is rooted in distinctions of units children seem to use and operations with/on these units—particularly number as an abstract, symbolized composite unit. We provide a task-generating platform game, depictions of each scheme, and tasks supportive of constructing it. We discuss the need to distinguish between tasks and child's cognitive conceptions, and to organize learning situations that (a) begin at and build on the child's available scheme, (b) geared to the next scheme in the sequence, and (c) link to intended math concepts.

INTRODUCTION

In this paper we propose a developmental framework that makes distinctions and links among schemes—conceptual structures and operations children construct to reason in multiplicative situations. We provide a set of tasks (problem situations) to promote construction of such schemes. Elaborating on Steffe et al.'s (Steffe & Cobb, 1998) seminal work, this framework synthesizes findings of our teaching experiments[i] with over 20 children who have disabilities or difficulties in mathematics. This empirically grounded framework contributes to articulating and promoting multiplicative reasoning—a key developmental understanding (Simon, 2006) that presents a formidable conceptual leap from additive reasoning for students and teachers (Harel & Confrey, 1994; Simon & Blume, 1994). In place of pedagogies that focus primarily on multiplication procedures, our framework can inform teaching for and studying of children's conceptual understandings. Such understandings provide a basis not only for promoting multiplication and division concepts and procedures but also for reasoning in place-value number systems, and in fractional, proportional, and algebraic situations (Thompson & Saldnha, 2003; Xin, 2008).

We contrast our stance on children's cognitive change and teaching that promotes it with the *Cognitively Guided Instruction* (CGI) approach (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998)). CGI grew out of research on children's solutions to addition and subtraction tasks. By asserting that "children's solution processes directly modeled the action or relationships described in the problem" (Carpenter, Hiebert, & Moser, 1983, p. 55), CGI researchers seemed to equate children's cognitive processes with tasks. In contrast, we argue for explicitly distinguishing between task features as adults conceive of them and schemes children bring forth for solving tasks. Consider a *Join* task such as, "We had 7 toys and got 4 more; how many toys we then had in all?"

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A child may solve such a task by *counting-all* 1s (**1**-2-3-…10-11), by *counting-on* (**7**; 8-9-10-11), or by using a *through-ten strategy* (7+3=10; 10+1=11). The latter two indicate the child understands number as a composite unit, hence preparedness for multiplicative reasoning, whereas the first does not. We concur with CGI's premise of the need to use children's ways of thinking in teaching. However, we disagree that the structure of a task as seen by an adult determines, in and of itself, the way a child makes sense of and acts to solve it. The next section presents the conceptual framework that underlies our synthesis.

CONCEPTUAL FRAMEWORK

Our framework builds on the core notion of *scheme*—a psychological construct for inferring into the mental realms of thinking and learning. von Glasersfeld (1995) depicted scheme as a tripartite mental structure: a *situation* (recognition template) that sets one's goal, an *activity* triggered to accomplish that goal, and a *result* expected to follow the activity. Tzur et al. (Tzur & Lambert, 2011; Tzur & Simon, 2004) further distinguished *effect* from 'goal' and 'result', asserting that effect can more precisely pertain to anticipated and actually noticed outcomes of a mental activity on/with certain 'objects'. As a person's mind 'runs' activities and regulates them by the goal, novel effects can be noticed, differentiated from anticipated ones, and related to the activity. An *activity-effect relationship* (*AER*) is conceived of as a sub-component of a scheme (2^{nd} and 3^{rd} parts). Existing or noticed *AERs* can be linked to a given scheme's situation, transferred to, and linked with other situations.

A mathematical *task* pertains to a pedagogical tool used to promote student learning, that is, advancing from current to intended schemes. Typically, a task consists of depictions of relationships among quantities, some given and some unknown, including a question for figuring out the latter. In recent years, tasks became a primary tool *through* which to foster mathematics learning, as opposed to a way of applying taught concepts *after* learning took place (NCTM, 2000; Watson & Mason, 1998). To solve a task, a child has to (a) assimilate it into an existing scheme's 'situation', (b) identify the quantities (mental objects) involved, (c) set a goal compatible with the question, and (d) initiate mental activities on those quantities that (in the child's mind) correspond to the depicted relationships.

A key construct for distinguishing multiplicative from additive reasoning is number as a *composite unit* (CU) (Steffe, 1992). To reason *additively* requires students to operate with number as a CU. Children establish this in situations that trigger a goal of determining the amount of 1s in a collection of items and the activity of counting, which involves iterating the unit of one to *compose* larger units (e.g, 1+1+1=3). Gradually, the nested nature of the resulting, composed quantity becomes explicit (e.g., [1+1+1]+1=4; +1=5; etc.). When number is conceived of as a CU, children can anticipate *decomposing* units into nested sub-units. For example, a child can think of '11-7=?' as '7+?=11', that is, a CU of 11 ('whole') of which she knows one part (7) and can find the other. Key to additive reasoning is that the referent unit is *preserved* (Schwartz, 1991): 11 *apples* - 7 *apples* = 4 *apples*.

Learning to reason multiplicatively requires a major conceptual shift—a coordination of operations *on* CUs (Behr, Harel, Post, & Lesh, 1994). Consider placing 2 apples into each of 3 baskets; 2 is one CU (*apples per basket*) and 3 is another (*baskets*). Multiplicative reasoning entails distributing one unit over items of another (2 apples per basket) and finding the total (goal) via a *coordinated counting activity*: 1 (basket) is *1-2* (apples), 2 (baskets) are *3-4* (apples), 3 (baskets) are *5-6* (apples). C*oordinated* counting entails *deliberately* keeping track of CUs while accruing the total of 1s based on the distributed CU (2 apples-per-basket). As this example indicates, in multiplicative reasoning the referent unit is *transformed* (Schwartz, 1991), and the product has to be conceptualized as a unit of units of units (Steffe, 1992): here, '6 apples' is a unit composed of 3 units (baskets) of 2 units (apples per basket). The *simultaneous count* of two CUs and the resulting *unit transformation* constitute the conceptual advance from additive reasoning.

A FRAMEWORK OF MULTIPLICATIVE SCHEMES AND TASKS

This section first describes tasks we used to promote students' construction of multiplicative schemes—revolving around the platform game, *Please Go and Bring for Me* (PGBM). Then, a six-scheme developmental framework is presented. This order helps to delineate teaching that can foster construction of the schemes while clearly separating between instructional tasks and children's thinking.

Tasks for Fostering Multiplicative Schemes

PGBM is an example of a task-generating platform game. It fosters multiplicative reasoning by engaging children in tasks conducive to carrying out and reflecting on double-counting activities. The basic form is played in pairs. Each turn partners switch roles—one playing a *sender* and the other a *bringer*. The *sender* begins by asking the *bringer* to produce, one at a time, towers composed of the same number of cubes. Once the *bringer* has produced the needed amount of same-size towers (e.g., 5 towers, 3 cubes each; denoted $5T_3$), the *sender* asks her four questions: (1) How many towers did you bring? (2) How many cubes are in each tower? (3) How many cubes are there in all? (4) How did you figure it out? Questions 1 & 2 orient student reflections on the CUs involved—to distinguish activities of producing/counting a set of CUs from counting 1s to produce each CU. Questions 3 & 4 foster coordinated counting CUs (e.g., raising one finger per tower) while accruing the total of cubes (e.g., 3-6-9-12-15) based on the size of the distributed CU (e.g., 3 cubes per tower).

When students become facile in playing PGBM with tangible objects (cubes and towers), we use two major variations to foster abstraction of coordinated counting. Variation (1) supports students' shift from operating on tangible objects to figural objects in which a substitute item stands for real objects the students attempt to quantify. Variation (2) supports students' shift from operating on figural objects, to abstractly symbolized objects, to mental objects. In (1) partners produce a given set, say $3T_4$, *cover the towers* (Fig. 1a), then answer the 4 questions. Initially, we let children use spontaneous ways of keeping track of CUs and 1s (e.g., count on fingers, tally marks, etc.) Later, we guide them to sketch towers in a gradually more abstract manner. They begin with tower diagrams comprising of single cubes, then sketch tower diagrams with a number indicating the tower's size, then a line-with-number represents tower, and finally just a number (Fig. 1b). Using these diagrams fosters a shift from attending to 1s that constitute a CU to the numerical value resulting from how each CUs was produced. In (2) partners *pretend* as if they were producing towers, but do not actually do so. As in (1), we guide students to sketch increasingly abstract diagrams, beginning with figural objects and progressing to abstractly symbolized 1s and CUs. When a student can anticipate the structure of the 1s and the CUs, this suggests she or he can operate on CUs as mental objects. Like in the Singapore approach (Ng & Lee, 2009), these variations foster students' advancement from acting on CUs as tangible objects, to tangible-but-invisible, to mental objects.

Within Variations (1) and (2) we use different amounts of towers and cubes to support students' productive participation. Initially, children use familiar numbers (2, 5, or 10 cubes per tower) and small sets (up to 6 towers). Then, we guide them to use more difficult numbers (towers of 3-4 cubes, and later of 6, 7, 8, or 9 cubes) and larger sets (up to 12 towers). When students operate on cubes/towers as figural objects, we introduce similar tasks in other contexts (e.g., How many cookies are in 5 bags, if each bag has 3 cookies?). In doing so, we support students' use of coordinated-counting to figure out the total of 1s (e.g., cubes, cookies) across situations constituted by a number of same-size CUs (e.g., towers, bags of cookies).

Building on Xin's (2008) work, we gradually introduce children to a single symbolic structure that ties both multiplication and division. We begin with: *Cubes in each tower x Number of towers = Total of Cubes* (Fig. 1c). As they solve tasks in different contexts, we replace it by: *Items in Each Group x Number of Groups = Total of Items*; and finally by: *Unit Rate x Number of Composite Units = Total of 1s*. This symbolic structure supports students' determination of the needed computation (multiplication or division). In a multiplication situation, the total of 1s is unknown. In a division situation, either the number of CUs or the number of 1s per CU is unknown.

Figure 1a: Covered Towers

Figure 1b: Tower modelling

______	x	______	=	______
Cubes in Each tower		Number of Towers		Total Cubes

Figure 1c: Equation modelling

A Six-Scheme Developmental Framework

This section describes each of six schemes that, combined, constitute the framework we propose about children's development of multiplicative reasoning. For each, we indicate what the scheme involves, provide a sample task linked to the scheme, explicate goals, activities, and results associated with developing the scheme, and articulate mathematics that the established scheme supports.

The **first scheme** a child may construct is termed *multiplicative Double Counting* (mDC, Woodward, et al., 2009). It involves recognizing a given number of CUs, each consisting of the same number of 1s. Typical tasks include Variations (1) and (2) of the PGBM platform game. The child's goal is to figure out the total of 1s in this 'set', and the activity is simultaneous (double) counting of CUs and 1s that constitute each CU. When established, mDC includes a child's anticipation that a total of items (say, 24 cookies) is a CU constituted of another CU (4 bags), each of which a CU itself (6 cookies). This scheme provides a basis for the strategic use of known facts to derive unknown ones (e.g., "7x5 is like 5 towers of 7, and I know it is 35 (cubes); so 7x6 is as if I brought one more unit of 7, hence it is the same as 35+7=42").

The **second scheme** is termed *Same Unit Coordination* (SUC). It involves operating additively on CUs without losing sight of each CU being both a unit in and of itself and composed of 1s. Typical tasks linked to this scheme involve *two* sets of CUs and a question to figure out sums of or differences between the sets. SUC tasks may ask: "You brought $7T_5$ and then I brought $4T_5$; How many towers do we have in all?" or "You brought $7T_5$; I brought a few more; Together, you and I have $11T_5$; how many towers did I bring?" The child's goal is to figure out the sum or difference of CUs (not of 1s), and the activity may be any of those a child has constructed for operating additively on 1s (counting-all, counting-on, through-ten, fact retrieval, etc.). Like with units composed of 1s, the key in this scheme is the child's conception of the embedded (nesting) of CU sets within a larger CU (e.g., a CU consisting of 11 units of 10 can be decomposed into 7 units of 10 + 4 units of 10). When established, SUC provides a basis for operating on specific CUs such as 10s, 100s, and 1000s in a place-value system (with contexts including distance, weight, money, etc.).

The **third scheme** is termed *Unit Differentiation and Selection* (UDS, McClintock, Tzur, Xin, & Si, 2011). It involves explicitly distinguishing operations on CUs from operations on 1s, and operating multiplicatively on the *difference* of 1s between two sets of CUs. Typical tasks include, "You have $7T_5$ and I have $4T_5$; how are our collections similar? Different? How many more cubes do you have?" (Note: Sets may differ in number of CUs, or in unit rate, or in both.) The child's goal is to specify the similarities and differences, and to figure out the difference in 1s between the two sets. The child's activity can include (a) operating multiplicatively on each set to find its total of 1s and then find the difference (*Total-First* strategy) or (b) finding the difference in CUs and then multiplying it by the unit rate (*Difference-First* strategy). We promote use and coordination of both. When established, UDS includes a situation recognized as two sets of CUs that can be similar or different with respect to quantities that constitute each set. UDS provides a basis for the distributive property of multiplication over addition (e.g., 7x5+4x5=5(7+4)) and for solving algebraic equations such as 7x-4x=15.

The **fourth scheme** is termed *Mixed-Unit Coordination* (MUC, Tzur, Xin, Si, Woodward, & Jin, 2009). After UDS has enabled distinguishing CUs from 1s, MUC involves operating on 1s to answer questions about CUs in two sets. Typical tasks include, "You have $7T_5$; I'll give you *10 more cubes*; if you put these 10 cubes in T_5, how many towers would you have in all?" (Note: The question can be, "How many cubes would you have in all?") The child's goal is to figure out the number of CUs (or of 1s) in a 'global' CU combined of both given quantities. To this end, the child's activity includes selection and coordination of the unit rate (e.g., 5) from the given set with a segmenting operation on the given number of 1s to yield the additional number of CUs (2 towers), and then adding this newly found set of CUs to the initially given set (2+7=9 towers). MUC includes a situation recognized as one set of CUs and another CU composed of 1s. MUC supports the segmenting of a CU of 1s based on a given unit rate, which is a precursor to partitioning a totality as required for division.

The **fifth scheme** is termed *Quotitive Division* (QD). It involves operating on a given CU of 1s (say, 28 cubes) in anticipation of the count of iterations of a sub-CU (T_4). Typical tasks include, "You have 28 cubes; pretend you'll take them back to the box in towers of 4 cubes each. How many towers will you take back?" The child's goal is to figure out how many sub-CUs constitute the given total, and the activity is mDC regulated for stoppage when accruing and given totals are equal. When established, a QD scheme reverses mDC. QD provides a basis for conceiving of division as an inverse operation to multiplication, and thus for using fact "families" of the latter to solve division problems in which the total and the size of each group is given. While playing a game in which children *posed* PGBM tasks, with conditions specified about the fit between the given totality and sub-CUs (e.g., you need to give me a total and a number of cubes in each tower so when I run out of cubes there will still be 2 cubes left), we also fostered a conceptual prerequisite for division with remainders.

The **sixth scheme** is termed *Partitive Division* (PD). Similar to QD, it involves recognizing a situation with a given totality of 1s. However, the other aspect of the situation a child must recognize is that a given number of sub-CUs requires accomplishing the goal of figuring out the equal-size of each. A typical task would be "You want to put 28 cubes in 4 equal towers. How many cubes will you have in each tower?" Initially, children may accomplish the goal by the activity of distributing all given 1s to each group one after another. Given constraints (e.g., "Do you think there would be more than one cube in each tower? Will 3 cubes work? Why?"), children with whom we worked began to anticipate that each round of distribution of 1s would yield a composite unit. They then could double-count to figure out the end result (unit rate) *without* carrying out the distribution—the essence of the PD scheme. PD provides a basis for seeing division as a twofold (QD/PD) inverse of multiplication, and corresponding algebraic operations with equations.

DISCUSSION

The developmental framework of schemes and tasks presented in this paper makes two main contributions. For research and theory building, it demonstrates how the stance that "*the task is not the child's thinking*" can be applied to children's learning of a foundational way of reasoning. Thus, studying transformations in schemes can be done via design and use of task sequences that occasion, but do not determine, children's spontaneous and/or prompted thought processes. Reflexively, task design can be guided by conceptual analysis of scheme components to increase the likelihood of promoting, and hence detecting, particular scheme transformations.

For teaching and teacher education, our framework provides general and content specific guidelines for promoting multiplicative reasoning in students and teachers. A key principle indicated by the framework is the need to analyze students' existing schemes. Such analysis supports using tasks that deliberately reactivate those schemes as a means to foster construction of more advanced schemes, while keeping in mind the gradual nature of such advances. For example, two 4th graders with whom we worked solved the task, "*Pretend* you have $9T_3$; together you and I have $14T_3$; how many T_3 do I have?" by counting-up on their fingers ("9; 10-11-12-13-14; so that's $5T_3$"). But when asked a structurally similar task (*adult's perspective*!) with $19T_3$ and $24T_3$, they had no idea how to proceed. One of them could later solve it after drawing the first set of CUs, whereas the other could only do so after producing all towers (tangible objects). Our framework provides a basis for designing tasks, and variations, that address such gradations and individual differences.

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ARGUMENTATION IN UNDERGRADUATE MATH COURSES :A STUDY ON PROOF GENERATION

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The purpose of this study is to analyze the complex argumentative structure in undergraduate mathematics classroom conversations by taking into consideration students' and teacher' utterances in the classroom using field-independent Toulmin's theory of argumentation . The analyses contributed to an emerging body of research on classroom conversations.

INTRODUCTION

Proof is central to university mathematics courses and widely agreed to be central to the activity of mathematicians. It is, however, a notoriously difficult concept for even undergraduate students to learn (Alcock & Simpson, 2004, 2005; Epp, 1998; Jones, 2000; Larsen & Zandieh, 2008; Leron, 1985; Mejia-Ramos & Inglis, 2009; Moore, 1994; Portnoy et al., 2006; Smith, 2006; Segal, 2000; Uhlig, 2002; Weber 2001, 2004). In the early years of research at the university level, proof generation is discussed within a framework of formal logic. Although formal mathematics builds on formal logic, formal logic does not seem adequate to analyse proof generation especially in classroom for two main reasons. First, students are in the process of developing logical thinking patterns, and so the thinking they express in classrooms includes many elements which a logical analysis would simply describe as ''illogical'' but which are nevertheless important to the future development of their thinking (Knipping, 2008). Second, no formal logic captures all of the nuance of natural language because formal logic is the study of inference with purely formal content. Formal logic is inadequate to capture some aspects of students' arguments in proof generation. As a result of these reasons, in recent years many researchers studying proof generation have conducted their studies using field-independent Toulmin model (1958) which has made great contributions to informal logic. As Toulmin's model is intended to be applicable to arguments in any field, it has provided researchers in mathematics education with a useful tool for research, including formal and informal arguments in classrooms (Knipping, 2008). Studies using Toulmin model focused on analyzing students' arguments and argumentations in proving processes in a classroom (Knipping, 2002, 2008; Krummheuer, 1995) and, individual students' arguments in proving processes (Pedemonte, 2007). Toulmin himself noted that his ideas has no finality. Indeed his model has been reshaped in various ways, his claims have been contested by some and in response reformulated by others, and some but not all aspects of his approach have been incorporated in applications in different domains (Hitchcock & Verheij, 2006).

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