MA 571. Problems for the first midterm.

The midterm will consist of three problems chosen from this list. (All of these problems are from topology qualifying exams.)

On the exam, it's not enough just to have the right idea or to say something that's approximately correct—you must give a solution which is mathematically correct and is explained in a clear and logical way to get full points.

Unless otherwise stated, you may use anything in Munkres's book in your solution—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

1. Let X_α be a family of topological spaces.

   For each α, let A_α be a subset of X_α.

   Prove that
   $$\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}.$$

2. Let X_α be a family of topological spaces.

   For each α, suppose that we are given a point b_α of X_α.

   Let $Y = \prod_\alpha X_\alpha$, with the product topology. Let $\pi_\alpha : Y \to X_\alpha$ be the projection.

   Prove that the set
   $$S = \{y \in Y \mid \pi_\alpha y = b_\alpha \text{ except for finitely many } \alpha\}$$
   is dense in Y (that is, its closure is Y).

3. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's.

   For each $n \geq 1$, let $y_n \in Y$ be the element $(1, 1, \ldots, 1, 0, 0, \ldots)$, with n 1's at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1's. **Prove** that the set $\{y_n\}_{n\geq 1} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

4. **Prove** that the countable product
   $$\prod_{n=1}^{\infty} \mathbb{R}$$
   (with the product topology) has the following property: there is a countable family $\mathcal{F}$ of neighborhoods of the point
   $$\mathbf{0} = (0, 0, \ldots)$$
   such that for every neighborhood V of $\mathbf{0}$ there is a $U \in \mathcal{F}$ with $U \subset V$.

   Note: the book proves that $\prod_{n=1}^{\infty} \mathbb{R}$ is a metric space, but you may not use this in your proof. Use the definition of the product topology.

5. Let X and Y be topological spaces and let $f : X \to Y$ be a function with the property that
$$f(\overline{A}) \subset \overline{f(A)}$$
for all subsets A of X.

   Prove that f is continuous.

6. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. **Prove** that
$$f(\overline{A}) \subset \overline{f(A)}$$
for all subsets A of X.

7. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let G_f (called the *graph* of f) be the subspace $\{ (x, f(x)) \mid x \in X \}$ of $X \times Y$. **Prove** that if Y is Hausdorff then G_f is closed.

8. Let X and Y be connected. **Prove** that $X \times Y$ is connected.

9. Let X be a topological space.

   Let $A \subset X$ be connected.

   Prove $\bar{A}$ is connected.

10. Let X be a topological space and let $f, g : X \to [0, 1]$ be continuous functions.

    Suppose that X is connected and f is onto.

    Prove that there must be a point $x \in X$ with $f(x) = g(x)$.

11. For any space X, let us say that two points are "inseparable" if there is no separation $X = U \cup V$ into disjoint open sets such that $x \in U$ and $y \in V$.

    Write $x \sim y$ if x and y are inseparable. Then $\sim$ is an equivalence relation (you don't have to prove this).

    Now suppose that every point of X has a connected neighborhood.

    Prove that each equivalence class of the relation $\sim$ is connected.

12. Let X be a connected space. Let $\mathfrak{U}$ be an open covering of X and let U be a nonempty set in $\mathfrak{U}$. Say that a set V in $\mathfrak{U}$ is *reachable from* U if there is a sequence
$$U = U_1, U_2, \ldots, U_n = V$$
of sets in $\mathfrak{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for each i from 1 to $n - 1$.

    Prove that every nonempty V in $\mathfrak{U}$ is reachable from U.