MA 571. Problems for the first midterm.

The midterm will consist of three problems chosen from the following three lists.

- Definitions and examples
- Assigned excercises and theorems from the book
- Qualifying exam problems

On the exam, it's not enough just to have the right idea or to say something that's approximately correct—you must give a solution which is mathematically correct and is explained in a clear and logical way to get full points.

Unless otherwise stated, you may use anything in Munkres's book in your solution—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

Definitions and examples

1. Give the definition of the subspace topology.
2. Give the definition of the product topology.
3. How does the product topology behave w.r.t. the subspact topology and the order topology?
4. Give the definition of a quotient space. Give an example of a quotient space of a Hausdorff space that is T_1 but not Hausdorff.
5. Give the definition of (local) (path) connectedness/compactness.
6. Give an example of a space that is connected, but not locally connected.
7. Give an example of a space that is connected, but not path connected.
8. Give an alternative definition of local path connectedness.
9. Give the definition of a quotient map. Give a quotient map that is open, but not closed, and one that is neither open nor closed.
10. Give the defintition of compact, limit point compact and sequentially compact. What are the relations?

Theorems and excercises

1. Prove that a function to a product space is continuous if and only if its components are.
2. Prove that a subspace is closed if and only if it contains all its limit points.
3. Prove that the projection maps for a product are open maps.
4. Prove that $Bd(A) = \emptyset$ if and only if A is open and closed.
5. Prove that a path–connected space is connected.
6. Prove that a closed subset of a compact space is compact.
7. Prove that a compact subset of a Hausdorff space is closed. (give a counterexample in the non–Hausdorff space.
8. Prove that $\mathbb{R}^\omega$ is not connected in the box topology.
9. Show that the diagonal map is not continuous in the box topology, but it is in the product topology.
10. Prove the sequence lemma.
11. Give an example of a surjecive map of spaces that is not a quotient map.
12. Prove that the image of a connected/compact set is connected/compact.

Qualifying exam problems

1. Let X_α be a family of topological spaces.

   For each α, let A_α be a subset of X_α.

   Prove that
   $$\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}.$$

2. Let X_α be a family of topological spaces.

   For each α, suppose that we are given a point b_α of X_α.

   Let $Y = \prod_\alpha X_\alpha$, with the product topology. Let $\pi_\alpha : Y \to X_\alpha$ be the projection.

   Prove that the set
   $$S = \{y \in Y \mid \pi_\alpha y = b_\alpha \text{ except for finitely many } \alpha\}$$
   is dense in Y (that is, its closure is Y).

3. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's.

   For each $n \geq 1$, let $y_n \in Y$ be the element $(1, 1, \ldots, 1, 0, 0, \ldots)$, with n 1's at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1's. **Prove** that the set $\{y_n\}_{n \geq 1} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

4. **Prove** that the countable product

$$\prod_{n=1}^{\infty} \mathbb{R}$$

   (with the product topology) has the following property: there is a countable family $\mathcal{F}$ of neighborhoods of the point

$$\mathbf{0} = (0, 0, \ldots)$$

   such that for every neighborhood V of $\mathbf{0}$ there is a $U \in \mathcal{F}$ with $U \subset V$.

   Note: the book proves that $\prod_{n=1}^{\infty} \mathbb{R}$ is a metric space, but you may not use this in your proof. Use the definition of the product topology.

5. Let X and Y be topological spaces and let $f : X \to Y$ be a function with the property that

$$f(\overline{A}) \subset \overline{f(A)}$$

   for all subsets A of X.

   Prove that f is continuous.

6. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. **Prove** that

$$f(\overline{A}) \subset \overline{f(A)}$$

   for all subsets A of X.

7. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let G_f (called the *graph* of f) be the subspace $\{\, (x, f(x)) \mid x \in X \,\}$ of $X \times Y$. **Prove** that if Y is Hausdorff then G_f is closed.

8. Let X and Y be connected. **Prove** that $X \times Y$ is connected.

9. Let X be a topological space.

   Let $A \subset X$ be connected.

   Prove $\bar{A}$ is connected.

10. Let X be a topological space and let $f, g : X \to [0, 1]$ be continuous functions.

    Suppose that X is connected and f is onto.

    Prove that there must be a point $x \in X$ with $f(x) = g(x)$.

11. For any space X, let us say that two points are "inseparable" if there is no separation $X = U \cup V$ into disjoint open sets such that $x \in U$ and $y \in V$.

    Write $x \sim y$ if x and y are inseparable. Then $\sim$ is an equivalence relation (you don't have to prove this).

    Now suppose that every point of X has a connected neighborhood.

    Prove that each equivalence class of the relation $\sim$ is connected.

12. Let X be a connected space. Let $\mathfrak{U}$ be an open covering of X and let U be a nonempty set in $\mathfrak{U}$. Say that a set V in $\mathfrak{U}$ is *reachable from* U if there is a sequence

    $$U = U_1, U_2, \ldots, U_n = V$$

    of sets in $\mathfrak{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for each i from 1 to $n-1$.

    Prove that every nonempty V in $\mathfrak{U}$ is reachable from U.

13. Let X be a Hausdorff space and let A and B be disjoint compact subsets of X. Prove that there are open sets U and V such that U and V are disjoint, $A \subset U$ and $B \subset V$.

14. Show that if Y is compact, then the projection map $X \times Y \to X$ is a closed map.

15. Let X be a compact space and suppose we are given a nested sequence of subsets

    $$C_1 \supset C_2 \supset \cdots$$

    with all C_i closed. Let U be an open set containing $\cap C_i$.

    Prove that there is an i_0 with $C_{i_0} \subset U$.

16. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. **Prove** that X is homeomorphic to a subspace of $\mathbb{R}^n$.

17. Let X be a compact metric space and let $\mathfrak{U}$ be a covering of X by open sets.

    Prove that there is an $\epsilon > 0$ such that, for each set $S \subset X$ with diameter $< \epsilon$, there is a $U \in \mathfrak{U}$ with $S \subset U$. (This fact is known as the "Lebesgue number lemma.")

18. Let S^1 denote the circle

    $$\{x^2 + y^2 = 1\}$$

    in $\mathbb{R}^2$. Define an equivalence relation on S^1 by

    $$(x, y) \sim (x', y') \Leftrightarrow (x, y) = (x', y') \text{ or } (x, y) = (-x', -y')$$

    (you do *not* have to prove that this is an equivalence relation). **Prove** that the quotient space $S^1/\sim$ is homeomorphic to S^1.

    One way to do this is by using complex numbers.

19. Let X be a compact Hausdorff space and let $f : X \to X$ be a continuous function. Suppose f is 1-1. **Prove** that there is a nonempty closed set A with $f(A) = A$.

20. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. **Prove** that the quotient space $\mathbb{R}^2/\!\sim$ is compact but not Hausdorff.

21. Let X be a locally compact Hausdorff space. **Explain** how to construct the one-point compactification of X, and **prove** that the space you construct is really compact (you do not have to prove anything else for this problem).

22. Show that if $\displaystyle\prod_{n=1}^{\infty} X_n$ is locally compact (and each X_n is nonempty), then each X_n is locally compact and X_n is compact for all but finitely many n.