MA 571. Problems for the final.

The midterm will consist of problems chosen from the following three lists.

- Definitions and examples
- Assigned exercises and theorems from the book
- Qualifying exam problems

On the exam, it's not enough just to have the right idea or to say something that's approximately correct—you must give a solution which is mathematically correct and is explained in a clear and logical way to get full points.

Unless otherwise stated, you may use anything in Munkres' book in your solution—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

DEFINITIONS AND EXAMPLES

1. Give the definition of (locally) compact
2. Give the definition of (locally) path connected
3. Give the definition and comparisons between the different versions of compactness.
4. Give the definition of a Hausdorff, regular and normal space.
5. Give the definition of an m–manifold.
6. Give the definition of a coherent topology.
7. Give the definition of a complete metric space.
8. Give the definition of the uniform metric on Y^J
9. Give the definition of the topology of pointwise convergence, the topology of compact convergence and the compact–open topology.
10. Give the definition of the fundamental group and the group law of the fundamental group.
11. Give the definition of homotopic and path homotopic.
12. Give the definition of a simply connected space
13. Give the definition of a covering map.

14. Give an example of a covering.

15. Give an example of a local homeomorphism which is not a covering map.

16. Give the definition of the lifting correspondence.

17. Give the definition of a deformation retract.

18. Give the definition of the group of covering transformations and of a regular cover.

19. Give an example of a space that is not semi–locally connected.

Theorems and exercises

1. Prove that a closed subset of a compact space is compact.

2. Prove that a compact subset of a Hausdorff space is closed. (give a counterexample in the non–Hausdorff space.

3. Prove the sequence lemma.

4. Give the theorem that if X is locally path connected then the components and path components are the same.

5. Give the theorems of comparison of the topologies of function spaces

6. Show that for a compact Hausdorff space the composition map is continuous and give the definition and the theorem involving $\mathcal{C}(X \times Z, Y)$ and $\mathcal{C}(Z, \mathcal{C}(X, Y))$

7. Sketch a proof of how the composition of paths gives a group structure to homotopy classes of loops.

8. State the Brouwer fixed–point theorem.

9. Prove that homotopy equivalent spaces have isomorphic fundamental groups.

10. State the Path Lifting Lemma for covering maps.

11. State the Seifert–van–Kampen Theorem

12. State the Classification Theorem for surfaces.

13. State and prove that the fundamental group of an n–wedge of S^1s is $\mathbb{F}_n$.

Qualifying exam problems

1. Let X be a Hausdorff space and let A and B be disjoint compact subsets of X. Prove that there are open sets U and V such that U and V are disjoint, $A \subset U$ and $B \subset V$.

2. Show that if Y is compact, then the projection map $X \times Y \to X$ is a closed map.

3. Let X be a compact space and suppose we are given a nested sequence of subsets

$$C_1 \supset C_2 \supset \cdots$$

with all C_i closed. Let U be an open set containing $\cap C_i$.

Prove that there is an i_0 with $C_{i_0} \subset U$.

4. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. **Prove** that X is homeomorphic to a subspace of $\mathbb{R}^n$.

5. Let X be a compact metric space and let $\mathcal{U}$ be a covering of X by open sets.

Prove that there is an $\epsilon > 0$ such that, for each set $S \subset X$ with diameter $< \epsilon$, there is a $U \in \mathcal{U}$ with $S \subset U$. (This fact is known as the "Lebesgue number lemma.")

6. Let X be a locally compact Hausdorff space. **Explain** how to construct the one-point compactification of X, and **prove** that the space you construct is really compact (you do not have to prove anything else for this problem).

7. Show that if $\prod_{n=1}^{\infty} X_n$ is locally compact (and each X_n is nonempty), then each X_n is locally compact and X_n is compact for all but finitely many n.

8. Let X be a locally compact Hausdorff space, let Y be any space, and let the function space $\mathcal{C}(X, Y)$ have the compact-open topology.

Prove that the map

$$e : X \times \mathcal{C}(X, Y) \to Y$$

defined by the equation

$$e(x, f) = f(x)$$

is continuous.

9. Let I be the unit interval, and let Y be a path-connected space. Prove that any two maps from I to Y are homotopic.

10. Let X be a topological space and $f : [0, 1] \to X$ any continuous function. Define $\bar{f}$ by $\bar{f}(t) = f(1 - t)$. Prove that $f * \bar{f}$ is path-homotopic to the constant path at $f(0)$.

11. Let X be a topological space and let $x_0, x_1 \in X$. Recall that any path α from x_0 to x_1 gives a homomorphism $\hat{\alpha}$ from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ (you do not have to prove this).

Suppose that for every pair of paths α and β from x_0 to x_1 the homomorphisms $\hat{\alpha}$ and $\hat{\beta}$ are the same. **Prove** that $\pi_1(X, x_0)$ is abelian.

12. Let $p : E \to B$ be a covering map with B connected. Suppose that $p^{-1}(b_0)$ is finite for some $b_0 \in B$. Prove that, for every $b \in B$, $p^{-1}(b)$ has the same number of elements as $p^{-1}(b_0)$.

13. Let $p : E \to B$ be a covering map. Assume that B is connected and locally connected. Show that if C is a component of E, then $p|C : C \to B$ is a covering map.

14. Let B be a Hausdorff space.

    Let $p : E \to B$ be a covering map.

    Prove that E is Hausdorff.

15. Let $p : E \to B$ be a covering map. **Prove** that p takes open sets to open sets.

16. Let X be a topological space and let $f : X \to X$ be a homeomorphism for which $f \circ f$ is the identity map.

    Suppose also that each $x \in X$ has an open neighborhood V_x for which $V_x \cap f(V_x)$ is empty.

    Define an equivalence relation $\sim$ on X by: $x \sim y$ if and only if $x = y$ or $f(x) = y$. (You do **not** have to prove that this is an equivalence relation; this is the only place where the assumption that $f \circ f$ is the identity is used).

    (a) (5 points) **Prove** that the quotient map $q : X \to X/\sim$ takes open sets to open sets.

    (b) (9 points) **Prove** that q is a covering map. (You may use part (a) even if you didn't prove it.)

17. Let $p : E \to B$ be a covering map with E path-connected. Let $p(e_0) = b_0$.

    (a) Give the definition of the standard map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ constructed in Munkres (you do NOT have to prove that this is well-defined).

    (b) Suppose that α and β are two elements of $\pi_1(B, b_0)$ with $\phi(\alpha) = \phi(\beta)$. Prove that there is an element γ of $\pi_1(E, e_0)$ with $\beta = p_*(\gamma) \cdot \alpha$.

18. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let $x_0 \in X$ and let $y_0 = f(x_0)$.

    (a) (6 points) Give the definition of the function $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, including the proof that it is well-defined.

    (b) (10 points) Prove that if f is a covering map then f_* is one-to-one.

19. Let X be a path-connected space.

    Let x_0 and x_1 be two different points in X.

    Suppose that every path from x_0 to x_1 is path-homotopic to every other path from x_0 to x_1.

    Prove that X is simply-connected.

20. Let X and Y be topological spaces, let $x_0 \in X$, $y_0 \in Y$, and let $f : X \to Y$ be a continuous function which takes x_0 to y_0.

    Is the following statement true? If f is 1-1 then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is 1-1. Prove or give a counterexample (and if you give a counterexample justify it). You may use anything in Munkres's book.

21. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let $x_0 \in X$ and let $y_0 = f(x_0)$.

    Find an example in which f is onto but $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is not onto. **Prove** that your example really has this property. You may use any fact from Munkres.

22. Let D^2 be the unit disk $\{ x^2 + y^2 \leq 1 \}$ and let S^1 be the unit circle $\{ x^2 + y^2 = 1 \}$. Prove that S^1 is not a retract of D^2 (that is, prove that there is no continuous function $f : D^2 \to S^1$ whose restriction to S^1 is the identity function). You may use anything in Munkres for this.

23. Let X and Y be topological spaces and let $x \in X$, $y \in Y$.

    Prove that there is a 1-1 correspondence between

    $$\pi_1(X \times Y, (x, y))$$

    and

    $$\pi_1(X, x) \times \pi_1(Y, y).$$

    (You do **not** have to show that the 1-1 correspondence is compatible with the group structures.)

24. Let $p : Y \to X$ be a covering map, let $y \in Y$, and let $x = p(y)$.

    Let σ be a loop beginning and ending at x and let $[\sigma]$ be the corresponding element of $\pi_1(X, x)$.

    Let $\tilde{\sigma}$ be the unique lifting of σ to a path starting at y.

    Prove that if $[\sigma] \in p_* \pi_1(Y, y)$ then $\tilde{\sigma}$ ends at y.

25. **Definition.** If W is a space with base point w_0 and Z is a space with base point z_0, a map $f : W \to Z$ is said to be *based* if $f(w_0) = z_0$, and a homotopy $H : W \times I \to Z$ is said to be *based* if $H(w_0, t) = z_0$ for all t.

    Let X be a space with basepoint x_0 and let $u_0 = (1, 0)$ be the base point of S^1.

    Prove that there is a 1-1 correspondence between $\pi_1(X, x_0)$ and the based homotopy classes of based continuous maps $S^1 \to X$.

26. Let $p : \mathbb{R} \to S^1$ be the usual covering map (specifically, $p(t) = (\cos 2\pi t, \sin 2\pi t)$). Let $b_0 \in S^1$ be the point $(1, 0)$. Recall that the standard map

    $$\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$$

is defined by $\phi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is a lifting of f with $\tilde{f}(0) = 0$.

(a) (14 points) **Prove** that ϕ is 1-1.

(b) (14 points) **Prove** that ϕ is a group homomorphism.

27. Let S^1 be the circle

$$\{(x_1, x_2) \,|\, x_1^2 + x_2^2 = 1\}$$

in $\mathbb{R}^2$. Let $\mathbf{0}$ be the origin in $\mathbb{R}^2$.

Prove from the definitions that S^1 is a deformation retract of $\mathbb{R}^2 - \mathbf{0}$.

28. Let X be a topological space and let $x_0 \in X$.

Let U and V be open sets containing x_0, and suppose that the hypotheses of the Seifert-van Kampen theorem are satisfied (that is,

$$U \cup V = X,$$

and U, V, $U \cap V$ are path-connected).

Let $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$ and $j_2 : V \to X$ be the inclusion maps.

Suppose that $(i_1)_* : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is an isomorphism.

Prove, using the Seifert-van Kampen theorem, that there is an homomorphism

$$\Phi : \pi_1(X, x_0) \longrightarrow \pi_1(V, x_0)$$

for which $\Phi \circ (j_2)_*$ is the identity map of $\pi_1(V, x_0)$.

29. Let S^2 be the 2-sphere, that is, the following subspace of $\mathbb{R}^3$:

$$\{\, (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1 \,\}.$$

Let x_0 be the point $(0, 0, 1)$ of S^2.

Use the Seifert-van Kampen theorem to **prove** that $\pi_1(S^2, x_0)$ is the trivial group. You may use either of the two versions of the Seifert-van Kampen theorem given in Munkres's book. You will **not** get credit for any other method.

30. Let X be a topological space and let $x_0 \in X$.

Let U and V be open sets containing x_0, and suppose that the hypotheses of the Seifert-van Kampen theorem are satisfied (that is,

$$U \cup V = X,$$

and U, V, $U \cap V$ are path-connected).

Let $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$ and $j_2 : V \to X$ be the inclusion maps.

Suppose that $(i_1)_* : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is onto.

Prove, using the Seifert-van Kampen theorem, that $(j_2)_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ is onto.

31. Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $aabbcdc^{-1}d^{-1}$.

i) Calculate $H_1(X)$. (You may use any fact in Munkres, but be sure to be clear about what you're using.)

ii) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?