R. Kaufmann
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Problem Set 2

Problems

Problem 1: Check that indeed a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $F : \mathcal{C} \to \mathcal{D}^{op}$.

Problem 2: Show that $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

Problem 3: Write out the details that $Hom_{\mathcal{C}}(\cdot\,,\cdot)$ as a functor from $\mathcal{C} \times \mathcal{C} \to Set$ is contravariant in the first variable and covariant in the second variable. This means that for fixed X the functor given on objects Y as $Hom_{\mathcal{C}}(Y, X)$ is contravariant, and the functor given on objects Y as $Hom_{\mathcal{C}}(X, Y)$ is covariant. The first step is to give the definition of the functors on morphisms.

Problem 4: Check the identities for $\delta_i^n : [n-1] \to [n]$ and $\sigma_i^n : [n+1] \to [n]$, $i = 0, \dots, n$.

$$\begin{aligned}
\delta_i^{n+1}\delta_j^n &= \delta_{j+1}^{n+1}\delta_i^n && i \le j \\
\sigma_j^n\sigma_i^{n+1} &= \sigma_i^n\sigma_{j+1}^{n+1} && i \le j \\
\sigma_j^n\delta_i^{n+1} &= \delta_i^n\sigma_{j-1}^{n-1} && i < j \\
\sigma_j^n\delta_i^{n+1} &= id_n && i = j \text{ or } i = j+1 \\
\sigma_j^n\delta_i^{n+1} &= \delta_{i-1}^n\sigma_j^{n-1} && j+1 < i
\end{aligned}$$

Problem 5: Write the corresponding the identities for a simplicial set $X_\bullet : \Delta^{op} \to Set$. $d_i^n = X_\bullet(\delta_i^n), s_n^i := Y_\bullet(\sigma_i^n)$.

Problem 6: Using the first identity above show that for $d_n = \sum_{i=1}^n (-1)^i d_i^n$, $d^2 = 0$ as a map $Free_{\mathbb{Z}}(X_n) \to Free_{\mathbb{Z}}(X_{n-1})$