Ralph M Kaufmann

Background in Topology[1]

1. Background, Idea and Motivation

Let's consider a space like $\mathbf{R}$, $\mathbf{R}^2$ or $\mathbf{R}^3$. It is made up of points and in that respect it is a set – the set of its points. But there is more structure. We know that it is a vector space and we have the norm of vectors which induces a distance for points. This distance allows us to say that points are close together. This is done e.g. in the $\epsilon - \delta$ criterion for continuity. More precisely given a point x in say $\mathbf{R}^2$ we can consider the open ball centered at x of radius $\epsilon > 0$ $B_{\epsilon,x} := \{y \in \mathbf{R}^2 : |x - y| < \epsilon\}$. We know from calculus that a set $U \subset \mathbf{R}^2$ is open if $\forall x \in U \exists \epsilon$ s.t. $B_{\epsilon,x} \subset U$. In words there is an open ball around each point of U which sits fully inside U.

The usefulness of this concept has been demonstrated in the various theorem in calculus where openness appears as a condition, such as Clairaut's Theorem, Green's Theorem etc.

More importantly looking at the definition of continuity or the differentiability of a function at a point we see that not only the function at the point is involved, but the function plus a small neighborhood of it, i.e. a small ball around the point.

A generalization of these concepts is given in the following.

2. Topological Spaces

The concept of a topological space is the observation that a space is made up of points, which is nothing but a set, and the extra data of what subsets of the space should be regarded as open.

2.1. **Definition.** *A topology* on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1) $\emptyset$ and X are in $\mathcal{T}$.
2) The union of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A *topological space* is a set X together with a topology $\mathcal{T}$ on X.

The elements of $\mathcal{T}$ are called the *open sets.*

2.2. **Definition.** Given a topological space $(X, \mathcal{T})$, a set A is called *closed* if $X \setminus A$ is open.

2.3. **Caveats.**

1) There may be sets which are neither open nor closed!
2) There are sets which are open and closed. By the condition 1) both X and $\emptyset$ are open as well as closed. There might be more sets like this.

2.4. **Exercises.**

1) Show that the set of open sets of $\mathbf{R}^2$ and $\mathbf{R}^3$ (for that matter of $\mathbf{R}^n$) as defined in section 1 is a topology.
2) Give examples of open, closed and sets that are neither open nor closed in $\mathbf{R}^2$ with the above topology.
3) Give a collection of open subsets in $\mathbf{R}^2$ with the above topology whose intersection is not open.
4) Show that for any set there are two topologies given by:
   a) The so–called trivial topology: $\mathcal{T} = \{\emptyset, X\}$
   b) The so–called discrete topology in which $\mathcal{T}$ is the collection of *all* subsets of X.

   These are the minimal and maximal possible topologies.

Sometimes it is easier to specify a set that "generates" the open sets rather than specifying all open sets. The formal definition for this is given by a basis for the topology.

2.5. **Definition.** If X is a set, a *basis for a topology on* X is a collection $\mathcal{B}$ of subsets of X called basis elements s.t.

1) For each $x \in X$, there is at least one basis element B containing x.
2) If x belongs to the intersection of two basis elements B_1 and B_2 then there is a basis element B_3 containing x, s.t. $B_3 \subset B_1 \cap B_2$

If $\mathcal{B}$ is a basis for a topology on X the *topology generated by* $\mathcal{B}$ is the topology in which a set U is open if it satisfies the property that for any $x \in U$ there is a $B \in \mathcal{B}$ s.t. $x \in B$ *and* $B \subset U$.

2.6. **Exercises.**

1) Show that the sets $B_{\epsilon,x}$ of section 1 give a basis for a topology on say $\mathbf{R}^2$.
2) Show that the set of points of a set X form a basis for the discrete topology. (Hence the name. It is for instance the natural topology for a finite set or for a discrete set such as $\mathbf{Z}$.)
3) Show that the topology generated by a basis is indeed a topology.

3. SUBSPACE TOPOLOGY

A natural question one can ask is the following: If I have a set X together with a topology on it and a subset $Y \subset X$, is there a natural topology on Y? The answer is that one should take the open sets of X and intersect them with Y.

3.1. **Definition.** Let X be a topological space with topology $\mathcal{T}$ and Y be a subset of X. We call the collection $\mathcal{T}_Y := \{Y \cap U : U \in \mathcal{T}\}$ the *subspace topology.*

3.2. **Exercises.**

1) Show that the subspace topology is a topology.
2) Show that if $\mathcal{B}$ is a basis for $\mathcal{T}$ then $\mathcal{B}_Y := \{B \cap Y | B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.

4. THE METRIC TOPOLOGY

If we are looking at a space where we know the distance between any two points, we can build the balls $B_{\epsilon,x}$ and use them as a basis for a topology. This works as follows.

4.1. **Definition.** A *metric* on a set X is a function

$$d : X \times X \mapsto \mathbf{R}$$

having the following properties:

1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$ (Positivity).
2) $d(x, y) = d(y, x)$ (Symmetry).
3) $d(x, y) + d(y, z) \geq d(x, z)$ (Triangle inequality).

4.1.1. *Remarks.* The metric sometimes is also called a distance function. It takes two points and produces a real number – the distance between them. This distance should be positive and the only point at distance 0 from a given point x is the point itself. The symmetry states that the distance between two points does not depend on the order of the two points. Finally, the triangle inequality states that the sum of two sides of a triangle are greater or equal to the third side. (Caveat: actually we do not know what "sides" are in this context, but we do know the distance i.e. the length of what might be a side.)

4.2. **Exercise.** Show that the norm for vectors on $\mathbf{R}^3$ induces a metric. I.e. $d(P,Q) := |\vec{OP} - \vec{OQ}| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$ where (x_1,y_1,z_1) are the coordinates of P and (x_2,y_2,z_2) are the coordinates of Q.

4.3. **Definition.** Given a set X together with a distance function, we set for $x \in X$ and $\epsilon \in \mathbf{R}$

$$B_{\epsilon,x} := \{y : d(x,y) < \epsilon\}$$

and call it the *ϵ–ball centered at x.*

4.4. **Definition/Lemma.** If d is a metric on the set X, then the collection of all ϵ–balls $B_{x,\epsilon}(d)$ is a basis for a topology called the *metric topology.*

4.5. **Exercise (optional).** Prove the Lemma part, i.e. that the balls provide a basis.

5. Homeomorphisms

When looking at finite sets an obvious structure is their cardinality, and one tends to look at them as being equivalent if they have the same cardinality. Rephrased one can find a bijection (a one–to–one correspondence) between the two sets. This works as well for any type of set.

So we tend to think that two sets X and Y are about the same if there is a bijection between them, i.e., there is a function $f : X \to Y$, s.t. it has an inverse $f^{-1} : Y \to X$ with the property $f \circ f^{-1} = id$ and $f^{-1} \circ f = id$ where id is the identity map $id(x) = x$. Notice you need both conditions.

One can say that functions relate the two sets as sets. The question is now, if one includes topologies into the picture, how to relate the topologies of spaces.

The next definition is the key!

5.1. **Definition.** A function of a topological space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{T}')$ is called *continuous* if for all $V \in \mathcal{T}'$ $f^{-1}(V) \subset \mathcal{T}$ (i.e. the *inverse* image of an open set is open.)

Recall $f^{-1}V := \{x \in X : f(x) \in V\}$

Note if the topology of Y is given by a basis it is sufficient to show that the inverse image of the basis elements are open, since the union of open sets is open and

$$f^{-1}(V) = f^{-1}(\bigcup_{v \in V} B(v)) = \bigcup_{v \in V} f^{-1}(B(v))$$

where $B(v)$ is a basis element containing v and lying inside V whose existence is guaranteed.

5.2. **Exercise.** In the case of $X = Y = \mathbf{R}$ with the metric topology ($d(x,y) := |x-y|$) show that the above condition is equivalent to the $\epsilon - \delta$ condition you know from calculus.

When dealing with topological spaces we would like our function to be continuous and we consider spaces to be about the same if there is a bijection between them where both the function and the inverse are continuous, in this way the topological structures are preserved.

5.3. **Definition.** A function f of a topological space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{T}')$ is called *a homeomorphism* if it is bijective, continuous and its inverse f^{-1} is also continuous.

In the case that the function is not bijective, but only injective, we can make it bijective by restricting the function to its image making it surjective.

5.4. **Definition.** A *topological imbedding* is an injective continuos map between topological spaces $f : X \to Y$ s.t. the restriction of f to the image $f' : X \to f(X)$ is a homeomorphism.

5.5. **Remark.** The condition 2 for a regular surface amounts to saying that the map x is a topological imbedding.

5.6. **Exercises.**

1) Show that the following sets are homeomorphic: $\mathbf{R}, (0, \infty), (0, 1)$!
2) (harder/optional) Show that for any fixed R the map $f : S^1 \to \mathbf{R}^2, f(\theta) = (R\cos(\theta), R\sin(\theta))$ is a topological imbedding where we take $S^1 \subset \mathbf{R^2}$ to be given by the equation $r = 1$ in polar coordinates together with the subspace topology. Conclude that all circles are homeomorphic and thus the curvature is not preserved under homeomorphism.