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Re-Gauging Groupoid, Symmetries and Degeneracies for Graph Hamiltonians and Applications to the Gyroid Wire Network

Ralph M. Kaufmann, Sergei Khlebnikov and
Birgit Wehefritz-Kaufmann

Abstract. We study a class of graph Hamiltonians given by a type of quiver representation to which we can associate (non)-commutative geometries. By selecting gauging data, these geometries are realized by matrices through an explicit construction or a Kan extension. We describe the changes in gauge via the action of a re-gauging groupoid. It acts via matrices that give rise to a noncommutative 2-cocycle and hence to a groupoid extension (gerbe). We furthermore show that automorphisms of the underlying graph of the quiver can be lifted to extended symmetry groups of re-gaugings. In the commutative case, we deduce that the extended symmetries act via a projective representation. This yields isotypical decompositions and super-selection rules. We apply these results to the primitive cubic, diamond, gyroid and honeycomb wire networks using representation theory for projective groups and show that all the degeneracies in the spectra are consequences of these enhanced symmetries. This includes the Dirac points of the G(yroid) and the honeycomb systems.

1. Introduction

We study a class of graph Hamiltonians given by a type of quiver representation to which we can associate (non)-commutative geometries. Our particular focus are symmetries in these geometric realizations especially those coming from the symmetries of the graph. Via considering a re-gauging group(oid) action, we can show that the classical graph symmetries lead to enhanced (centrally extended) symmetries which are realized as projective representations in the commutative case.

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The physical motivation for considering such systems stems from considering wire systems on the nano-scale where the presence of higher dimensional irreps in the decomposition of the above symmetries leads to degeneracies in the spectrum. After giving the general arguments, we apply them to the primitive cubic (P), diamond (D), gyroid (G) and the honeycomb wire systems. Here, we are especially motivated by understanding the electronic properties of a novel material [1] based on the G(yroid) geometry [2]. We expect that our considerations can also be applied to other graph-based setups, such as those coming from quiver representations, e.g., in field theory, or the coordinate changes in cluster algebras and varieties.

Mathematically, the initial data we start from are a finite graph $\bar{\Gamma}$ together with a separable Hilbert space $\mathscr{H}_v$ for each vertex v of the graph and a unitary morphism for each oriented edge, such that the inverse oriented edge corresponds to the inverse morphism. Algebraically these data correspond to a groupoid representation in separable Hilbert spaces, as we explain in Sect. 2.1. In this situation, as we derive, there is an associated Hamiltonian acting on the direct sum of all the Hilbert spaces $\mathscr{H}_v$.

To obtain a matrix representation of the Hamiltonian, one has to fix some additional gauge data. The gauge data consist of a rooted spanning tree and an order on the vertices. With this choice in place, each edge corresponds to a loop and we can represent an isometry associated to an edge by an element of the C^*-algebra $\mathscr{A}$ generated by the morphisms corresponding to the loops of $\bar{\Gamma}$ at a fixed base point, cf. Sect. 2.1.4. Via pull-back this also yields a matrix representation of the Hamiltonian in $M_k(\mathscr{A})$ where k is the number of vertices of $\bar{\Gamma}$.

From the noncommutative geometry point of view, the C^*-algebra $\mathscr{A}$ represents a space. If $\mathscr{A}$ is commutative (since $\mathscr{A}$ is unital), this space can be identified as a compact Hausdorff space X such that the C^*-algebra of complex valued continuous functions $C^*(X)$ is isomorphic to $\mathscr{A}$. In the applications, we consider X as the momentum space, which in the commutative case is the n-dimensional torus $X = T^n = (S^1)^{\times n}$ and in the noncommutative case $\mathscr{A}$ is the noncommutative n-torus $\mathbb{T}^n_\Theta$ for a fixed value of Θ, that in physical situations is given by a background B-field. See below Sect. 2.2.

Concrete extended symmetry groups are constructed via a lift of the action of the underlying graph symmetries $\mathrm{Sym}(\bar{\Gamma})$ on this data as re-gaugings. The lift of the classical symmetries is rather complicated and proceeds in several steps:

(1) We first establish that the different matrix realizations of the Hamiltonian given by choosing different rooted spanning trees and orders are all linked by gauge transformations—see Theorem 3.3. The specific gauge transformations that arise form the re-gauging groupoid $\mathcal{G}$. It acts transitively on the set of all the matrix Hamiltonians that can be obtained from the decorated graph by all different choices of data. Using category theory, these realizations are just Kan extensions given by pushing forward to the graph obtained by contracting the spanning tree.

(2) We then show that the gauge transformations can be represented as conjugation with matrices with coefficients in $\mathscr{A}$. We prove that these matrices lead to a noncommutative 2-cocycle. This in turn gives rise to a groupoid extension of $\mathcal{G}$. Geometrically this corresponds to a gerbe.
(3) In the commutative case (Sect. 3.5), we furthermore show that these matrices give a *projective* representation of the re-gauging groupoid. Just like in ordinary theory of projective representations this means that there is a *bona fide* representation of a central extension of this groupoid.
(4) In the commutative setup, if we fix a point $p \in X$ and evaluate the matrix Hamiltonian with coefficients in $\mathscr{A}$ at p we obtain a matrix Hamiltonian with coefficients in $\mathbb{C}$ that we denote by $H(p) \in M_k(\mathbb{C})$. In this way, we can think of X as the base of a family of finite-dimensional Hamiltonians. Likewise, the re-gauging actions give a groupoid representation in matrices $M_k(\mathbb{C})$ which commute with $H(p)$. The stabilizer groups of a particular fixed Hamiltonian are the sought after enhanced symmetry groups.
(5) For applications, this leaves the problem of identifying the points p and the stabilizer groups or at least subgroups. To address the latter question, we establish that the automorphism group $Sym(\bar{\Gamma})$ of the graph induces re-gaugings, by pushing forward the spanning tree and the order of the vertices. In this way, the symmetries of the graph give rise to a subgroupoid of $\mathcal{G}$. Going through the construction outlined above, we can restrict to this subgroupoid and see that at a fixed point of the re-gauging action we get a projective representation of the stabilizer subgroup which leads to possible higher dimensional irreps and thus band sticking.
(6) To identify points of X—which we take to be T^n for concreteness—where such enhanced symmetry groups can occur, we show that under certain assumptions, that hold in all cases of our initial physical interest, the operations of the symmetry group of the graph $Sym(\bar{\Gamma})$ via re-gaugings action on the base torus T^n (Theorems 3.10 and 3.14).[1] At points $t \in T^n$ with non-trivial stabilizer groups, we automatically get a projective representation of these stabilizer subgroups of the automorphism groups of the underlying graph, which commutes with the Hamiltonian. Hence, we get isotypical decompositions, which can give us non-trivial information about the spectrum using the arguments above.

We wish to point out that this approach is broader than that of considering classical symmetries of decorated graphs and in the commutative case generalizes the extensive analysis of [3], see Sect. 3.1 for details.

We apply all these considerations to the cases of the PDG wire networks and the honeycomb lattice; see Sect. 4. Here, the graph $\bar{\Gamma}$ arises physically as the quotient graph of a given (skeletal) graph $\Gamma \subset \mathbb{R}^n$ by a maximal translation group $L \simeq \mathbb{Z}^n$. Each edge of the quotient graph is decorated with a partial

[1] We say lift to the base here, since any action on the base gives rise to an action on the parameterized family of Hamiltonians, but it is not clear that any such action comes from one on the base.

isometry operator of translation in the direction of any lift of that edge to the graph Γ. A Harper-like Hamiltonian is constructed from these isometries; in the simplest version (tight-binding approximation), it is simply the summation over them. This Hamiltonian along with the symmetries of the given material is the main input into the noncommutative geometry machine, which constructs a C^*-algebra that encodes relevant information about the system out of this data.

One main objective is to analyze and understand the branching behavior or stated otherwise the locus of degenerate eigenvalues. The motivation is that in solid-state physics such degenerate eigenvalues may lead to novel electronic properties, as is the case, for instance, with the Dirac points in graphene [4]. The key observations are that (a) non-Abelian extended symmetry groups by themselves can force degeneracies via higher dimensional—i.e., > 1—irreducible representations and (b) any enhanced symmetries, also Abelian ones, give rise to super-selection rules. The latter ones can facilitate finding the spectrum considerably, since the Hamiltonian eigenspace decomposition has to be compatible with the isotypical decomposition of the representation.

Complementary to this group theoretic approach, there is another one via singularity theory, which is contained in [5]. Our main result for the PDG and honeycomb networks is that both approaches yield the same classification of degeneracies in the commutative case. Namely, at all degenerate points, which were analytically classified in [5], there is an enhanced symmetry group arising from graph automorphisms in the above way which forces the degeneracy. Here, the surprising fact is that we find *all* the degeneracies and degenerate points through the projective representations of (subgroups of) $\mathrm{Sym}(\bar{\Gamma})$ given by re-gaugings.

Here, the G-wire network corresponding to the double Gyroid which was our original motivator is the most interesting case. As shown in [5], there are exactly two points with triple degeneracies and two points with degeneracy $(2, 2)$, that is two doubly degenerate eigenvalues. The automorphism group of the graph is $\mathbb{S}_4$. The representation theory becomes very pretty in this case. There are two fixed points $(0, 0, 0)$ and (π, π, π) on T^3 under the whole $\mathbb{S}_4$ action. The projective representation is just the ordinary representation of $\mathbb{S}_4$ by 4×4 permutation matrices at the point $(0, 0, 0)$ which is known to decompose into the trivial and a three-dimensional irreducible representation which forces the triple degeneracy. This result was also found by [3], where an initially different system was considered that results in the same spectrum.

At the point (π, π, π), things become slightly more interesting. There is a projective representation of $\mathbb{S}_4$, but we can show that this projective representation corresponds to an extension which is isomorphic to the trivial extension. Hence, after applying the isomorphisms, we have a representation of $\mathbb{S}_4$ and there is a trivial and a 3-dim irrep, giving the second triple degeneracy. This elucidates the origin of the symmetry stated in [3]. Notice that the classical symmetries of decorated graphs would only yield an $\mathbb{S}_3$ symmetry at this point, which cannot explain a triple degeneracy as there are no 3-dim irreps for this group.

Things really get interesting at the two points $(\pm\pi/2, \pm\pi/2, \pm\pi/2)$. Here, the stabilizer group is A_4. The projective representation gives rise to an extension which we show to be isomorphic to the non-trivial double cover $2A_4$ of A_4 aka. 2T, the binary tetrahedral group or $SL(2,3)$. Using the character table, we deduce that the representation decomposes into two 2-dim irreps forcing the two double degeneracies.

These are completely novel results. We wish to point out that one absolutely needs the double cover as A_4 itself has no 2-dim irreps and hence the projective extension is essential.

We also use the fact that the diagonal of T^3 is fixed by a cyclic subgroup C_3 of A_4 to determine the spectrum analytically. Here, we use the superselection rules.

For the D and honeycomb case, we show that the degenerate points which are well known in the honeycomb case and were computed for D in [6] are all detected by enhanced symmetries. These however, yield Abelian representations and hence we have to use the arguments of the type (b), that is superselection rules, to show that the eigenvalues are degenerate over these points. Similar results to ours have now also been independently found for the D case in [7] using different methods.

2. General Setup

In this section, we show how to construct the C^*-algebra $\mathscr{A}$ and the Hamiltonian $H \in M_k(\mathscr{A})$ mentioned in the introduction from a graph representation of a finite graph $\bar{\Gamma}$ with k vertices. Furthermore, we embed a copy $\mathscr{A}$ into $M_k(\mathscr{A})$ and define $\mathscr{B}$ to be the subalgebra generated by H and $\mathscr{A}$ under this embedding. The pair $\mathscr{A} \hookrightarrow \mathscr{B}$ is the basic datum for our noncommutative geometry.

2.1. Groupoid Graph Representations in Separable Hilbert Spaces

Given a finite graph $\bar{\Gamma}$, we define a groupoid representation of $\bar{\Gamma}$, an association of a separable Hilbert space $\mathscr{H}_v$ for each $v \in V(\bar{\Gamma})$ and an isometry $U_{\vec{e}} : \mathscr{H}_v \to \mathscr{H}_w$ for each directed edge $\vec{e}$ from v to w.

These data indeed determine a unique functor $(\mathscr{H}, U)$ from the path groupoid of $\bar{\Gamma}$ to the category of separable Hilbert spaces. The path groupoid $\mathcal{P}_{\bar{\Gamma}}$ (or $\mathcal{P}$ for short) of $\bar{\Gamma}$ is the category whose objects are the vertices of $\bar{\Gamma}$ and whose morphisms are generated by the oriented edges, where the inverse of a morphism given by $\vec{e}$ is the one given by $\overleftarrow{e}$. Notice that we are looking at the morphisms generated by the oriented edges, this means that $\mathrm{Hom}_{\mathcal{P}}(v, w)$ is the set of paths along oriented edges from v to w modulo the relation that going back and forth along an edge is the identity. Composition is only allowed if the first oriented edge terminates at the beginning of the second oriented edge. This is why we only obtain a groupoid and not a group.

In particular, this lets one view the fundamental group $\pi_1(\bar{\Gamma}, v_0)$ in two equivalent fashions. First as the topological π_1 of the realization of a graph, and second as the group $\mathrm{Hom}_{\mathcal{P}}(v_0, v_0) = \mathrm{Aut}(v_0)$ where $\mathrm{Hom}_{\mathcal{P}}$ are the morphisms in the path groupoid $\mathcal{P}$.

The collection of automorphisms in $\mathcal{P}$ forms a subgroupoid $\mathcal{L}$. It is the disjoint union $\mathcal{L} = \amalg_{v \in V(\bar{\Gamma})} \pi_1(\bar{\Gamma}, v)$. These are the classes of free loops on $\bar{\Gamma}$.

2.1.1. Hamiltonian, Symmetries and the C^*-Geometries. Given a groupoid representation as above, set $\mathscr{H} = \bigoplus_{v \in V(\bar{\Gamma})} \mathscr{H}_v$ and define H by

$$H = \sum_{e \in E(\bar{\Gamma})} (U_{\vec{e}} + U_{\overleftarrow{e}}) \in B(\mathscr{H}) \tag{1}$$

where $B(\mathscr{H})$ is the C^* algebra of bounded operators on $\mathscr{H}$.

We let $\mathscr{A}$ be the *abstract* C^*-algebra which $\pi_1(\bar{\Gamma}, v_0)$ generates in $B(\mathscr{H})$ via the representation. This is a bit subtle, as the concrete algebra depends on the choice of base point v_0. We will use the notation $\mathscr{A}_{v_0} := U(\pi_1(\bar{\Gamma}, v_0)$ to emphasize this.[2]

Of course any two choices of a base vertex give isomorphic algebras but there is no preferred isomorphism between them. In fact, any path γ from v to w induces an isomorphism of $\pi_1(\bar{\Gamma}, w)$ to $\pi_1(\bar{\Gamma}, v)$ by conjugation with γ. This induces an isomorphism $\hat{U}_\gamma : \mathscr{A}_w \to \mathscr{A}_v$. In the physical situation of wire networks, we are interested in Sect. 2.2, there is however a global identification of these algebras which comes from the embedding of the system into Euclidean space. Algebraically we realize this as extra coherence isomorphisms $\alpha_{*v} : \mathscr{A}_v \xrightarrow{\sim} \mathscr{A}$ with inverse $\alpha_{v*} := \alpha_{*v}^{-1}$.

The direct sum of the α_{v*} gives a representation α of $\mathscr{A}$ into $\tilde{A} := U(\mathcal{L}) = \bigoplus_{v \in V(\bar{\Gamma})} \mathscr{A}_v \subset B(\mathscr{H})$. The algebra $\mathscr{B}$ is now the sub C^*-algebra generated by H and $\alpha(\mathscr{A})$. We also set $\tilde{B} = U(\mathcal{P}_{\bar{\Gamma}}) \subset B(\mathscr{H})$.

The C^*-geometry we are interested in is the inclusion $\mathscr{A} \to \mathscr{B}$. We call the system commutative, if $\mathscr{B}$ (and hence $\mathscr{A}$) is commutative. We call the situation *fully commutative* if in addition for any edge $\vec{e}$ from v to w: $\alpha_{*v} \hat{U}_{\vec{e}} \alpha_{w_*} = id$. Iterating this conditions allows one to deduce the condition to arbitrary edge paths.

In the wire-network case, the condition to be fully commutative corresponds to the case of zero magnetic field.

Notation If we choose a fixed base point v_0, we will tacitly use the isomorphism $\alpha_{v_0 *}$ to identify $\mathscr{A}$ and $\mathscr{A}_{v_0}$.

2.1.2. Matrix Hamiltonian. If we fix a rooted spanning tree and an order on the vertices, we can identify $\mathscr{H} \simeq \mathscr{H}_{v_0}^k$ via a unitary U as follows and H becomes equivalent to a matrix in $M_k(\mathscr{A})$.

A spanning tree τ is by definition a contractible subgraph of $\bar{\Gamma}$ which contains all the vertices of $\bar{\Gamma}$. It is rooted if one of the vertices is declared the root; denote it by v_0. To obtain a honest $k \times k$ matrix, we also have fix an

[2] Here and below for any subgroupoid $\mathcal{P}'$ of $\mathcal{P}$ we denote the C^*-subalgebra of $B(\mathscr{H})$ generated by the morphisms of $\mathcal{P}'$ via U by $U(\mathcal{P}')$.

order $<$ on all the vertices, where we insist that the root is the first vertex in this order. That is we fix a rooted ordered spanning tree $\boldsymbol{\tau} := (\tau, v_0, <)$.

For each vertex v of $\bar{\Gamma}$, there is a unique shortest path in τ to v_0. This defines a choice of fixed isomorphism $U^{\tau}_{vv_0} : \mathscr{H}_{v_0} \to \mathscr{H}_v$ by translations along the edge path, see Sect. 3.2 for details. Assembling these maps gives the desired isomorphism $\mathscr{H} \simeq \mathscr{H}^k_{v_0}$. Pulling back H to $\mathscr{H}^k_{v_0}$ using this isomorphism, we obtain *a* matrix version $H_{\boldsymbol{\tau}}$ where we include the subscript to stress that this matrix depends on the choice of rooted ordered spanning tree $\boldsymbol{\tau}$.

To fix the notation, which we will need later, we give the full details: let v_i be the i—the vertex in the enumeration $<$. Then, we obtain a matrix $H_{\boldsymbol{\tau}}$ using the isomorphisms $U^{\tau}_{v_iv_0}$ which are defined as follows. Let $v_0, w_1, \dots, w_k, v_i$ be the sequence of vertices along the unique shortest path $\gamma^{\tau}_{v_iv_0}$ from v_0 to v_i in τ, then

$$U^{\tau}_{v_iv_0} = U_{v_iw_k}U_{w_kw_{k-1}} \cdots U_{w_2w_1}U_{w_1v_0} = U(\gamma^{\tau}_{v_iv_0}) \tag{2}$$

and $U^{\tau}_{v_0v_i} = (U^{\tau}_{v_iv_0})^*$. Given the choice of $\boldsymbol{\tau}$, we get the corresponding matrix Hamiltonian as

$$H_{\boldsymbol{\tau}} : \bigoplus_{i=1}^{k} \mathscr{H}_{v_0} \xrightarrow{\oplus_i U^{\tau}_{v_0v_i}} \bigoplus_i \mathscr{H}_{v_i} = \mathscr{H} \xrightarrow{H} \mathscr{H} = \bigoplus_i \mathscr{H}_{v_i} \xrightarrow{\oplus_i U^{\tau}_{v_iv_0}} \bigoplus_i \mathscr{H}_{v_0} \tag{3}$$

Of course, all the $\mathscr{H}_{\boldsymbol{\tau}}$ are equivalent, although not canonically. For the equivalence, one has to choose a path from one root to the other. We will exploit this fact extensively below.

2.1.3. $\mathscr{A}$ Weighted Graph. After having fixed an initial spanning tree, the matrix Hamiltonian has a different description.To each edge $\vec{e}$ from v to w, we can associate $wt(\vec{e}) := U^{\tau}_{v_0w}U_{\vec{e}}U^{\tau}_{vv_0} \in U(\mathscr{A})$. That is, we can regard $\bar{\Gamma}$ as having a weight function on ordered edges with weights in $U(\mathscr{A})$. If e is an edge of τ, then with this definition $wt(\vec{e}) = wt(\overleftarrow{e}) = 1$.

An alternative way of viewing this data is as a certain type of quiver representation, we will comment on this more below.

2.1.4. Weights as a Representation of the Fundamental Group. For any finite graph $\bar{\Gamma}$, the Euler Characteristic is $\chi(\bar{\Gamma}) = |V(\bar{\Gamma})| - |E(\bar{\Gamma})| = 1 - b_1$ where b_1 is the rank of $H_1(\bar{\Gamma})$ which is the same as the "number of loops". More precisely, if $\bar{\Gamma}$ is connected, $\pi_1(\bar{\Gamma}) = \mathbb{F}_{b_1}$ that is the free group in b_1 generators. In the applications, b_1 is the rank of the lattice of translational symmetries.

One way to view a rooted spanning tree (τ, v_0) is to think of it as fixing a base point v_0 and a set of symmetric generators/basis for $\pi_1(\bar{\Gamma}, v_0) = \mathrm{Hom}_{\mathcal{P}}(v_0, v_0)$. Topologically after contracting the spanning tree, one is left with a wedge of S^1s. There are b_1 of these, one for each non-contracted edge. Each simple loop around one of the S^1s gives a generator. Picking one generator per loop gives a basis.

Without doing the contraction, the correspondence on $\bar{\Gamma}$ itself is given by all the (ordered) edges not contained in τ. To each such ordered edge $\vec{e}$ from v to w, we associate the loop starting at v_0 going along τ to v then traversing $\vec{e}$ to w and afterwards returning to v_0 along τ. Again picking both orientations gives a symmetric set of generators of the free group while picking only one orientation per edge fixes a basis. Any edge in the spanning tree corresponds to the unit, that is the class of a constant loop.

In this language, the weight function wt is a representation of π_1 lifted to the edges of the graph by the above correspondence. Thus, as long as the base point v_0 stays fixed, the changes of spanning tree can be viewed as a change-of-basis of $\pi_1(\bar{\Gamma}, v_0)$. If v_0 moves, say to v_0', then, as usual, any path from v_0' to v_0 gives an isomorphism taking $\pi_1(\bar{\Gamma}, v_0')$ to $\pi_1(\bar{\Gamma}, v_0)$. Both types of isomorphisms will play a role later in the symmetry group actions.

2.1.5. Non-Degeneracy and Toric Non-Degeneracy. We call a groupoid representation non-degenerate, if the images of the basis of the free group given by the construction above are independent unitary generators of $\mathscr{A}$ and call it toric non-degenerate if $\mathscr{A}$ is isomorphic to the noncommutative torus $\mathbb{T}^{b_1}_{\Theta}$.

Notice that if $\mathscr{A}$ is commutative and non-degenerate, then $\mathscr{A} \simeq \mathbb{T}^{b_1}$, the C^* algebra of the torus of dimension b_1.

2.1.6. Hamiltonian and $\mathscr{A}$ from a Weighted Graph. Alternatively to starting with a groupoid representation, one can also start with an $\mathscr{A}$-weighted graph. It is in this representation that we can understand the re-gauging groupoid $\mathcal{G}$.

Fix a finite connected graph $\bar{\Gamma}$, a rooted ordered spanning tree $\boldsymbol{\tau}$ of $\bar{\Gamma}$ such that the root of τ is the first vertex, a unital C^* algebra $\mathscr{A}$, and a morphism $wt : \{\text{Directed edges of } \bar{\Gamma}\} \to \mathscr{A}$ which satisfies

1. $wt(\overrightarrow{e}) = wt(\overleftarrow{e})^*$ if $\overrightarrow{e}$ and $\overleftarrow{e}$ are the two orientations of an edge e.
2. $wt(\overrightarrow{e}) = 1 \in \mathscr{A}$ if the underlying edge e is in the spanning tree.

In general, if wt is as above and it satisfies the first condition, we will call it a weight function (with values in $\mathscr{A}$) and if it satisfies both conditions, a weight function compatible with the spanning tree.

By Gel'fand–Naimark–Segal representability, we realize $\mathscr{A} \subset B(\mathscr{H}_{v_0})$ for a separable Hilbert space $\mathscr{H}_{v_0}$. Here, $\mathscr{H}_{v_0}$ is the Hilbert space constructed by the Segal part of the theorem.

We shall also postulate that $\mathscr{A}$ *is minimal*, which means that it is the C^*-algebra generated by the $wt(\overrightarrow{e})$ where $\overrightarrow{e}$ runs through the directed edges of $\bar{\Gamma}$. This makes the terminology of Sect. 2.1.5 applicable. Also, we see that this is again just a lift of a representation of $\pi_1(\bar{\Gamma}, v_0)$ to the edges of $\bar{\Gamma}$ using the spanning tree τ.

Given this data, let k be the number of vertices of $\bar{\Gamma}$. We will enumerate the vertices $v_0, \ldots, v_{k-1}$ according to their order; v_0 being the root. Given this data, the Hamiltonian $H = H(\bar{\Gamma}, \boldsymbol{\tau}, w)$ is the matrix $H = (H_{ij})_{ij} \in M_k(\mathscr{A})$ whose entries are:

$$H_{ij} = \sum_{\text{directed edges } \overrightarrow{e} \text{ from } v_i \text{ to } v_j} wt(\overrightarrow{e}) \tag{4}$$

It acts naturally on $\mathscr{H} := \mathscr{H}_0^k$. In this sense, the weighted graph encodes both the Hamiltonian *and* the symmetry algebra $\mathscr{A}$.

In the general noncommutative case, this is *not quite enough* for the whole theory, as we do not recover the action of $\mathscr{A}$ on $\mathscr{H}$ and the connection between the action of H and that of $\mathscr{A}$. Recall that the action of $\mathscr{A}$ on $\mathscr{H} = \mathscr{H}_{v_0}^k = \bigoplus_{v\in V(\bar{\Gamma})} \mathscr{H}_{v_0}$ is given on each summand $\mathscr{H}_{v_0}$ corresponding to v by pulling back the action from $\mathscr{H}_v$. That is, the true action is a conjugated action.

In the *commutative case*, this is not an issue as the representation is exactly the diagonal representation.

2.1.7. Geometry in the Commutative Case. If $\mathscr{B}$ is commutative (and hence also $\mathscr{A}$),[3] then there is a geometric version of these algebras which can be understood as the spectra of a family of Hamiltonians over a base. We have the following inclusion of commutative C^* algebras $i : \mathscr{A} \hookrightarrow \mathscr{B}$, by Gel'fand representation theorem of the commutative Gel'fand–Naimark theorem this gives us a surjection of compact Hausdorff spaces[4] $\pi : Y \to X$ where $C(X) \simeq \mathscr{A}$ and $C(Y) \simeq \mathscr{B}$. The correspondence is given via characters. Namely, a character is a C^*-homomorphism $\chi : \mathscr{A} \to \mathbb{C}$. The characters are by definitions the points of X. Vice versa any point $t \in X$ determines a character ev_t : $C^*(X) \to \mathbb{C}$ via evaluation. That is any $f \in C^*(X)$ is sent to $f(t) \in C$. Given a character χ on $\mathscr{A}$, we can lift it to a C^*-morphism $\hat{\chi} : M_k(\mathscr{A}) \to M_k(\mathbb{C})$ by applying it in each matrix entry.

Thus, any point $t \in X$ represented by the character χ determines a Hamiltonian $\hat{\chi}(H) \in M_k(\mathbb{C})$ via $\hat{\chi}$, namely

$$(\hat{\chi}(H))_{ij} = \chi(H_{ij}) \tag{5}$$

Thus, we get a family of Hamiltonians $H(t)$ parameterized over the base. One can furthermore check, see [2], that π is a branched cover over X with $\pi^{-1}(t) = \mathrm{spec}(H(t))$.

2.2. Physical Example: PDG and Honeycomb Wire Networks

The PDG examples are based on the unique triply periodic constant mean curvature (CMC) surfaces where the skeletal graph is symmetric and self-dual. Physically, in the P (primitive), D (diamond) and G (Gyroid) case, one starts with a "fat" or thick version of this surface, which one can think of as an interface. A solid-state realization of the "fat" Gyroid aka. double Gyroid has recently been synthesized on the nano-scale [1]. The structure contains three components, the "fat" surface or wall and two channels. Urade et al. [1] have also demonstrated a nanofabrication technique in which the channels are filled with a metal, while the silica wall can be either left in place or removed. This yields two wire networks, one in each channel. The graph we consider and call Gyroid graph is the skeletal graph of one of these channels. The graph Hamiltonian we construct algebraically below is the tight-binding Harper-like Hamiltonian for one channel of this wire network. The 2D analog is the honeycomb lattice underlying graphene. Graph theoretically the quotient graph for the honeycomb is the 2D version of that of the D surface, but as we

[3] This is for instance the case in the applications if the magnetic field vanishes.
[4] Both $\mathscr{A}$ and $\mathscr{B}$ are unital.

showed, the behavior, such as the existence of Dirac points [5], is more like that of the Gyroid surface.

To formalize the situation, we regard the skeletal graph of one channel as an embedded graph $\Gamma \subset \mathbb{R}^n$. The crystal structure gives a maximal translational symmetry group L which is a mathematical lattice, i.e., isomorphic to $\mathbb{Z}^n$, s.t. $\bar{\Gamma} := \Gamma/L$ is a *finite* connected graph. The vertices of this quotient graph are the elements in the primitive cell.

The Hilbert space $\mathscr{H}$ for the theory is $\ell^2(V(\Gamma))$, where $V(\Gamma)$ are the vertices of Γ. This space splits as $\bigoplus_{v \in V(\bar{\Gamma})} \mathscr{H}_v$ where for each vertex $v \in V(\bar{\Gamma})$, $\mathscr{H}_v = \ell^2(\pi^{-1}(v))$ where $\pi : \Gamma \to \bar{\Gamma}$ is the projection. All the spaces $\mathscr{H}_v$ are separable Hilbert spaces and hence isomorphic. Furthermore, if $\overrightarrow{e}$ is a directed edge from v to w in $\bar{\Gamma}$, then it lifts uniquely as a vector to $\mathbb{R}^n$, which we denote by the same name. Moreover, for each such vector, there is a naturally associated translation operator $T_{\overrightarrow{e}} : \mathscr{H}_v \to \mathscr{H}_w$, by the usual action of space translations on functions. We also allow for a constant magnetic field $B = 2\pi\hat{\Theta}$ where $\hat{\Theta} = \sum \theta_{ij} \mathrm{d}x_i \wedge \mathrm{d}x_j$ is a constant 2-form given by the skew-symmetric matrix $\Theta = (\theta_{ij})_{ij}$. If $\Theta \neq 0$ then the translations become magnetic translations or Wannier operators $U_{\overrightarrow{e}} : \mathscr{H}_v \to \mathscr{H}_w$; see, e.g., [8]. These operators are still unitary and give partial isometries when regarded on $\mathscr{H}$ via projection and inclusion, which we again denote by the same letter. The Harper-like Hamiltonian is then defined by Eq. (1).

Likewise, L acts by magnetic translations. If $\overrightarrow{\lambda}$ is a vector in L then $U_{\overrightarrow{\lambda}}$ sends each $\mathscr{H}_v$ to itself and the diagonal action gives an action on $\mathscr{H}$. Since L is a lattice, the representation it generates is given by n linearly independent unitaries. The commutation relations among the Wannier operators amount to the fact that the representation is a copy of $\mathbb{T}^n_\Theta$, the noncommutative n-torus, see [2] or [8]. The direct sum yields the global symmetry representation $\alpha : \mathbb{T}^n_\Theta \to B(\mathscr{H})$. This representation and H by definition generate the Bellissard–Harper C^*-algebra $\mathscr{B}$.

We see that this is an example of the general setup where the groupoid representation of $\bar{\Gamma}$ is given by the $U_{\overrightarrow{e}}$. Here, $\mathscr{A} = \mathbb{T}^n_\Theta$ and the Hamiltonian associated to the graph is the Harper-like Hamiltonian. If $B = 0$ then the situation is fully commutative. Notice that $\mathscr{A}$ being commutative just means that the fluxes though the L lattice are 0. If $\mathscr{B}$ is commutative, this means the fluxes cancel according to the entries of H. If there are vertices with more than one edge, potentially the situation is commutative, but not fully commutative.

The original example of the Harper Hamiltonian corresponds to the quotient graph which has one vertex and two loops. Here, the graph Γ is the square lattice and L is $\mathbb{Z}^2 \subset \mathbb{R}^2$ and the Harper Hamiltonian is given by $H = U_{\overrightarrow{e_1}} + U_{\overleftarrow{e_1}} + U_{\overrightarrow{e_2}} + U_{\overleftarrow{e_2}}$ where e_1 and e_2 are the two loops and $\overrightarrow{e_i}, \overleftarrow{e_i}$ are their two orientations. This is the 2D analog of the P graph and the geometry for the quantum Hall effect [8,9]. In fact, in all dimensions, there is the example of $\mathbb{Z}^n \subset \mathbb{R}^n$ with $\mathbb{Z}^n$ acting by translations. We call this the Bravais case. The quotient graph is a bouquet of S^1s or a petal graph with one vertex and n loops.

2.2.1. Geometric Construction in the Commutative Toric Non-Degenerate Case. If we are in the fully commutative toric non-degenerate case, there is a nice geometric construction yielding the Hilbert spaces and operators due to [10,11]. By basic covering space theory [12], every finite graph $\bar{\Gamma}$ when endowed with the standard topology, e.g., as a CW complex, has a maximal Abelian cover $\hat{\Gamma}$. This means that $\hat{\Gamma}$ is a topological space with a is free action of the maximal Abelian quotient of the fundamental group, $\mathbb{Z}^{b_1} = \pi(\bar{\Gamma}, v_0)/[\pi(\bar{\Gamma}, v_0), \pi(\bar{\Gamma}, v_0)]$, such that $\hat{\Gamma}/\mathbb{Z}^{b_1} = \bar{\Gamma}$. Following [10,11], the abstract space $\hat{\Gamma}$ can actually be embedded as a graph $\Gamma \subset \mathbb{R}^{b_1} \simeq H_1(\bar{\Gamma}, \mathbb{R})$ with $\mathbb{Z}^{b_1}$ acting on the ambient $\mathbb{R}^{b_1}$ inducing the covering action on Γ.[5] The analysis of the last paragraph then applies to Γ yielding $\mathscr{H}_{v_0}$ and the action of $\mathscr{A}$ geometrically rather than abstractly via the GNS theorem.

2.2.2. Rational Flux. It θ is rational, which corresponds to rational magnetic flux per unit cell, then the rotation algebra $\mathbb{T}_\Theta$ actually has a matrix representation and the framework can be rewritten in terms of matrices with commutative entries at the expense of considering a smaller translational lattice. This procedure is explained in [8] and the references therein. This will be considered in future work.

3. Symmetries

To deal with the symmetries of the graph, it is helpful to first fix the notation as sometimes questions become subtle.

3.1. Classical Symmetries

A graph Γ is described by a set of vertices V_Γ and a set of edges E_Γ together with incidence relations ∂ where for each edge e, $\partial(e) = \{v, w\}$ is the unordered set of the two vertices it is incident to. A directed edge is given by an order on this pair. Hence, for each edge e, there are two ordered edges by the orders (v, w) and by (w, v). We usually denote these two edges by $\overrightarrow{e}$ and $\overleftarrow{e}$. The set of all oriented edges is called E_Γ^{or}.

An isomorphism ϕ of two graphs Γ and Γ' is a pair of bijections (ϕ_V, ϕ_E) $\phi_V : V_\Gamma \to V_{\Gamma'}$ and $\phi_E : E_\Gamma \to E_{\Gamma'}$. The compatibility is that the incidence conditions are preserved: if $\partial(e) = \{v, w\}$ then $\partial(\phi_E(v)) = \{\phi_V(v), \phi_V(w)\}$. Notice the ϕ also induces a map of oriented edges, the orientation of the edge $\phi_E(e)$ given by $(\phi_V(v), \phi_V(w))$, if the orientation of e is (v, w).

We will treat isomorphism classes of graphs from now on. Fixing an isomorphism class of a graph still allows for automorphisms. These are given as follows. Fix a representative of Γ then an automorphism is a pair of compatible maps (ϕ_V, ϕ_E); $\phi_V : V_\Gamma \to V_\Gamma$ and $\phi_E : E_\Gamma \to E_\Gamma$.

Example 3.1. Let us illustrate this for the graphs corresponding to the PDG and honeycomb cases which are given in Fig. 1; see Sect. 2.2 for details about

[5] This graph is called canonical placement and has an energy minimizing property. This theory also works for smaller free Abelian covers.

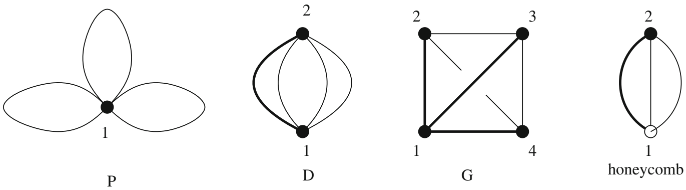

FIGURE 1. The graphs P, D and G and honeycomb together with the preferred spanning tree and order

the corresponding wire networks. We fix once and for all an isomorphism class of these graphs and then consider their automorphisms using the representatives given in the figure.

For the P case, there is only one vertex hence $\phi_V = id$ is the only possibility. However, there is an $\mathbb{S}_3$ action permuting the three loop edges.

The D graph has the possibility of switching the two vertices and freely permuting the three edges. This gives the automorphism group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{S}_3$. The honeycomb similarly has automorphism group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{S}_3$.

For the Gyroid, there is an $\mathbb{S}_4$ worth of potential choices for ϕ_V. Now, all these choices extend uniquely to the edges, since there is exactly one edge between each distinct pair of vertices and hence the symmetry group is exactly $\mathbb{S}_4$.

3.1.1. Pushing Forward Spanning Trees and Orders. Given a pair (Γ, τ) of a graph and a rooted spanning tree, we define the action of isomorphisms and automorphisms by push-forward. That is an isomorphism between (Γ, τ) and (Γ', τ') is an isomorphism from Γ to Γ' such that ϕ_V maps the root of τ to the root of τ' and ϕ restricted to τ is an isomorphism onto τ'.

If we have not already specified a spanning tree on Γ', we can extend any isomorphism ϕ from (Γ, τ) to it by push-forward. This means that we push-forward all the vertices and the edges of the spanning tree τ to Γ': $E_{\tau'} := \phi_E(E_\tau)$ and likewise push-forward the root.

In particular, $\mathrm{Aut}(\Gamma)$ acts on the set of spanning trees of a fixed graph Γ. This action is not transitive in general and may have fixed points.

If there is an order on all the vertices, then the isomorphisms are asked to be compatible with this order and auto- and isomorphisms can be extended by pushing forward the order.

Example 3.2. In the cases of PDG and the honeycomb, it is a transitive action.

For the G graph, the action is not fixed point free, there is an $\mathbb{S}_3$ subgroup fixing a given spanning tree.

For the P graph, the action is fixed point free, while for the D and the honeycomb although the action is transitive, there are again stabilizers. For the honeycomb, the group fixing a spanning tree is the $\mathbb{S}_2 = \mathbb{Z}/2\mathbb{Z}$ interchanging the two other edges, with both vertices fixed, while in the D case it is an $\mathbb{S}_3$ action interchanging the edges which are not part of the spanning tree.

As an example of a non-transitive action consider the triangle graph, with one edge doubled. That is three vertices $1, 2, 3$ with one edge between 1 and 2, one edge between 1 and 3 and two edges between 2 and 3.

3.1.2. Classical vs. Extended Symmetries. The final piece of data we consider on a graph is a weight function wt as in Sect. 2.1.6. Weight functions naturally pull back via ϕ_E, that is $\phi_E^*(wt)(\vec{e}) = wt(\phi_E(\vec{e}))$. Using that ϕ_E is an isomorphism, one can push-forward by pulling back along ϕ_E^{-1}. A natural choice of iso- or automorphism for graphs with weight functions is to demand that the weight functions agree: $wt(\vec{e}) = wt(\phi_E(\vec{e}))$.

One could call these symmetries *classical symmetries of the weighted graph.* These are the kinds of symmetries that were for instance considered by [3]. These symmetries are rather restrictive, for instance the normalization condition (2.1.6) might not be compatible with the new, pushed forward spanning tree.

We will consider symmetries of the *underlying graph*, not of the weighted graph. The weights are taken care of by re-gauging. The idea of re-gauging is to re-establish condition (2.1.6) by passing to an equivalent, re-gauged weight function.[6] To this end, we utilize an extended symmetry group which allows for phase factors at the vertices. The details are given below.

3.2. Gauging

We will now consider the relationship between the different matrices $H_{\boldsymbol{\tau}}$ and $H_{\boldsymbol{\tau}'}$ for different gauging data, which represent H via different isomorphisms. Here, and in the following, we use the notation: $\boldsymbol{\tau} = (\tau, v_0, <)$, $\boldsymbol{\tau}' = (\tau', v_0', <')$, etc.

There are basically three situations: First, $\tau = \tau'$ as rooted trees and only the order changes. Second, $\tau = \tau'$ but $v_0 \neq v_0'$ and third, the trees just do not coincide, i.e., at least one edge is different. In the first case, the isomorphisms are simply changed by permutations of the factors $\mathscr{H}_{v_0}$. This means that the difference between the two matrix Hamiltonians is simply conjugation by M_σ, the standard permutation matrix. Since v_0 is fixed to be the first element, the permutation actually lies in the subgroup $\mathbb{S}_{k-1} \subset \mathbb{S}_k$ fixing the first element. $\mathbb{S}_{k-1}$ then acts simply transitively on the orders.

In the second and third case, the situation is more complicated. Of course the second type of change is related to an action of $\mathbb{S}_k$, but things are not that simple, since there is a change of the space that $H_{\boldsymbol{\tau}'}$ acts on. If the tree moves, then we have to also change the weight functions to make them compatible with the new spanning tree.

Taking the point of view of π_1's, the first type of transformation is just a permutation of the basis. But when we move the base point, we move to an isomorphic group. In doing this, we effectively use the path groupoid and not just the fundamental group.

---

[6] One way to see the failure to preserve the condition (2) of Sect. 2.1.6 is to view the change of spanning tree as a change of basis of π_1 or rather an isomorphism of π_1s via the path groupoid. This point of view is at the basis of the proof of Theorem 3.10 below.

3.2.1. Gauging in Groupoid Representations. As discussed in Sect. 2.1, the Harper-like Hamiltonian *before* fixing a spanning tree can be thought of as a certain type of groupoid representation.

For such representation, we can re-gauge it to an equivalent representation by acting with any of choice automorphisms of the $\mathscr{H}_v$, that is the group $\times_{v\in\Gamma}\mathrm{Aut}(\mathscr{H}_v)$. Picking an element ϕ in this group is the same as the assignment $v \mapsto \phi(v) \in \mathrm{Aut}(\mathscr{H}_v)$. The operators $U_{\vec{e}} : \mathscr{H}_v \to \mathscr{H}_w$, where $\vec{e} = (v, w)$ get re-gauged to $\phi(w)U_{\vec{e}}\phi^{-1}(v)$. Again, one has to be careful with the indexing of the direct sums. Since there is no natural order, there is a natural $\mathbb{S}_k$ action by permutations which interacts with the diagonal re-gaugings via the wreath product.

In our situation, since we have Hilbert spaces, we can look at unitary equivalences and restrict the automorphisms to be unitary. Note that the gauge group is smaller than the full group of unitary equivalences $U(\mathscr{H})$.

Also, choosing an identification of all the isomorphic separable Hilbert spaces $\mathscr{H}_v$ with some fixed $\mathscr{H}_{v_0}$ we can take the re-gaugings to live in the unitary operators on $\mathscr{H}_{v_0}$.

In this situation, the gauge group becomes $G = U(\mathscr{A})^k \wr \mathbb{S}^k$, where $\wr$ denotes the wreath product. It acts on the orders, the weight functions and on the Hamiltonians by conjugation and permutation just as above.

3.2.2. Spanning Tree Re-Gauging.

Proposition 3.3. *Given two ordered rooted spanning trees* $\boldsymbol{\tau}$ *and* $\boldsymbol{\tau}'$*, there is a matrix* $M \in M_k(\mathscr{A})$ *with* $MM^* = M^*M = id$ *such that* $MH_{\boldsymbol{\tau}}M^* = U^{\tau}_{v_0v_0'}H_{\boldsymbol{\tau}'}U^{\tau}_{v_0'v_0}$. *Moreover,* M *is an element of the gauge group.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccccccc}
\bigoplus_{i=1}^k \mathscr{H}_{v_0} & \xrightarrow{\bigoplus_i U^{\tau}_{v_0v_i}} & \bigoplus_i \mathscr{H}_{v_i} = \mathscr{H} & \xrightarrow{H} & \mathscr{H} = \bigoplus_i \mathscr{H}_{v_i} & \xrightarrow{\bigoplus_i U^{\tau}_{v_iv_0}} & \bigoplus_i \mathscr{H}_{v_0} \\
\downarrow{\scriptstyle \bigoplus U^{\tau}_{v_0v_0'}} & & \downarrow{\scriptstyle \sigma} & & {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \bigoplus U^{\tau}_{v_0v_0'}} \\
\bigoplus_{i=1}^k \mathscr{H}_{v_0'} & \xrightarrow{\bigoplus_i U^{\tau'}_{v_0'v_i'}} & \bigoplus_i \mathscr{H}_{v_i'} = \mathscr{H} & \xrightarrow{H} & \mathscr{H} = \bigoplus_i \mathscr{H}_{v_i'} & \xrightarrow{\bigoplus_i U^{\tau'}_{v_i'v_0'}} & \bigoplus_i \mathscr{H}_{v_0'}
\end{array} \tag{6}$$

We see that if $i' = \sigma(i)$ and $j' = \sigma(j)$ so that $v'_{i'} = v_i$:

$$\begin{aligned}
(H_{\boldsymbol{\tau}'})_{i'j'} &= U^{\tau'}_{v_0'v'_{i'}}H_{v'_{i'},v'_{j'}}U^{\tau'}_{v'_{j'},v_0'} \\
&= U^{\tau}_{v_0'v_0}(U^{\tau}_{v_0v_0'}U^{\tau'}_{v_0'v'_{i'}}U^{\tau}_{v_iv_0})(U^{\tau}_{v_0v_i}H_{v_iv_j}U^{\tau}_{v_jv_0})(U^{\tau}_{v_0v_j}U^{\tau'}_{v'_{j'},v_0'}U^{\tau}_{v_0'v_0})U^{\tau}_{v_0v_0'} \\
&= U^{\tau}_{v_0'v_0}\phi^*_{i'}(H_{\boldsymbol{\tau}})_{ij}\phi_{j'}U^{\tau}_{v_0v_0'}
\end{aligned}$$

With $\phi_{j'} = U^{\tau}_{v_0v_j}U^{\tau'}_{v'_{j'},v_0'}U^{\tau}_{v_0'v_0} \in U(\mathscr{A})$. So that if $\Phi = \mathrm{diag}(\phi_{i'})$ and M_σ is the permutation matrix of σ which moves the order $<$ to $<'$ then $M = \Phi^* M_{\sigma^{-1}}$ and $M^*M = id$. □

We choose to place M on the left of the Hamiltonian so as to get a left action later on.

Remark 3.4. Unraveling the definition given in Eq. (2), we can express the matrix Φ as a re-gauging by the following iterative procedure. We start at the root of τ' and choose $\phi(v_0') = id$. Assume we have already assigned weights to all vertices at distance i from v_0' and let w be a vertex at distance $i+1$. Then, there is a unique v at distance i which is connected to w along a unique directed edge $\vec{e}$ of the spanning tree τ'. Set $\phi(w) = wt(\vec{e})\phi(v) \in U(\mathscr{A})$. Then, $\Phi = \mathrm{diag}_{v_i' \in \tau'}(\phi(v_i'))$.

Of course, the form of M depends on the initial choice of $\phi(v_0) = id$, which amounts to using the iso $U^{\tau}_{v_0 v_0'}$ to pull-back the matrix. Any other choice of iso will differ by an element of $\mathscr{A}$ which is then the value of ϕ on v_0. This plays a crucial role later.

3.2.3. Commutative Case And Reduced Gauge Group. In the commutative case, we can fix a character $\chi : \mathscr{A} \to \mathbb{C}$ and then under $\hat{\chi}$ all matrices become $U(1)$ valued and all the Hilbert spaces $\mathscr{H}_v$ become identified with $\mathbb{C}$. In this case, we can identify the gauge group action with an action of $U(1)^{\times V_\Gamma}$ on $U(1)$-valued weight functions, using $\lambda = \chi \circ \phi$. For every oriented edge $\vec{e}$ from v to w, the re-gauged weights are

$$wt'(\chi(\vec{e})) := \lambda(v)\chi(wt(\vec{e}))\bar{\lambda}(w)$$

Notice that we have taken the indexed unordered product. If we fix an order of the vertices, then the group $\mathbb{S}_k$ acts on the vertices as well and the full gauge group which acts on the Hamiltonians by conjugation is the wreath product $G = U(1) \wr \mathbb{S}_k$.

We see that the constant functions λ act trivially and hence to get a more effective action we can quotient by the diagonal $U(1)$ action and consider the reduced gauge group $\bar{G} := G/U(1)$, where $U(1)$ is diagonally embedded in $U(1)^k$ and $\mathbb{S}_k$ acts trivially.

Abstractly $U(1)^k/U(1) \simeq U(1)^{k-1}$, to make this explicit, we can choose a section of $\bar{G} \to G$. Our choice $\phi(v_0) = 1$ is just such a choice of a section. The action of $\mathbb{S}_k$ on the remaining $k-1$ factors is then more involved, however. It is still a semi-direct product, but not a wreath product any more. This has practical relevance in the Gyroid case.

The proof of the theorem above then boils down to the fact that a rooted spanning tree uniquely fixes a unique gauge transformation as follows. We let $\lambda(\mathrm{root}) = 1$ by the global gauge $U(1)$. Now, the weight on each vertex of the tree is fixed iteratively by the condition that $\lambda wt(e) = 1$. The whole set of weights then gives a diagonal unitary matrix and taking the product with the appropriate permutation, we obtain the matrix M.

3.3. Re-Gauging Groupoid $\mathcal{G}$, Representations, Cocycles and Extensions

To keep track of all the re-gaugings and ultimately find the extended symmetries, we introduce the following abstract groupoid $\mathcal{G}$. It has rooted ordered spanning trees $\boldsymbol{\tau}$ of $\bar{\Gamma}$ as objects and a unique isomorphism between any two such pairs. If the two pairs coincide, the isomorphism is the identity map.

Having fixed the representation $(\mathscr{H}, U)$, there is an induced representation ρ of $\mathcal{G}$ which also takes values in separable Hilbert spaces. On objects it is given by $\rho(\boldsymbol{\tau}) = \mathscr{H}^k_{v_0}$, v_0 being the base point of τ. For a re-gauging morphism $g : \boldsymbol{\tau} \to \boldsymbol{\tau}'$ we set $\rho(g) = U^{\tau}_{v_0' v_0} M$ for the M of Proposition 3.3. Plugging into the definitions, one checks that indeed $\rho(g)\rho(h) = \rho(gh)$ for composable g and h.

To find the symmetry groups, we will however need to consider only the matrix "M" part of ρ. This is not a representation, but gives rise to a noncommutative 2-cocycle and moreover, this cocycle can be lifted to the groupoid level.

3.4. Induced Structures and Cocycles

To understand the cocycle, let us first consider the "U"—part of ρ. For this, we notice that there is a functor $p : \mathcal{P}_{\mathcal{G}} \to \mathcal{P}_{\bar{\Gamma}}$ from the path space of $\mathcal{G}$ to that of $\bar{\Gamma}$. It is given by $p(\boldsymbol{\tau}) = v_0$ and $p(\boldsymbol{\tau} \xrightarrow{g} \boldsymbol{\tau}') = \gamma^{\tau}_{v_0' v_0}$, the shortest path of Sect. 2.1.2. We can now compose with $(\mathscr{H}, U)$ and obtain $\nu := p \circ (\mathscr{H}, U)$ on objects and morphisms, i.e., for $g : \boldsymbol{\tau} \to \boldsymbol{\tau}'$ we have $\nu(g) = U^{\tau}_{v_0' v_0} : \mathscr{H}_{v_0} \to \mathscr{H}_{v_0'}$. This is not a representation of $\mathcal{G}$, but for g as above and $h : \boldsymbol{\tau}' \to \boldsymbol{\tau}''$ it satisfies

$$\nu(h)\nu(g) = \nu(hg)C^{-}(h, g), \quad \text{with } C^{-}(h, g) := U^{\tau}_{v_0 v_0''} U^{\tau'}_{v_0'' v_0'} U^{\tau}_{v_0' v_0} \in \mathscr{A}_{v_0} \tag{7}$$

For the M part of ρ, the relevant cocycle will actually be the inverse of C^{-}, see also Sect. 3.4.2 below. Explicitly, $C(h, g) = C^{-}(h, g)^{-1} = U^{\tau}_{v_0 v_0'} U^{\tau'}_{v_0' v_0''} U^{\tau}_{v_0'' v_0}$. For three composable morphisms $\boldsymbol{\tau} \xrightarrow{g} \boldsymbol{\tau}' \xrightarrow{h} \boldsymbol{\tau}'' \xrightarrow{k} \boldsymbol{\tau}'''$, one obtains the following equation for C by plugging in:

$$C(h, g)C(k, hg) = \nu(g)^{-1}\nu(h)^{-1}\nu(k)^{-1}\nu(khg) =: C(k, h, g) \tag{8}$$

And if we denote conjugation of x by y with an upper left index ${}^{y}x = yxy^{-1}$ to keep with standard notation [13–15], we find the cocycle equation

$${}^{\nu(g)^{-1}}C(k, h)C(kh, g) = C(h, g)C(k, hg) \tag{9}$$

One can also lift the cocycle C to a cocycle l with values in $\mathcal{L}$:

$$l(h, g) = \gamma^{\tau}_{v_0 v_0'} \gamma^{\tau'}_{v_0' v_0''} \gamma^{\tau}_{v_0'' v_0} \in \pi_1(\bar{\Gamma}, v_0) \tag{10}$$

l satisfies the analogous equation to (8) with ν replaced by p: $p(l(h, g)) = C(h, g)$.

3.4.1. Matrix Version and Cocycle. To do calculations, it is preferable to work with a matrix representation of the groupoid action. The problem is that although the groupoid associates a matrix to each re-gauging, these matrices all act in different spaces. To make everything coherent one has to use pull-backs. Explicitly, for $\boldsymbol{\tau} \xrightarrow{g} \boldsymbol{\tau}'$ we set $Mat(g) := M_g := M \in \mathscr{A}_{v_0}$ of Proposition 3.3. If we have another re-gauging $\boldsymbol{\tau}' \xrightarrow{h} \boldsymbol{\tau}''$ then we cannot directly multiply the matrices M_g and M_h as they have coefficients in different algebras. We therefore define the product $M_h \circ_{\boldsymbol{\tau}} M_g := U^{\tau}_{v_0 v_0'} M_h U^{\tau}_{v_0' v_0} M_g$.[7] A straightforward calculation shows:

Proposition 3.5. *$M_h \circ_{\boldsymbol{\tau}} M_g = C(h,g) M_{hg}$ with the same cocycle $C(h,g)$ as above.* ☐

Again, by a straightforward calculation:

Lemma 3.6. *If $\mathscr{A}$ is commutative, then the product is independent of the choice of pull-back $U^{\tau}_{v_0 v}$. Defining the product using conjugation by any $U(\gamma)$ with γ a contractible path from v_0 to v_0' will give the same result.* ☐

Corollary 3.7. *If the situation is fully commutative, we can use the $\alpha(*v)$ to pull back all the matrices M_g to matrices with coefficients in $\mathscr{A}$. Then, the multiplication above simply becomes matrix multiplication in $M_k(\mathscr{A})$.* ☐

3.4.2. Groupoid Cocycles and Extension. The data of ν and C as well as p and l technically yield a crossed noncommutative groupoid 2-cocycle [13,14,16]. To get one of the standard forms of the cocycle, e.g., that of [13], we will have to transform the pairs (p,l) and (ν, C) a bit. It turns out that everything is more natural in the opposite groupoid Γ^{op} of the groupoid Γ. This is because we are actually *re*-gauging. On the groupoid level, define $p^{\mathrm{op}} : \mathcal{G}^{\mathrm{op}} \to \mathcal{P}_{\bar{\Gamma}}$ and $l^{\mathrm{op}} : \mathcal{G}_1^{\mathrm{op}}{}_t \times_s \mathcal{G}_1^{\mathrm{op}} \to \mathcal{L}$

$$p^{\mathrm{op}}(g^{\mathrm{op}}) := \gamma^{\tau}_{v_0 v_0'}, \quad l^{\mathrm{op}}(g^{\mathrm{op}}, h^{\mathrm{op}}) := l(h,g) \tag{11}$$

And similarly hitting the above maps with $(\mathscr{H}, U)$, we get $\nu^{\mathrm{op}} : \mathcal{G}^{\mathrm{op}} \to \tilde{B}$ and $C^{\mathrm{op}} : \mathcal{G}_1^{\mathrm{op}}{}_t \times_s \mathcal{G}_1^{\mathrm{op}} \to \tilde{A}$

$$\nu^{\mathrm{op}}(g^{\mathrm{op}}) := \nu(g)^{-1}, \quad C^{\mathrm{op}}(g^{\mathrm{op}}, h^{\mathrm{op}}) := C(h,g) \tag{12}$$

Now, $\mathcal{L}$ is a $\mathcal{P}_{\bar{\Gamma}}$ crossed module via the inclusion $i : \mathcal{L} \to \mathcal{P}_{\bar{\Gamma}}$ and the conjugation action $\Phi : \mathcal{P}_{\bar{\Gamma}} \times \mathcal{L} \to \mathcal{P}_{\bar{\Gamma}} : (\gamma, l) \mapsto \gamma l \gamma^{-1}$. Analogously, $\tilde{A}$ is a $\tilde{B}$ crossed module via inclusion and conjugation action.

Proposition 3.8. *The pair $(p^{\mathrm{op}}, l^{\mathrm{op}})$ are an element of $C^2_{\mathcal{P}_{\bar{\Gamma}}}(\mathcal{G}, \mathcal{L})$ that is a $\mathcal{P}_{\bar{\Gamma}}$-crossed $\mathcal{G}$ 2-cocycle with values in $\mathcal{L}$. Likewise, the pair $(\nu^{\mathrm{op}}, C^{\mathrm{op}})$ are an element of $C^2_{\tilde{B}}(\mathcal{G}^{\mathrm{op}}, \tilde{A})$.*

Groupoid extension By general theory, [13,16,17] the noncommutative cocycle (p,l) gives rise to a groupoid extension (Σ, b) over $\mathcal{P}_{\bar{\Gamma}}$

$$\Sigma : 1 \to \mathcal{L} \to \hat{\mathcal{G}} \to \mathcal{G} \to 1 \quad b : \hat{\mathcal{G}} \to \mathcal{P}_{\bar{\Gamma}} \tag{13}$$

Remark 3.9. It is this extension via Mat that gives rise to the projective representation of $\mathrm{Sym}(\bar{\Gamma})$ in the commutative case. In the noncommutative case, the geometry begins to look like a gerbe geometry. This fits with the non-commutativity being given by a 2-form B-field. We leave this for further study.[8]

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[7] Notice that here $U^{\tau}_{v_0' v_0}$ is taken to be a "scalar" that is it acts as the $k \times k$ diagonal matrix $\mathrm{diag}(U^{\tau}_{v_0' v_0}, \ldots, U^{\tau}_{v_0' v_0}) : \mathscr{H}^k_{v_0} \to \mathscr{H}^k_{v_0'}$.

[8] The constructions we have presented have a more high-brow explanation in terms of Kan extensions. The main observation is that each spanning tree τ gives a functor from F_τ:

3.4.3. Re-Gaugings Induced by Graph Symmetries. Given a symmetry ϕ, aka. automorphism of the graph $\bar{\Gamma}$, and a fixed choice of a rooted ordered spanning tree $\boldsymbol{\tau}$, we can push- forward all the data contained in $\boldsymbol{\tau}$ with ϕ to obtain another rooted ordered spanning tree $\boldsymbol{\tau}'$. This means that for every $H_{\boldsymbol{\tau}}$ any automorphism ϕ gives rise to a re-gauging of $H_{\boldsymbol{\tau}}$ to $H_{\boldsymbol{\tau}'}$. Abstractly, for every element $\boldsymbol{\tau}$ of $\mathcal{G}_0$, we have a map of $\mathrm{Sym}(\bar{\Gamma}) \to \mathcal{G}_1$ and in total a map $\mathrm{Sym}(\bar{\Gamma}) \times \mathcal{G}_0 \to \mathcal{G}_1 : (\phi, \boldsymbol{\tau}) \mapsto \boldsymbol{\tau}'$ where $\mathcal{G}_1$ are the morphisms in $\mathcal{G}$ and $\mathcal{G}_0$ are the underlying objects. These re-gaugings will give rise to the irreducible representations forcing the degeneracies in the spectrum, that is the singular points of the cover $Y \to X$ of Sect. 2.1.7, as we now explain.

3.4.4. Lifts to Automorphisms. One interesting question for any given re-gauging is if there are automorphisms ψ of $\mathscr{A}$ such that

$$\hat{\psi}(H_{\boldsymbol{\tau}}) = U^{\tau}_{v_0 v_0'} H_{\boldsymbol{\tau}'} U^{\tau}_{v_0' v_0} \tag{14}$$

where, again, $\hat{\psi}$ is ψ applied to the entries. This is the type of enhanced, extended symmetry we will use in the commutative case.

One way such a symmetry can arise is by a re-gauging induced by an automorphism ϕ of $\bar{\Gamma}$. A stricter requirement that is easier to handle is that not only the matrix coefficients of the Hamiltonian transform into each other, but rather already the weight functions. This avoids dealing with sums of weights. We say a re-gauging induced by an automorphism ϕ of $\bar{\Gamma}$ is *weight liftable by an automorphism* ψ of $\mathscr{A}$ if $\psi(wt(\vec{e})) = wt'(\phi(\vec{e}))$, where wt' is the re-gauged weight function for the pushed forward spanning tree.

Theorem 3.10. *Given an automorphism ϕ of $\bar{\Gamma}$, there is at most one weight lift by an automorphism ψ of the re-gauging induced by ϕ. On the generators $wt(\vec{e})$, e not a spanning tree edge, the putative map is fixed by the condition $\psi(wt(\vec{e})) := wt'(\phi(\vec{e}))$, where wt' is the re-gauged weight function.*

Furthermore, the $\psi(wt(\vec{e}))$ again generate $\mathscr{A}$ and hence whether ψ indeed defines an automorphism only needs to be checked on the generators $wt(\vec{e})$.

Lastly, ψ is induced by a base change of $\pi_1(\bar{\Gamma})$.

Proof. Let wt' be the re-gauged weights after moving from $\boldsymbol{\tau}$ to $\boldsymbol{\tau}'$. If an automorphism ψ of $\mathscr{A}$ that lifts ϕ exists, then it satisfies $wt'(\phi(\vec{e})) = \psi(wt(\vec{e}))$. After fixing an orientation for each edge, the $wt(\vec{e})$ generate, we see that the morphism is already fixed, since by assumptions the $wt(\vec{e})$ generate $\mathscr{A}$.

To show that the $\psi(wt(\vec{e}))$ are generators, we will prove the last statement first. As discussed in Sect. 2.1.4, wt gives a representation ρ of $\pi_1(\bar{\Gamma}, v_0)$ and wt' gives a representation ρ' of $\pi_1(\bar{\Gamma}, v_0')$ if v_0 is the root of τ and $v_0' = \phi(v_0)$ is the root of τ', the pushed forward spanning tree. In τ, there is a canonical

Footnote 8 continued
$\mathcal{P}\bar{\Gamma} \to \mathcal{P}_{\bar{\Gamma}/\tau}$ where $\bar{\Gamma}/\tau$ is result of contracting τ. The re-gauging groupoid compares all the left Kan extensions $\mathrm{Lan}_{\tau}(\mathscr{H}, U) : \mathcal{P}_{\bar{\Gamma}/\tau} \to \mathrm{Hilb}$, where Hilb is the category of separable Hilbert spaces.

shortest path $p^{\tau}_{v_0 v_0'}$ from v_0' to v_0. Conjugating by this path gives an isomorphism $P : \pi_1(\bar{\Gamma}, v_0') \to \pi_1(\bar{\Gamma}, v_0)$. This is in essence the definition of the path groupoid of $\bar{\Gamma}$. Let $l^{\tau}(\vec{e})$ be the loop associated to $\vec{e}$ using τ as a spanning tree, see Sect. 2.1.4, then $wt(\vec{e}) = \rho(l^{\tau}(\vec{e}))$. It follows from the definition of the re-gauging that

$$\psi(\rho(l^{\tau}(\vec{e}))) = \psi(wt(\vec{e})) = wt'(\phi(\vec{e})) = \rho'(l^{\tau'}(\phi(\vec{e}))) = \rho(P(l^{\tau'}(\phi(\vec{e}))))$$

so that ψ is induced by the chance of basis $l^{\tau}(\vec{e}) \to P(l^{\tau'}(\phi(\vec{e})))$ in $\pi_1(\bar{\Gamma}, v_0)$. From this, it follows that the $\psi(wt(\vec{e}))$ generates. □

Corollary 3.11. *If the groupoid representation is non-degenerate, so that $\mathscr{A}$ is generated by the $wt(\vec{e})$ and each non-spanning-tree edge gives a linearly independent generator, then the morphism ψ above is well defined as a linear morphism.*

If there are no relations among the generators, e.g., in the case $\mathscr{A} = \mathbb{T}^n$ the commutative algebra of the torus, then every automorphism is weight liftable, i.e., ψ from above is well defined as an algebra homomorphism. □

We will use the corollary in Sect. 3.5 to define the enhanced symmetry groups in the commutative toric non-degenerate case.

3.5. Enhanced Symmetries in the Commutative Case

We will concentrate on the commutative case in the following. One physical feature that makes the noncommutative theory more complicated is that conjugating H with elements from $\mathscr{A}$ usually does not leave it invariant. This is of course the starting point for considering the C^*-algebra $\mathscr{B}$ which contains all these conjugates.

3.5.1. Extension. In the commutative case, $U(\mathcal{L})$ is a commutative group and the 2-cocycle defines a central extension $\tilde{\mathcal{G}}$ of $\mathcal{G}$ by $U(\mathcal{L})$. We can consider the action of this central extension, since the action of $U(\mathcal{L})$ commutes with the Hamiltonians, permutations and the re-gaugings in this case. If we are moreover in the fully commutative case, then using the diagonal embedding of $\mathscr{A}$ we can even make the cocycle take values in $U(\mathscr{A})$ and hence obtain the central extension.

$$1 \to U(\mathscr{A}) \to \tilde{\mathcal{G}} \to \mathcal{G} \to 1 \tag{15}$$

Then, ρ does give a groupoid representation of $\tilde{\mathcal{G}}$.

Remark 3.12. There is a nice geometric interpretation of this in the case of wire networks. Here, the group $U(\mathscr{A})$ corresponds to translations along the lattice L. One can identify the vertices of $\bar{\Gamma}$ with the elements in a chosen primitive cell and likewise one can arrange the spanning tree edges to be inside this cell. When we are re-gauging, we move the base point along the spanning tree edges. After doing this several times, the new root can lie outside the original primitive cell. The cocycle then measures the displacement of the new cell relative to the old cell in terms of an element $\lambda \in L$, more precisely it is just U_{λ}.

3.5.2. Enhanced Symmetry Group. To find degeneracies in the spectrum, we use the characters and then look for fixed points under the induced groupoid action. Using the language of Sect. 2.1.7, given a point $\chi \in \mathscr{A}$, we get a map $\hat{\chi} : \mathrm{Ham}_0 \to M_k(\mathbb{C})$. There is then an induced action of the groupoid on $\hat{\chi}(\mathrm{Ham}_0)$, by pushing forward with this character. It can now happen that $\hat{\chi}(H_{\boldsymbol{\tau}}) = \hat{\chi}(H_{\boldsymbol{\tau}'})$, that is $H_{\boldsymbol{\tau}}(t) = H_{\boldsymbol{\tau}'}(t)$, for the point $t \in X$ corresponding to χ.

For each element $H(t) \in \hat{\chi}(\mathrm{Ham}_0)$, we get its stabilizer group $St(H(t))$ under the induced groupoid action. This is the image of the transitive action of the groupoid on the fiber of $\hat{\chi}$ over $H(t)$. We can identify $St(H(t))$ with the image of that subgroupoid. If this group is not trivial, which means that the fiber is not just a point, we call this group the enhanced symmetry group of $H(t)$. It is realized by re-gaugings, that is conjugation by specific matrices which form a projective representation of the stabilizer group as we presently discuss.

3.5.3. Super-Selection Rules, Projective Representation and Degeneracies. If $St(H) \neq 1$ then this means that the set of all matrices $\hat{\chi}(\rho(g))$ for $g \in St(H(t))$, where we identified g with its defining element in $\mathcal{G}$, all commute with the Hamiltonian $H(t)$ and hence each one and all of them together give super-selection rules. This of course is already a great help in finding the spectrum.

Since ρ is only a groupoid representation of $\tilde{\mathcal{G}}$, we get that $\hat{\chi} \circ \rho$ is a representation of an extension of $St(H(t))$. If we are in the fully commutative case, this extension is central and gives rise to a *projective* representation of $St(H(t))$.

$$1 \to U(1) \to \tilde{St}(H(t)) \to St(H(t)) \to 1 \tag{16}$$

Here, we pulled back with the diagonal embedding, i.e., $U(1)$ is embedded as scalars, viz. diagonal matrices.

To apply the general arguments of representation theory, we will be interested in the class of this extension. These extensions are classified up to isomorphism by $H^2(St(H(t)), U(1))$ [18,19].

We give a brief definition of this cohomology group, as it is important for our calculations (see, e.g., [20]). Let G be a group and A be an Abelian group, which we also write multiplicatively.[9] Set $C^i(G, A) := \mathrm{Map}(G^{\times i}, U(1))$ these are the ith cochains. There is a general differential $d : C^i \to C^{i+1}$ with $d^2 = 0$. We will need the formulas for it on 1- and 2-cochains. If $\lambda \in C^1(G, A)$ then $d\lambda(g, h) = \lambda(g)\lambda(h)\lambda(gh)^{-1}$ and if $c \in C^2(G, A)$ then $dc(g, h, k) = c(h, k)c^{-1}(gh, k)c(g, hk)c(g, h)^{-1}$. Set $Z^2(G, A) := \ker(d : C^2(G, A) \to C^3(G, A))$ and $B^2(G, A) := \mathrm{Im}(d : C^1(G, A) \to C^2(G, A))$. Notice that an element $c \in C^2$ is in Z^2 precisely means that c satisfies the cocycle condition (9) in the Abelian case, where the conjugation action is trivial. Now, $B^2(G, A) \subset Z^2(G, A)$ and $H^2(G, A) := Z^2(G, A)/B^2(G, A)$.

What this means is that we can move to an isomorphic extension using a rescaling $\lambda \in C^1(St(H(t)), U(1))$. Another interesting concrete question is if a given homology class $[c]$ can be represented by a cocycle in a subgroup

[9] We consider A to be a G-module with the trivial action.

of A. This is especially interesting if the subgroup is finite. In our concrete calculations for the Gyroid, we will use for instance $\mathbb{Z}/2\mathbb{Z}$ and this will lead us to consider double covers.

In general, if we identify that the projective action of $St(H(t))$ is isomorphic to an action of a finite group extension $\tilde{St}(H(t))$, then we can use this representation to decompose $\mathbb{C}^k$ into its isotypical decompositions with respect to this group action. If the group is non-Abelian, then there is a chance that some of the irreducible representations in the decomposition are higher dimensional, which implies degeneracies of the order of these dimensions. Again this is present for the Gyroid.

3.5.4. Geometric Lift of the Groupoid Action. To understand the (projective) group action, geometrically in the commutative case, one lifts the action on the Hamiltonians to an action on the underlying geometric space. We will now for concreteness fix $\mathscr{A} = \mathbb{T}^n$,[10] that is the groupoid representation is commutative, toric non-degenerate, as is the case in all crystal examples we consider: PDG, Bravais and Honeycomb.

Finding lifts then means that one considers the commutative diagram

$$\begin{array}{ccc} T^n & \xrightarrow{\;H_{\boldsymbol{\tau}}\;} & \hat{\chi}(\mathrm{Ham}_0) \\ \Big\downarrow{\scriptstyle \Psi^{\boldsymbol{\tau}}_{\boldsymbol{\tau}'}} & \searrow^{H_{\boldsymbol{\tau}'}} & \Big\downarrow{\scriptstyle \Phi^{\boldsymbol{\tau}}_{\boldsymbol{\tau}'}} \\ T^n & \xrightarrow[\;H_{\boldsymbol{\tau}}\;]{} & \hat{\chi}(\mathrm{Ham}_0) \end{array} \tag{17}$$

where the dotted morphism is the lift to be constructed and $H_{\boldsymbol{\tau}}$ is the map $t \mapsto H_{\boldsymbol{\tau}}(t) := \hat{\chi}(H)$ if χ is the character corresponding to t.

The existence of these lifts is not guaranteed in general, and indeed there are examples of re-gaugings that cannot be lifted. A non-liftable example can be produced from the cube graph obtained from the Gyroid graph by quotienting out by the the simple cubic lattice, see [2]. We will show that all lifts stemming from automorphisms of the underlying graph do lift.

Looking at the diagram (17), one consequence of this action is that it lets us pinpoint Hamiltonians with enhanced symmetry group. Using Corollary 3.11 and translating it to the geometric side, we obtain

Proposition 3.13. *Let $(\bar{\Gamma}, wt)$ be a toric non-degenerate weighted graph. In the commutative case, the automorphism group of $\bar{\Gamma}$ lifts via the gauging action to an automorphism group of $\mathbb{T}^n$. That is we get a morphism $\mathrm{Aut}(\bar{\Gamma}) \to \mathrm{Aut}(\mathbb{T}^n)$.*

If a point $t \in T^n$ is a fixed point of a lift of an element $g \in \mathcal{G}$, then, the re-gauging is an enhanced symmetry for the corresponding Hamiltonian, that is $\hat{\chi}(\rho(g))$ commutes with the Hamiltonian $H_{\boldsymbol{\tau}}(t)$. □

Summarizing these results:

Theorem 3.14. *If $(\bar{\Gamma}, wt)$ is commutative and toric non-degenerate, then a stabilizer sub-group G_t of $t \in T^n$ under the induced action of $\mathrm{Aut}(\bar{\Gamma})$ on T^n*

---

[10] Recall that $T^n = (S^1)^{\times n}$ is the n-torus and $\mathbb{T}^n = C^*(T^n)$.

leads to an enhanced symmetry group $St(H(t))$ for the Hamiltonian $H(t)$. This group also has a projective representation via the matrices $\hat{\chi}(\rho(g))$. □

We can exploit the representation theory of this group to get information about degeneracies.

Example 3.15. For commutative toric non-degenerate groupoid representations of symmetric graphs, the re-gaugings by re-orderings are always representable via an automorphism of the graph. If $\sigma \in \mathbb{S}_{k-1}$ permutes the vertices of the spanning tree leaving the root fixed, then the re-gauging lifts as the reordering of the generators and possibly taking $*$ of them. The matrices are just the usual permutation matrices of $\mathbb{S}_{k-1} \subset \mathbb{S}_k$ acting on the last $k-1$ copies of $\mathbb{C}$ in $\mathbb{C}^k$.

Remark 3.16. In the commutative case, the representation of $\pi_1(\bar{\Gamma}, v_0)$ factors through its Abelianization $H_1(\bar{\Gamma})$.

4. Calculations and Results for Wire Networks

In this paragraph, we perform the calculation for the PDG and honeycomb graphs of Fig. 1. These correspond to wire networks as reviewed in Sect. 2.2. In all these situations, Theorem 3.14 applies. The upshot of the following calculations together with the analysis of [5] is:

Summary 4.1. *In all the examples PDG and honeycomb, all the fixed points come from fixed points of re-gauging by enhanced graph symmetries as defined in Sect. 3.5 via the mechanism explained in Sect. 3.4.3. Moreover, the fixed points, stabilizer groups, their extensions and the decomposition into irreps for the case of the Gyroid are given in Table 1 and the ones for the other surfaces in Sects. 4.2, 4.3 and 4.4.*

For the calculations, we note that we are in the fully commutative case and hence Corollary 3.7 applies. Furthermore, the graphs all have transitive symmetry groups, so that we only have to calculate for one source $\boldsymbol{\tau}$ and can then transport the results by push-forward to any other.

Table 1. Possible choices of parameters (a, b, c) leading to non-Abelian enhanced symmetry groups and degenerate eigenvalues of H

a, b, c	Group	Iso class of extension	Type	Dim of irreps	Eigenvalues λ
$(0, 0, 0)$	$\mathbb{S}_4$	$\mathbb{S}_4$	Trivial	1, 3	$\lambda = -1$ three times $\lambda = 3$ once
(π, π, π)	$\mathbb{S}_4$	$\mathbb{S}_4$	Trivializable cocycle	1, 3	$\lambda = 1$ three times $\lambda = -3$ once
$(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ $(\frac{3\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{2})$	A_4	$2A_4$	Isomorphic extension	2, 2	$\lambda = \pm\sqrt{3}$ twice each

4.1. Gyroid

The graph $\bar{\Gamma}$ in the Gyroid case is the full square. It has symmetry group $\mathbb{S}_4$. With the Gyroid weights, the graph is faithful and hence the action can be lifted to an action on the torus. It acts transitively on all ordered spanning trees. Such a spanning tree is fixed by specifying a root and the order. The subgroup of $\mathbb{S}_3$ acts transitively on all orders. The matrices of this subgroup action are just the permutation matrices acting on the last three copies of $\mathbb{C}$ in $\mathbb{C}^4$ and the lift of the $\mathbb{S}_3$ action on the generators A, B, C of $\mathbb{T}^3$ is given by the permutation action.

We fix an initial rooted spanning tree and order as in Fig. 3.

4.1.1. Action on T^n.

The action of $\mathbb{S}_4$ on T^3 is fixed once we know the action of the generators $(12), (23)$ and (34).

The action of (23) is graphically calculated in Fig. 2, from which one reads off $\Psi((23))(A, B, C) = (A^*, C^*, B^*)$. Here, (A, B, C) is the notation for the initially chosen basis of $\mathbb{T}^3$.

In the graphical calculation, we first write down the graph together with the initial spanning tree and order. We then push-forward the spanning tree and the order. For this, we keep the vertices and edges as well as the weights fixed. We then (if necessary) give the re-gauging parameters by writing them next to the respective vertices and (if necessary) perform the re-gauging. Finally, we move the vertices and edges, so that they coincide with their pre-images to read off the morphism on the generators given by Theorems 3.10 and 3.14.

A similar calculation shows that $\Psi((34))(A, B, C) = (B^*, A^*, C^*)$. A consequence is that the cycle $(234) = (23)(34)$ acts as $\Psi((234))(A, B, C) = \Psi((23))(B^*, A^*, C^*) = (B, C, A)$ and is the cyclic permutation.

The action of (12) is more complicated as the root is moved. For this, we calculate graphically, see Fig. 3, and read off Ψ as: $(A, B, C) \mapsto (A^*, B^*, ACB)$.

This allows us to compute fixed points and stabilizer groups. We will first concentrate on non-Abelian stabilizer groups. There are only two fixed points under the full $\mathbb{S}_4$ action and these are $(1, 1, 1)$ and $(-1, -1, -1)$. The

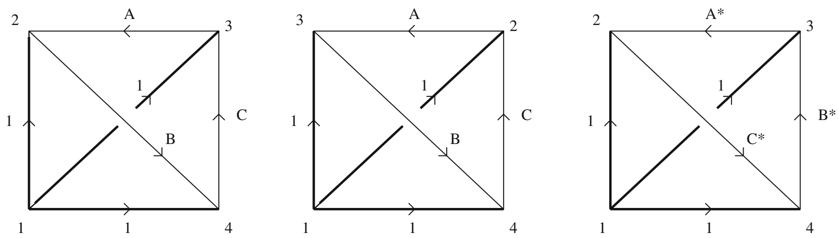

FIGURE 2. Calculation of the action of (23) on T^3. The original graph, the pushed forward order and the move into the old position to read off the morphism

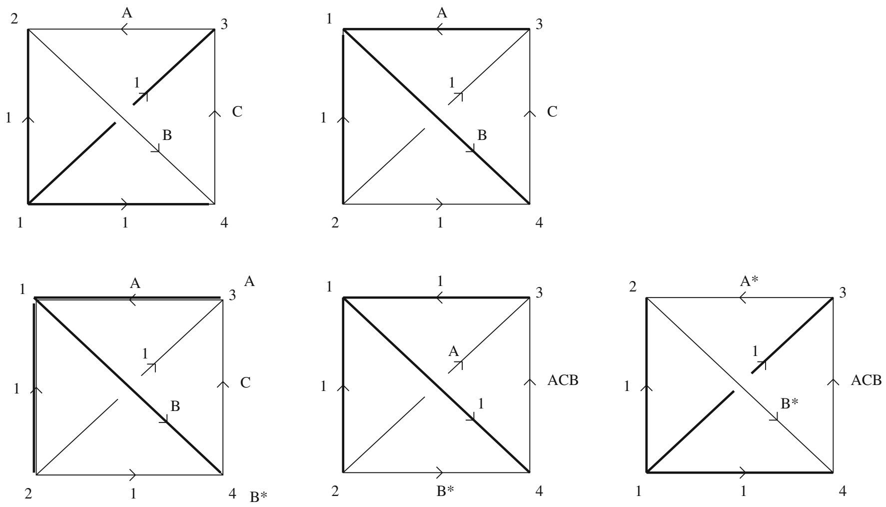

FIGURE 3. Calculation of the action of (12) on T^3

group A_4, the subgroup of all even permutations, is the stabilizer group of the two points (i,i,i) and $(-i,-i,-i)$. One can readily check that these are the only non-Abelian stabilizer groups. The other possibility would be $\mathbb{S}_3$, but a short calculation shows that anything that is stabilized by any $\mathbb{S}_3$ subgroup is stabilized by all of $\mathbb{S}_4$.

4.1.2. Representations. We collect together the matrices M needed for further calculation. Again, we fix our initial ordered rooted spanning tree as before.

Using short-hand notation, the matrices for the re-gauging induced by the transpositions $(12),(13),(14)$ from the initial spanning tree to the pushed forward one are

$$\rho_{12}=\begin{pmatrix}0&1&&\\1&0&&\\&&A&\\&&&B^*\end{pmatrix},\quad \rho_{13}=\begin{pmatrix}0&&1&\\&A^*&&\\1&&0&\\&&&C\end{pmatrix},$$

$$\rho_{14}=\begin{pmatrix}0&&&1\\&B&0&\\&0&C^*&\\1&&&0\end{pmatrix}$$

The calculation for ρ_{12} can be read off from Fig. 3. For this, we read off the matrix Φ from the re-gauging parameter and the matrix M_σ is given by the permutation we are considering. The other calculations are similar. All other transpositions, viz. those not involving 1, simply yield permutation matrices as there is no re-gauging involved. It is convenient to also have the following matrices as a reference:

$$\rho_{(12)(34)} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & A \\ & & B^* & 0 \end{pmatrix}, \quad \rho_{(14)(23)} = \begin{pmatrix} & & & 1 \\ & & B & \\ & C^* & & \\ 1 & & & \end{pmatrix}$$

and finally

$$\rho_{(123)} = \begin{pmatrix} 0 & 0 & 1 & \\ 1 & 0 & 0 & \\ 0 & A & 0 & \\ & & & B^* \end{pmatrix}$$

4.1.3. The Point $(0, 0, 0)$. At $(0, 0, 0)$, the matrices $\rho_{12}, \rho_{23}, \rho_{34}$ give the usual representation of $\mathbb{S}_4$ on $\mathbb{C}^4$. As is well known, this representation decomposes into the trivial representation and an irreducible three-dimensional representation. This means that there is an at least threefold degenerate eigenvalue λ. Since the trace of H is zero, we also know that the eigenvalues satisfy $\mu = -3\lambda$. Plugging in $(1, 1, 1, 1)$, which spans the trivial representation, we see that $\mu = 3$ and $\lambda = -1$.

4.1.4. The Point (π, π, π). In this case, the matrices $\rho_{12}, \rho_{23}, \rho_{34}$ only give a projective representation. As one can check $\rho_{12}\rho_{23}\rho_{12} = -\rho_{13}$ while $\rho_{23}\rho_{12}\rho_{23} = \rho_{13}$ for instance. Define the 1-cocycle λ by $\lambda(\sigma) = (-1)$ if 1 appears in a cycle of length >1 and 1 else. So that $\lambda((12)) = \lambda((13)) = \lambda((123)) = -1$ while $\lambda((23)) = \lambda((24)) = \lambda((234)) = 1$. Then, one calculates that $\tilde{\rho} := \rho \circ \lambda$ has a trivial cocycle c and thus ρ is isomorphic to a true linear representation of $\mathbb{S}_4$. Checking the characters, one sees again that in this case the irreducible components of $\tilde{\rho}$, which also commute with H are again the one-dimensional trivial representation and the three-dimensional standard representation. The trivial representation is spanned by $(-1, 1, 1, 1)$. The eigenvalues are then readily computed to be 1 with multiplicity 3 and -3 with multiplicity 1.

Remark 4.2. We would like to remark that the choice of λ amounts to choosing a different gauge for the root vertex, namely -1 instead of 1.

Remark 4.3. Notice that already in this case, even though there is no projective extension, our enhanced gauge group is necessary. Without it there would only be an $\mathbb{S}_3$ action, those elements which involve no re-gauging. This smaller symmetry group is, however, not powerful enough to force the triple degeneracy, as $\mathbb{S}_3$ has no irreducible 3-dim representation.

4.1.5. The Point $(\pi/2, \pi/2, \pi/2)$ and $(-\pi/2, -\pi/2, -\pi/2)$. These points are similar to each other. We will treat the first one in detail. Again, we have only a projective representation of A_4 aka. the tetrahedral group T. Namely, $\rho_{(12)(23)}\rho_{(13)(24)} = -i\rho_{(14)(23)}$. Again we can scale by a 1-cocycle λ. This time $\lambda(id) = 1$, $\lambda((ij)(kl)) = i$, $\lambda(ijk) = 1$ if $1 \notin \{i, j, k\}$, and $\lambda((ijk)) = i$ if $1 \in \{i, j, k\}$. The resulting representation $\tilde{\rho} = \rho \circ \lambda$ is then still a projective representation, but is it a representation of the unique non-trivial $\mathbb{Z}/2\mathbb{Z}$ extension of A_4, which goes by the names $2T, 2A_4, SL(2, 3)$ or the binary tetrahedral group. This group is well known. It is presented by generators s and t with the

Table 2. Character table of $2 \cdot A_4$ [21], where $\omega = \mathrm{e}^{\frac{2\pi i}{3}}$

Representative	1	-1	s^3	t^2	s^2	t	s
Elts in conj. class	1	1	6	4	4	4	4
Order	1	2	3	3	4	4	6
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	ω^2	ω^2	ω
χ_3	1	1	1	ω^2	ω	ω	ω^2
χ_4	2	-2	0	-1	-1	1	1
χ_5	2	-2	0	$-\omega$	$-\omega^2$	ω^2	ω
χ_6	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2
χ_7	3	3	-1	0	0	0	0

relations $s^3 = t^3 = (st)^2$. In $SL(2,3)$ (that is the special linear group of 2×2 matrices over the field with three elements $\mathbb{F}_3$), one can choose $s = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ and $t = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$.

For $2A_4$ using a set theoretic section $\wedge$ of the extension sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 2A_4 \underset{\wedge}{\overset{\longrightarrow}{\longleftarrow}} A_4 \longrightarrow 1 \tag{18}$$

and z as a generator for $\mathbb{Z}/2\mathbb{Z}$, we can pick $s = z\widehat{(123)}, t = z\widehat{(234)}$ as generators. Now, we can check the character table, Table 2, and find that the representation $\tilde{\rho}$ over the complex numbers decomposes as the sum of two irreducible two-dimensional representations $\chi_5 \oplus \chi_6$. In fact, these are the two representations into which the unique real irreducible four-dimensional representation of complex type splits over $\mathbb{C}$.

The explicit computation for the representation

$$\tilde{\rho}(s) = -\lambda((123))\rho_{(123)} = \begin{pmatrix} 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\tilde{\rho}(t) = -\lambda((234))\rho_{(234)} = -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{19}$$

is as follows. Suppose the $\tilde{\rho} = \bigoplus_{i=1}^{7} a_i \rho_i$, where ρ_i is the irrep with character χ_i. Now, $\mathrm{tr}(id) = 4, \mathrm{tr}(-1) = -4$, using the character table this implies that the coefficients $a_1 = a_2 = a_3 = a_7 = 0$ and furthermore $(*)\, a_4 + a_5 + a_6 = 2$. We furthermore have that $\mathrm{tr}(s) = -1$ so that $a_4 + \omega a_5 + \omega^2 a_6 = -1$ which together with (*) implies that $a_4 = 0, a_5 = a_6 = 1$. This fixes the decomposition into irreps. As a double check, one can verify that the rest of the equations are also satisfied.

So indeed we find that $(\pi/2, \pi/2, \pi/2)$ is a point with two eigenvalues with degeneracy 2. It is not hard to find (e.g., using the results of Sect. 4.1.6) that these eigenvalues are $\pm\sqrt{3}$.

The analysis of the complex conjugate point $(-\pi/2, -\pi/2, -\pi/2)$ is analogous.

We would briefly like to connect these results to [5]. There it was shown that these four points are the only singular points in the spectrum and that the two double crossing points are Dirac points.

4.1.6. Super-Selection Rules and Spectrum Along the Diagonal. To illustrate the power of the super-selection rules, we consider the action of the cyclic group generated by (234). One can easily see that the fixed point set in the T^3 is the diagonal $t = (a, a, a)$. The matrices $\hat{\chi}\rho$ actually give a bona fide representation of $\mathbb{C}^4$. This is the representation of C_3 given by cyclicly permuting the last three factors of $\mathbb{C}$. The action decomposes into irreps as follows: $\mathbb{C}^4 = \text{triv} \oplus \text{triv} \oplus \omega \oplus \bar{\omega}$. Where ω is the one-dimensional representation given by $\rho((123)) = \omega = \exp(2\pi i/3)$. The two trivial representations are spanned by $v_1 = (1, 0, 0, 0)$ and $v_2 = (0, 1, 1, 1)$, while the representation ω is spanned by $w = (0, 1, \omega, \bar{\omega})$ and $\bar{\omega}$ by $\bar{w}$.

Although we cannot extract information about the degeneracies from this it helps greatly in determining the eigenvalues, since there are two irreps with multiplicity one each giving a unique one-dimensional eigenspace for the Hamiltonian. Hence, we immediately get two eigenvalues. Plugging w and $\bar{w}$ into $H(t)$ one reads off

$$\lambda_1 = \omega \exp(ia) + \bar{\omega} \exp(-ia) \quad \lambda_2 = \bar{\omega} \exp(ia) + \omega \exp(-ia) \tag{20}$$

The sum of the two trivial representations gives a two-dimensional isotypical component. Therefore, we have to diagonalize H inside this eigenspace. It is interesting to note that at the special points it is exactly this flexibility that is needed to allow for crossings.

To determine the two remaining eigenvalues λ_3 and λ_4, we apply H to $\vec{v} = xv_1 + yv_2 = (x, y, y, y)$. The eigenvalue equation $H\,\vec{v} = \lambda\,\vec{v}$ leads to the equations $3y = \lambda x$ and $x + y(\exp(ia) + \exp(-ia)) = \lambda y$. Fixing $x = 3$ this gives the quadratic equation $\lambda^2 - 2\cos(a)\lambda - 3 = 0$ which has the two solutions

$$\lambda_{3,4} = \cos(a) \pm \sqrt{\cos^2(a) + 3} \tag{21}$$

This gives the spectrum along the diagonal which is given in Fig. 4. The calculation only involves the classical symmetries without re-gauging.

In [3], the authors also assert that numerically they only found singular points in the spectrum along the diagonal. The fact that the arising candidates for Dirac points are indeed such points and the analytic proof that indeed there are no other singular points in the spectrum is contained in [5].

4.2. The P Case

There is nothing much to say here. There is only the root of the spanning tree which is unique. The $\mathbb{S}_3$ action permutes the edges and their weights. This

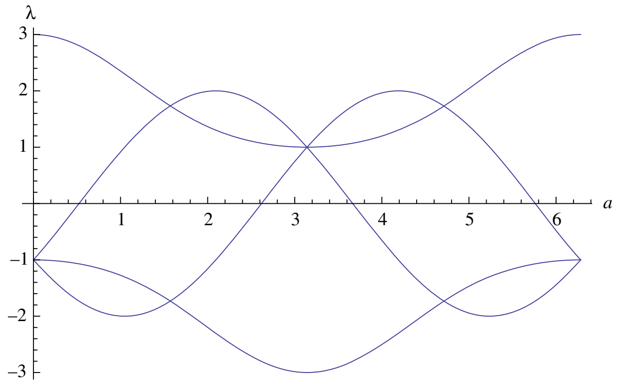

FIGURE 4. Spectrum of H along the diagonal in T^3

yields the permutation action on the T^3. There is no non-trivial cover and the eigenvalues remain invariant.

4.3. The D Case

Here, things again become interesting. Permuting the two vertices, we obtain eight fixed points if $a, b, c \in \{1, -1\}$. The matrix for this transposition is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This gives super-selection rules and we know that $v_1 = (1, 1)$ and $v_2 = (-1, 1)$ are eigenvectors. The eigenvalues being $1 + a + b + c$ and $-(1 + a + b + c)$ at these eight points.

We can also permute the edges with the $\mathbb{S}_4$ action. In this case, the $\mathbb{S}_3$ action leaving the spanning tree edge invariant acts as a permutation on (a, b, c). The relevant matrices, however, are just the identity matrices and the representation is trivial. The transposition (12), however, results in the action $(a, b, c) \mapsto (\bar{a}, \bar{a}b, \bar{a}c)$ on T^3, see Fig. 3. So to be invariant we have $a = 1$, but this implies that ρ_{12} is the identity matrix. Invariance for (13) and (14) and the three cycles containing 1 are similar. But, if we look at invariance under the element (12)(34) we are lead to the equations $a = \bar{a}, b = \bar{a}c, c = \bar{a}b$ This has solutions $a = 1, b = c$, for these fixed points again we find only a trivial action. But for $a = -1, b = -c$, these give rise to the diagonal matrix $\mathrm{diag}(1, -1)$ and hence eigenvectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, but looking at the Hamiltonian, these are only eigenvectors if it is the zero matrix $H(a, b, c) = 0$. Indeed, the conditions above imply $1 + a + b + c = 0$. Similarly, we find a $\mathbb{Z}/2\mathbb{Z}$ group for (13)(24) and (14)(23) yielding the symmetric equations $b = -1$, $a = -c$ and $c = -1, a = -b$. These are exactly the three circles found in [6].

Going to bigger subgroups of $\mathbb{S}_4$, we only get something interesting if the stabilizer group G_t contains precisely two of the double transposition above. That is the Klein four group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The invariants are precisely the intersection points of the three circles given by $a = b = -1$ and $c = 1$ and its cyclic permutations.

To find the two-dimensional irreducible representations, we look at different Klein four groups embedded into $\mathbb{Z}/2\mathbb{Z} \times \mathbb{S}_4$. If we denote the elements of $\mathbb{Z}/2\mathbb{Z}$ by $+, -$, then we first look at $(+, id)$, $(+, (12)(34))$, $(-, (13)(24))$, $(-, (14)(23))$. The element $(-, (13)(24))$ is the composition of edge permutation $(13)(24)$ together with the switching of the vertices. It gives the equation $a = b\bar{c}$ for fixed points, while the fixed points of $(-, (14)(23))$ satisfy $a = \bar{b}c$. Combining these equations with the ones for $(12)(23)$ above, we find again the solutions $a = 1, b = c$ and $a = -1, b = -c$. The difference, however, is that the representation in the case $a = -1$ is the irreducible projective representation of the Klein group corresponding to the irreducible 2-dim representation of its twofold cover given by the quaternion group $\pm 1, \pm i, \pm j, \pm k$. For $a = 1$, the irreps are one-dimensional and give no new information. Using the different embeddings of the Klein group, we find the 2-dim irreps on the three circles above responsible for the degeneration of the eigenvalues. These are three lines of double degenerate eigenvalue 0. They are not Dirac points since there is one free parameter accordingly the fibers of the characteristic map of [5] are one-dimensional which implies that the singular point is not isolated.

4.4. The Honeycomb Case

This is very similar to the D story. The vertex interchange renders the fixed points $a = \pm 1, b = \pm 1$ which have eigenvectors v_1, v_2 as above and eigenvalues $1+a+b$ and $-(1+a+b)$, respectively. The irreps of the C_3 action are triv$\oplus\omega$.

As far as the edge permutations are concerned, the interesting one is the cyclic permutation (123) which yields the equations

$$a = \bar{b}, \quad b = \bar{b}a$$

for fixed points. Hence, $a^3 = 1$. We get non-trivial matrices at the two points $(\omega, \bar{\omega})$ and $(\bar{\omega}, \omega)$. At these points, e_1, e_2 are eigenvectors with eigenvalue 0 and $H = 0$, since $1 + a + b = 1 + \omega + \bar{\omega} = 0$.

Denoting the elements of $\mathbb{Z}/2\mathbb{Z}$ again by $+, -$, there is an embedding of $\mathbb{S}_3 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{S}_3$ given by $(12) \mapsto (-, (12)), (23) \mapsto (-, (23))$. Notice that $(123) \mapsto (+, (123))$. It is then an easy-check that the equations for the fixed points are satisfied exactly by $(\omega, \bar{\omega})$ and $(\bar{\omega}, \omega)$. The representation is a projective representation of $\mathbb{S}_3$ cohomologous to the 2-dim irreducible representation of $\mathbb{S}_3$.

The fixed points are exactly the Dirac points of graphene and the symmetry above forces the degeneracies.

5. Conclusion

By considering re-gaugings, we have found the symmetry groups fixing the degeneracies of the PDG and honeycomb families of graph Hamiltonians. The symmetries we used were those induced by the automorphisms of the underlying graphs. In our specific examples, all the graphs were highly symmetric, and hence had large automorphism group. Here, we stress that our symmetries are extended symmetries and not just the classical ones. The most instructive and interesting case is the action of the binary tetrahedral group giving rise to the Dirac points in the Gyroid network. Note that as dimension-0 objects, the Dirac points for the Gyroid are codimension-3 defects in T^3, rather than codimension-2 defects in T^2 as for the honeycomb lattice, which describes graphene. Nevertheless, one may expect that they too lead to special physical properties.

There are several questions and research directions that tie into the present analysis.

It would be interesting to find concrete examples of lifts of re-gaugings either in the noncommutative case or in the case of re-gaugings not induced by graph symmetries. One place where we intend to look for the former is in the noncommutative case of PDG and the honeycomb as we aim to probe the noncommutative/commutative symmetry mentioned in [6].

We are furthermore interested in how these symmetries behave under deformations of the Hamiltonian and if they are topologically stable. A physically important type of deformations is those corresponding to periodic (in space) lattice distortions that describe crystals with lower spatial symmetry than those considered here. Such distortions may occur, for instance, during synthesis of the structure [1]. Codimension-3 Dirac points, such as those of the Gyroid network, are especially interesting in this respect: they can be viewed as magnetic monopoles in the parameter space [22] and as such are expected to be topologically stable. This makes the physics associated with such points immune to periodic lattice distortions.

Finally, it seems that on the horizon there are connections between our theory and two other worlds. The first being quiver representations in general and the second being cluster algebras. The connection to the first is inherent in the subject matter, while the connection to the second needs some work. The point is that in our transformations, we change several variables at a time. Nevertheless, the re-gauging groupoid can be viewed as a sort of mutation diagram. We plan to investigate these intriguing connections in the future.

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Ralph M. Kaufmann and Birgit Wehefritz-Kaufmann
Department of Mathematics
Purdue University
West Lafayette, IN 47907, USA
e-mail: rkaufman@math.purdue.edu;
ebkaufma@math.purdue.edu

Sergei Khlebnikov and Birgit Wehefritz-Kaufmann
Department of Physics and Astronomy
Purdue University
West Lafayette, IN 47907, USA
e-mail: skhleb@physics.purdue.edu

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