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Stringy K-theory and the Chern character

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Abstract. We construct two new G-equivariant rings: $\mathcal{K}(X, G)$, called the *stringy K-theory* of the G-variety X, and $\mathcal{H}(X, G)$, called the *stringy cohomology* of the G-variety X, for any smooth, projective variety X with an action of a finite group G. For a smooth Deligne–Mumford stack $\mathcal{X}$, we also construct a new ring $\mathsf{K}_{\text{orb}}(\mathcal{X})$ called the *full orbifold K-theory* of $\mathcal{X}$. We show that for a global quotient $\mathcal{X} = [X/G]$, the ring of G-invariants $K_{\text{orb}}(\mathcal{X})$ of $\mathcal{K}(X, G)$ is a subalgebra of $\mathsf{K}_{\text{orb}}([X/G])$ and is linearly isomorphic to the "orbifold K-theory" of Adem-Ruan [AR] (and hence Atiyah-Segal), but carries a different "quantum" product which respects the natural group grading.

We prove that there is a ring isomorphism $\mathcal{C}\mathbf{h} : \mathcal{K}(X, G) \to \mathcal{H}(X, G)$, which we call the *stringy Chern character*. We also show that there is a ring homomorphism $\mathfrak{Ch}_{\text{orb}} : \mathsf{K}_{\text{orb}}(\mathcal{X}) \to H^{\bullet}_{\text{orb}}(\mathcal{X})$, which we call the *orbifold Chern character*, which induces an isomorphism $Ch_{\text{orb}} : K_{\text{orb}}(\mathcal{X}) \to H^{\bullet}_{\text{orb}}(\mathcal{X})$ when restricted to the sub-algebra $K_{\text{orb}}(\mathcal{X})$. Here $H^{\bullet}_{\text{orb}}(\mathcal{X})$ is the Chen–Ruan orbifold cohomology. We further show that $\mathcal{C}\mathbf{h}$ and $\mathfrak{Ch}_{\text{orb}}$ preserve many properties of these algebras and satisfy the Grothendieck–Riemann–Roch theorem with respect to étale maps. All of these results hold both in the algebro-geometric category and in the topological category for equivariant almost complex manifolds.

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We further prove that $\mathscr{H}(X, G)$ is isomorphic to Fantechi and Göttsche's construction [FG,JKK]. Since our constructions do not use complex curves, stable maps, admissible covers, or moduli spaces, our results greatly simplify the definitions of the Fantechi–Göttsche ring, Chen–Ruan orbifold cohomology, and the Abramovich–Graber–Vistoli orbifold Chow ring.

We conclude by showing that a K-theoretic version of Ruan's Hyper-Kähler Resolution Conjecture holds for the symmetric product of a complex projective surface with trivial first Chern class.

Contents

1. Introduction

The first main result of this paper is the construction of two new G-Frobenius algebras $\mathscr{H}(X, G)$ and $\mathscr{K}(X, G)$, called the *stringy cohomology of X* and the *stringy K-theory of X*, respectively, where X is a manifold with an action of a finite group G. The rings of G-invariants of these algebras bear some resemblance to equivariant cohomology and equivariant K-theory, but they carry different information and generally produce a more refined invariant than their equivariant counterparts.

The most important part of these constructions is the multiplication, which is defined purely in terms of the the G-equivariant tangent bundle TX restricted to various fixed point loci of X.

While our stringy K-theory $\mathscr{K}(X, G)$ is an entirely new construction, we prove that our stringy cohomology $\mathscr{H}(X, G)$ is equivalent to Fantechi and Göttsche's construction of stringy cohomology [FG,JKK]. Since our definition avoids any mention of complex curves, admissible covers, or moduli spaces, it greatly simplifies the computations of stringy cohomology and allows us to give elementary proofs of associativity and the trace axiom.

Because our constructions are completely functorial, an analogous construction yields the *stringy Chow ring of X*, which we denote by $\mathscr{A}(X, G)$. The algebra $\mathscr{A}(X, G)$ is a *pre-G-Frobenius algebra*, a generalization of a G-Frobenius algebra which allows the ring to be of infinite rank and the metric to be degenerate.

The second main result of this paper is the introduction of a new *stringy Chern character* $\mathfrak{Ch} : \mathcal{K}(X, G) \to \mathcal{H}(X, G)$. We prove that $\mathfrak{Ch}$ is a *ring* isomorphism which preserves all of the properties of a pre-G-Frobenius algebra except those involving the metric.

The third main result of this paper is the introduction of two new *orbifold K-theories*. The first we call *full orbifold K-theory* and is defined for a general almost complex orbifold (or a smooth Deligne–Mumford stack). We denote it by $\mathsf{K}_{\mathrm{orb}}(\mathcal{X})$. The second algebra is defined when $\mathcal{X} = [X/G]$ is a global quotient by a finite group as the algebra of invariants $\mathcal{K}(X, G)^G$ of the stringy K-theory of X. We denote this algebra by $K_{\mathrm{orb}}(\mathcal{X})$ and call it the *small orbifold K-theory of* $\mathcal{X}$. It is linearly isomorphic to the construction of Adem and Ruan [AR], but our construction possesses a different, "quantum," product. We show there is a natural homomorphism of algebras $\mathsf{K}_{\mathrm{orb}}(\mathcal{X}) \xrightarrow{\pi^*} K_{\mathrm{orb}}(\mathcal{X})$, and an *orbifold Chern character* $\mathfrak{Ch}_{\mathrm{orb}} : \mathsf{K}_{\mathrm{orb}}(\mathcal{X}) \to H^{\bullet}_{\mathrm{orb}}(\mathcal{X})$ which, like the stringy Chern character, is a ring homomorphism which preserves all of the properties of a Frobenius algebra that do not involve the metric. In the special case that the orbifold is a global quotient $\mathcal{X} = [X/G]$, the orbifold Chern character induces an *isomorphism* $Ch_{\mathrm{orb}} : K_{\mathrm{orb}}(\mathcal{X}) \to H^{\bullet}_{\mathrm{orb}}(\mathcal{X})$ which agrees with that induced by the stringy Chern character on the rings of G-invariants.

Our results are initially formulated and proved in the algebro-geometric category, with Chow rings and algebraic K-theory, but they also hold in the topological category, with cohomology and topological K-theory (see Sect. 10) for almost complex manifolds with a G-equivariant almost complex structure. In fact, these algebraic structures depend only upon the homotopy class of the G-equivariant almost complex structure. Our results can also be generalized to equivariant stable complex manifolds (see Remark 10.2).

1.1. Notation and conventions. Unless otherwise specified, we assume throughout the paper that all cohomology rings have coefficients in the rational numbers $\mathbb{Q}$. Also, unless otherwise specified, all groups are finite and all group actions are left actions.

The stack (or orbifold) quotient of a variety (or manifold) X by G will be denoted $[X/G]$ and the coarse moduli space (i.e., underlying space) of this quotient will be denoted X/G.

The conjugacy class of any element g in a group G will be denoted $[\![g]\!]$ and the commutator $aba^{-1}b^{-1}$ of two elements $a, b \in G$ is denoted $[a, b]$.

1.2. Background and motivation. We now describe part of our motivation for studying stringy K-theory. For convenience, we assume throughout this subsection that the coefficient ring is $\mathbb{C}$ rather than $\mathbb{Q}$.

Let Y be a projective, complex surface such that $c_1(Y) = 0$. For all n, consider the product Y^n with the symmetric group S_n acting by permuting its factors. The quotient orbifold $[Y^n/S_n]$ is called the symmetric product of Y. Let $Y^{[n]}$ denote the Hilbert scheme of n points on Y. The morphism $Y^{[n]} \to Y^n/S_n$ is a crepant resolution of singularities and is, furthermore, a hyper-Kähler resolution [Rua]. Fantechi and Göttsche [FG] proved that there is a ring isomorphism $\psi' : H^\bullet_{\text{orb}}([Y^n/S_n]) \to H^\bullet(Y^{[n]})$, where $H^\bullet(Y^{[n]})$ is the ordinary cohomology ring (see also [Kau05,Uri]).[1]

The previous example is a verification, in a special case, of the following conjecture of Ruan [Rua], which was inspired by the work of string theorists studying topological string theory on orbifolds.

Conjecture 1.1 (Cohomological hyper-Kähler resolution conjecture). Suppose that $\tilde{V} \to V$ is a hyper-Kähler resolution of the coarse moduli space V of an orbifold $\mathcal{V}$. The ordinary cohomology ring $H^\bullet(\tilde{V})$ of $\tilde{V}$ is isomorphic (up to discrete torsion) to the Chen–Ruan orbifold cohomology ring $H^\bullet_{\text{orb}}(\mathcal{V})$ of $\mathcal{V}$.

Let us return again to the example of the symmetric product. The algebra isomorphism $\psi' : H^\bullet_{\text{orb}}([Y^n/S_n]) \to H^\bullet(Y^{[n]})$ suggests that there should exist a K-theoretic analogue $K_{\text{orb}}([Y^n/S_n])$ of $H^\bullet_{\text{orb}}([Y^n/S_n])$, a stringy Chern character isomorphism $Ch_{\text{orb}} : K_{\text{orb}}([Y^n/S_n]) \to H^\bullet_{\text{orb}}([Y^n/S_n])$, and an algebra isomorphism $\psi : K_{\text{orb}}([Y^n/S_n]) \to K(Y^{[n]})$, such that the following diagram commutes:

$$\begin{array}{ccc} K_{\text{orb}}\big([Y^n/S_n]\big) & \xrightarrow{Ch_{\text{orb}}} & H^\bullet_{\text{orb}}\big([Y^n/S_n]\big) \\ \psi\big\downarrow & & \psi'\big\downarrow \\ K(Y^{[n]}) & \xrightarrow{\mathbf{ch}} & H^\bullet(Y^{[n]}). \end{array} \tag{1.1}$$

In this paper we construct an *orbifold K-theory* analogous to the Chen–Ruan orbifold cohomology, and we construct an orbifold Chern character which is a ring isomorphism (see Theorem 9.8). This leads us to pose the following K-theoretic analogue of the Ruan conjecture.

Conjecture 1.2 (K-theoretic hyper-Kähler resolution conjecture). Suppose that $\tilde{V} \to V$ is a hyper-Kähler resolution of the coarse moduli space V of an orbifold $\mathcal{V}$. The ordinary K-theory $K(\tilde{V})$ of the resolution $\tilde{V}$ is isomorphic (up to discrete torsion) to the (small) orbifold K-theory $K_{\text{orb}}(\mathcal{V})$ of $\mathcal{V}$.

The method that Fantechi and Göttsche use to prove their result involves the construction of a new ring $\mathcal{H}(X, G)$, which we call *stringy cohomology*, associated to any smooth, projective manifold X with an action by a finite group G. They show that for a global quotient orbifold $\mathcal{X} := [X/G]$,

[1] In fact, they proved that the isomorphism holds over $\mathbb{Q}$, provided that the multiplication on $H^\bullet_{\text{orb}}([Y^n/S_n])$ is twisted by signs. This sign change can be regarded as a kind of discrete torsion (see Sect. 10.3 for more details).

the Chen–Ruan orbifold cohomology $H^{\bullet}_{\mathrm{orb}}(\mathcal{X})$ is isomorphic to the ring of invariants $\mathcal{H}(X, G)^G$ of the stringy cohomology.

Their construction suggests that a similar construction in K-theory should be possible and that the two constructions might be related by a stringy Chern character.

1.3. Summary and discussion of main results. We will now briefly describe the main results and constructions of the paper.

Let X be a smooth, projective variety with an action of a finite group G. For each $m \in G$ we denote the fixed locus of m in X by X^m, and we let

$$I_G(X) := \coprod_{m \in G} X^m \subset X \times G$$

denote the *inertia variety* of X. The inertia variety $I_G(X)$ should not be confused with the inertia *orbifold*, or inertia *stack*, $\coprod_{[\![g]\!]}[X^g/Z_G(g)]$, where the sum runs over conjugacy classes $[\![g]\!]$ in G. Note that the G-variety $I_G(X)$ contains $X = X^1$ as a connected component.

As a G-graded G-module, the *stringy Chow ring* $\mathcal{A}(X, G)$ *of* X is the Chow ring of $I_G(X)$, i.e.,

$$\mathcal{A}(X, G) = \bigoplus_{g \in G} \mathcal{A}_g(X) = \bigoplus_{g \in G} A^{\bullet}(X^g).$$

The inertia variety has a canonical G-equivariant involution $\sigma : I_G(X) \to I_G(X)$ which maps X^m to $X^{m^{-1}}$ via

$$\sigma : (x, m) \mapsto (x, m^{-1}) \tag{1.2}$$

for all m in G. We define a pairing $\eta_{\mathcal{A}}$ on $\mathcal{A}(X, G)$ by

$$\eta_{\mathcal{A}}(v_1, v_2) := \int_{[I_G(X)]} v_1 \cup \sigma^* v_2$$

for all v_1, v_2 in $\mathcal{A}(X, G)$.

In a similar fashion, we define the *stringy K-theory* $\mathcal{K}(X, G)$ *of* X, as a G-graded G-module, to be the K-theory of the inertia variety, i.e.,

$$\mathcal{K}(X, G) = \bigoplus_{g \in G} \mathcal{K}_g(X) = \bigoplus_{g \in G} K(X^g).$$

We define a pairing $\eta_{\mathcal{K}}$ on $\mathcal{K}(X, G)$ by

$$\eta_{\mathcal{K}}(\mathcal{F}_1, \mathcal{F}_2) := \chi\big(I_G(X), \mathcal{F}_1 \otimes \sigma^* \mathcal{F}_2\big)$$

for all $\mathcal{F}_1, \mathcal{F}_2$ in $\mathcal{K}(X, G)$, where $\chi(I_G(X), \mathcal{F})$ denotes the Euler characteristic of $\mathcal{F} \in K(I_G(X))$.

The definition of the multiplicative structure on $\mathcal{A}(X, G)$ and $\mathcal{K}(X, G)$ requires the following new constructions.

Definition 1.3. Define $\mathcal{S}$ in $K(I_G(X))$ (the rational K-theory) to be such that for any m in G, its restriction $\mathcal{S}_m$ in $K(X^m)$ is given by

$$\mathcal{S}_m := \mathcal{S}|_{X^m} := \bigoplus_{k=0}^{r-1} \frac{k}{r} W_{m,k}, \tag{1.3}$$

where r is the order of m, and $W_{m,k}$ is the eigenbundle of $W_m := TX|_{X^m}$ such that m acts with eigenvalue $\exp(2\pi ki/r)$.

The virtual rank $a(m)$ of $\mathcal{S}_m$ is called the *age of m* and is a locally constant $\mathbb{Q}$-valued function on X^m.

Remark 1.4. It is worth pointing out that $\mathcal{S}$ would remain the the same if, in the definition of $\mathcal{S}_m$, the restriction of TX to X^m were replaced by the normal bundle of X^m in X. For this reason, the construction of $\mathcal{S}$ and the construction of stringy cohomology and stringy K-theory still works over stable complex manifolds.

For any triple $\mathbf{m} := (m_1, m_2, m_3)$ in G^3 such that $m_1 m_2 m_3 = 1$, we let $X^{\mathbf{m}} := X^{m_1} \cap X^{m_2} \cap X^{m_3}$, where X^{m_i} is regarded as a subvariety of X.

Definition 1.5. Define the element $\mathcal{R}(\mathbf{m})$ in $K(X^{\mathbf{m}})$ by

$$\mathcal{R}(\mathbf{m}) := TX^{\mathbf{m}} \ominus TX\big|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \mathcal{S}_{m_i}\big|_{X^{\mathbf{m}}}. \tag{1.4}$$

It is central to our theory, but not at all obvious, that $\mathcal{R}(\mathbf{m})$ is actually represented by a vector bundle on $X^{\mathbf{m}}$. In general, the only way we know how to establish this key fact is through our proof in Sect. 8, which uses the Eichler trace formula (a special case of the holomorphic Lefschetz theorem) to show that $\mathcal{R}(\mathbf{m})$ is equal to the obstruction bundle $R^1\pi_*^G f^*(TX)$ arising in the Fantechi–Göttsche construction of stringy cohomology. However, once one knows that $\mathcal{R}(\mathbf{m})$ is always represented by a vector bundle, all of the properties of a pre-G-Frobenius algebra can be established (see Definition 3.2 for details). We first use $\mathcal{R}(\mathbf{m})$ to define the multiplication in $\mathcal{A}(X, G)$ and $\mathcal{K}(X, G)$ as follows. For all $i = 1, 2, 3$, let

$$\mathbf{e}_{m_i} : X^{\mathbf{m}} \to X^{m_i}$$

be the canonical inclusion morphisms, and define

$$\check{\mathbf{e}}_{m_i} := \sigma \circ \mathbf{e}_{m_i} : X^{\mathbf{m}} \to X^{m_i^{-1}},$$

where σ is the canonical involution (see (1.2)).

Definition 1.6. Given $m_1, m_2 \in G$, let $m_3 := (m_1 m_2)^{-1}$. For any $v_{m_1} \in \mathcal{A}_{m_1}(X)$ and $v_{m_2} \in \mathcal{A}_{m_2}(X)$, we define the *stringy product (or multiplication)* of v_{m_1} and v_{m_2} in $\mathcal{A}(X, G)$ to be

$$v_{m_1} * v_{m_2} := \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}^*_{m_1} v_{m_1} \cup \mathbf{e}^*_{m_2} v_{m_2} \cup c_{\text{top}}\left(\mathcal{R}(\mathbf{m})\right)\big). \tag{1.5}$$

The product is then extended linearly to all of $\mathcal{A}(X, G)$.

We define the *stringy product on $\mathcal{K}(X, G)$* analogously.

Definition 1.7. Given $m_1, m_2 \in G$, let $m_3 := (m_1 m_2)^{-1}$. For any $\mathcal{F}_{m_1} \in \mathcal{K}_{m_1}(X)$ and $\mathcal{F}_{m_2} \in \mathcal{K}_{m_2}(X)$, we define the *stringy product* of $\mathcal{F}_{m_1}$ and $\mathcal{F}_{m_2}$ in $\mathcal{K}(X, G)$ to be

$$\mathcal{F}_{m_1} * \mathcal{F}_{m_2} := \check{\mathbf{e}}_{m_3 *}\left(\mathbf{e}^*_{m_1} \mathcal{F}_{m_1} \otimes \mathbf{e}^*_{m_2} \mathcal{F}_{m_2} \otimes \lambda_{-1}\left(\mathcal{R}(\mathbf{m})^*\right)\right), \tag{1.6}$$

and again the product is extended linearly to all of $\mathcal{K}(X, G)$.

The stringy Chow ring and stringy K-theory are almost G-Frobenius algebras, but they are generally infinite dimensional and have degenerate pairings. An algebra which satisfies essentially all of the axioms of a G-Frobenius algebra except those involving finite dimensionality and a non-degenerate pairing is called a *pre-G-Frobenius algebra* (see Definition 3.2 for details).

Our first main result is the following.

Main result 1 (see Theorems 4.6 and 4.7 for complete details). *For any smooth, projective variety X with an action of a finite group G, the ring $\mathcal{A}(X, G)$ is a $\mathbb{Q}$-graded, pre-G-Frobenius algebra which contains the ordinary Chow ring $A^\bullet(X) = \mathcal{A}_1(X)$ of X as a sub-algebra. In particular, $\mathcal{A}(X, G)$ is a G-equivariant, associative ring (generally non-commutative) with a $\mathbb{Q}$-grading that respects the multiplication and the metric.*

Similarly, the ring $\mathcal{K}(X, G)$ is a pre-G-Frobenius algebra which contains the ordinary K-theory $K(X) = \mathcal{K}_1(X)$ of X as a sub-algebra.

Unfortunately, the ordinary Chern character $\mathbf{ch} : \mathcal{K}(X, G) \to \mathcal{A}(X, G)$ does not respect the stringy multiplications. We repair this problem by defining the *stringy Chern character* $\mathfrak{Ch} : \mathcal{K}(X, G) \to \mathcal{A}(X, G)$ to be a deformation of the ordinary Chern character. That is, for every element $m \in G$ and every $\mathcal{F}_m \in \mathcal{K}_m(X)$ we define

$$\mathfrak{Ch}(\mathcal{F}_m) := \mathbf{ch}(\mathcal{F}_m) \cup \mathbf{td}^{-1}(\mathcal{S}_m) = \mathbf{ch}(\mathcal{F}_m) \cup (\mathbf{1} - c_1(\mathcal{S}_m)/2 + \cdots), \tag{1.7}$$

where $\mathcal{S}_m$ is defined in (1.3). This yields our second main result.

Main result 2 (see Theorem 6.1 and Theorem 6.3 for complete details). *The stringy Chern character $\mathfrak{Ch} : \mathcal{K}(X, G) \to \mathcal{A}(X, G)$ is a G-equivariant algebra isomorphism. Moreover, $\mathfrak{Ch}$ is natural and satisfies a form of the Grothendieck–Riemann–Roch theorem with respect to G-equivariant étale maps.*

It is natural to ask whether the rings of G-invariants of the stringy Chow ring and stringy K-theory are presentation independent, and if so, whether these rings can be constructed for orbifolds which are not global quotients of a variety by a finite group. The answer is yes in both cases. It is already known that the ring of G-invariants of the stringy Chow ring $\mathcal{A}(X, G)^G$ is isomorphic to the Abramovich–Graber–Vistoli orbifold Chow ring $A^{\bullet}_{\text{orb}}([X/G])$ of the quotient orbifold $[X/G]$.

The third main result of this paper has three parts: first, the construction of a new *full orbifold K-theory* $\mathsf{K}_{\text{orb}}(\mathcal{X})$ for smooth Deligne–Mumford stacks, and a second *small orbifold K-theory* $K_{\text{orb}}([X/G])$ for global quotients by finite groups; second, the construction of an orbifold Chern character $Ch_{\text{orb}} : K_{\text{orb}}(\mathcal{X}) \to A^{\bullet}_{\text{orb}}(\mathcal{X})$ which is a ring homomorphism; and third, a demonstration of the relations between the two theories.

Main result 3 (see Theorems 9.5 and 9.8 for complete details). *For a smooth Deligne–Mumford stack $\mathcal{X}$ satisfying the resolution property, the full orbifold K-theory $\mathsf{K}_{\text{orb}}(\mathcal{X})$ is a pre-Frobenius algebra. Moreover, there is a* full orbifold Chern character $\mathfrak{Ch}_{\text{orb}} : \mathsf{K}_{\text{orb}}(\mathcal{X}) \to A^{\bullet}_{\text{orb}}(\mathcal{X})$ *which, like the stringy Chern character, is a ring homomorphism which preserves all of the properties of a pre-Frobenius algebra that do not involve the metric.*

For a global quotient $\mathcal{X} = [X/G]$ by a finite group, G the small orbifold K-theory $K_{\text{orb}}(\mathcal{X})$ is also a pre-Frobenius algebra, independent of the choice of resolution. There is an orbifold Chern character $Ch_{\text{orb}} : K_{\text{orb}}(\mathcal{X}) \to A^{\bullet}_{\text{orb}}(\mathcal{X})$ *which is an algebra* isomorphism, *and there is a natural algebra homomorphism $\pi^* : \mathsf{K}_{\text{orb}}(\mathcal{X}) \to K_{\text{orb}}(\mathcal{X})$ making the following diagram commute:*

$$\begin{array}{ccccc} \mathsf{K}_{\text{orb}}(\mathcal{X}) & \xrightarrow{\pi^*} & \mathcal{K}(X, G)^G & = & K_{\text{orb}}(\mathcal{X}) \\ \downarrow{\scriptstyle \mathfrak{Ch}_{\text{orb}}} & & & & \cong \downarrow{\scriptstyle Ch_{\text{orb}}} \\ H^{\bullet}_{\text{orb}}(\mathcal{X}) & & = & & \mathcal{H}^{\bullet}(X, G)^G. \end{array}$$

As we mentioned above, all these results are proved initially in the algebro-geometric category, but we prove in Sect. 10 that their analogues in the topological category also hold. That is, we define stringy cohomology, stringy topological K-theory, orbifold cohomology, orbifold topological K-theory, and their corresponding Chern characters. We prove theorems analogous to the above for these topological constructions. Furthermore, we prove that stringy cohomology of a complex (or almost complex) orbifold $\mathcal{X}$ is equal to the G-Frobenius algebra $\mathcal{H}(X, G)$ described in [FG,JKK]. Similarly, we prove that the orbifold cohomology $H^{\bullet}_{\text{orb}}(\mathcal{X})$ of a complex (or almost complex) orbifold $\mathcal{X}$ is equal to its Chen–Ruan orbifold cohomology [CR1,AGV].

We conclude with an application of these results to the case of the symmetric product of a smooth, projective surface Y with trivial canonical bundle and verify that our K-theoretic hyper-Kähler resolution conjecture

(Conjecture 1.2) holds in this case; that is, $K_{\mathrm{orb}}([Y^n/S_n])$ is isomorphic to $K(Y^{[n]})$.

1.4. Directions for further research. These results suggest many different directions for further research. The first is to generalize to the case where G is a Lie group and to higher-degree Gromov–Witten invariants. This will be explored elsewhere. It would also be interesting to study stringy generalizations of the usual algebraic structures of K-theory, e.g., the Adam's operations and λ-rings. Another interesting direction would be to study stringy generalizations of other K-theories, including algebraic K-theory and higher K-theory. It would also be very interesting to find an analogous construction in orbifold conformal field theory, e.g., twisted vertex algebras and the chiral de Rham complex [FS]. Finally, it would be interesting to see if our results can shed light upon the relationship between Hochschild cohomology and orbifold cohomology [DE] in the context of deformation quantization.

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2. The ordinary Chow ring and K-theory of a variety

In this section, we briefly review some basic facts about the classical Chow ring, K-theory, and certain characteristic classes that we will need. Throughout this section, all varieties we consider will be smooth, projective varieties over $\mathbb{C}$.

Recall that a *Frobenius algebra* is a finite dimensional, unital, commutative, associative algebra with an invariant (non-degenerate) metric. To each smooth, projective variety X, one can associate two algebras, which are almost Frobenius algebras, namely, the *Chow ring* $A^\bullet(X)$ *of* X, and the *K-theory* $K(X)$ *of* X. These fail to be Frobenius algebras in that they may be infinite dimensional and their symmetric pairing may be degenerate. Both $A^\bullet(X)$ and $K(X)$ also possess an additional structure, which we call a *trace element*, which is closely related to the Euler characteristic. We call such algebras (with a trace element) *pre-Frobenius algebras* (see Definition 2.1).

Furthermore, there is an isomorphism of unital, commutative, associative algebras $\mathbf{ch} : K(X) \to A^\bullet(X)$ called the *Chern character*. The Chern character does not preserve the metric, but it does preserve the trace elements. We call such an isomorphism *allometric*. We will now briefly review these constructions in order to fix notation and conventions, referring the interested reader to [Ful,FL] for more details.

2.1. The Chow ring. The Chow ring of a smooth, projective variety X is additively a $\mathbb{Z}$-graded Abelian group $A^\bullet(X, \mathbb{Z}) = \bigoplus_{p=0}^{D} A^p(X, \mathbb{Z})$, where D is the dimension of X, and $A^p(X)$ is the group of finite formal sums of $(D - p)$-dimensional subvarieties of X, modulo rational equivalence.

In this paper we will always work with rational coefficients, and we write

$$A^\bullet(X) := A^\bullet(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The vector space $A^\bullet(X)$ is endowed with a commutative, associative multiplication which preserves the $\mathbb{Z}$-grading, arising from the intersection product, and possesses an identity element $\mathbf{1} := [X]$ in $A^0(X)$. The intersection product $A^p(X) \otimes A^q(X) \to A^{p+q}(X)$ is denoted by $v \otimes w \mapsto v \cup w$ for all p, q.

Given a proper morphism $f : X \to Y$ between two varieties, there is an induced pushforward morphism $f_* : A^\bullet(X) \to A^\bullet(Y)$. In particular, if Y is a point and $f : X \to Y$ is the obvious map, then one can define integration via the formula

$$\int_{[X]} v := f_*(v)$$

for all v in $A^\bullet(X)$. Whenever X is equidimensional of dimension D, the integral vanishes unless v belongs to $A^D(X)$. Define a symmetric, bilinear form $\eta_A : A^\bullet(X) \otimes A^\bullet(X) \to \mathbb{Q}$ via $\eta_A(v, w) := \int_{[X]} v \cup w$. Finally, we define a special element $\tau^A \in (A^\bullet(X))^*$ in the dual, the *trace element*

$$\tau^A(v) := \int_X (v \cup c_{\text{top}}(TX)),$$

where $c_{\text{top}}(TX)$ is the top Chern class of the tangent bundle TX. The integer $\tau^A(\mathbf{1})$ is the usual Euler characteristic of X.

Although the Chow ring resembles the cohomology ring in many ways, it is important to note that $A^\bullet(X)$ is generally infinite dimensional, and that the pairing η_A is often degenerate. This motivates the following definition.

Definition 2.1.[2] Consider a tuple $(R, *, \eta, \mathbf{1}, \tau)$ consisting of a commutative, associative algebra $(R, *)$ (possibly infinite dimensional) with unity $\mathbf{1} \in R$, a symmetric bilinear pairing η (possibly degenerate), and τ in R^*, called the *trace element*. We say that $(R, *, \eta, \mathbf{1}, \tau)$ is a *pre-Frobenius algebra* if the pairing η is multiplicatively invariant:

$$\eta(r * s, t) = \eta(r, s * t)$$

for all $r, s, t \in R$.

Every Frobenius algebra $(R, *, \eta, \mathbf{1})$ has a *canonical trace element* $\tau(v) := \mathrm{Tr}_R(L_v)$ where L_v is left multiplication by v in R. We call $(R, *, \eta, \mathbf{1}, \tau)$ the *canonical pre-Frobenius algebra structure* associated to the Frobenius algebra $(R, *, \eta, \mathbf{1})$.

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[2] We thank the referee for help clarifying these details.

Proposition 2.2 (see [Kle, §1] and [Ful, §19.1]). *Let $A^\bullet(X)$ be the Chow ring of an irreducible, smooth, projective variety X.*

(1) *The triple $(A^\bullet(X), \cup, \mathbf{1}, \eta_A, \tau^A)$ is a pre-Frobenius algebra graded by $\mathbb{Z}$.*
(2) *If $f : X \to Y$ is any morphism, then the associated pullback morphism $f^* : A^\bullet(Y) \to A^\bullet(X)$ is a morphism of commutative, associative algebras graded by $\mathbb{Z}$.*
(3) *(Projection formula) For any proper morphism $f : X \to Y$, if $\alpha \in A^\bullet(X)$ and $\beta \in A^\bullet(Y)$, we have*

$$f_*(\alpha \cup f^*(\beta)) = f_*(\alpha) \cup \beta.$$

2.2. K-theory. $K(X; \mathbb{Z})$ is additively equal to the free Abelian group generated by isomorphism classes of (complex algebraic) vector bundles on X, modulo the subgroup generated by

$$[E] \ominus [E'] \ominus [E''] \tag{2.1}$$

for each exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0. \tag{2.2}$$

Here $\ominus$ denotes subtraction and $\oplus$ denotes addition in the free Abelian group. We define

$$K(X) := K(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The multiplication operation, also denoted by $\otimes$, taking $K(X) \otimes K(X) \to K(X)$ is the usual tensor product $[E] \otimes [E'] \mapsto [E \otimes E']$ for all vector bundles E and E'. We denote the multiplicative identity by $\mathbf{1} := [\mathcal{O}_X]$.

Given a proper morphism $f : X \to Y$ between two smooth varieties, there is an induced pushforward morphism $f_* : K(X) \to K(Y)$ given by $f_*([E]) = \sum_{i=0}^{D} (-1)^i R^i f_* E$, where D is the relative dimension of f. In particular, if Y is a point and $f : X \to Y$ is the obvious map, then the Euler characteristic of $v \in K(X)$ is the pushforward

$$\chi(X, v) = f_*(v).$$

If X is irreducible, we define a symmetric bilinear form $\eta_K : K(X) \otimes K(X) \to \mathbb{Q}$ via $\eta_K(v, w) := \chi(X, v \otimes w)$.

While $K(X)$ does not have a $\mathbb{Z}$-grading like $A^\bullet(X)$, it does have a *virtual rank (or augmentation)*. That is, for each connected component U of X, there is a surjective ring homomorphism $\mathbf{vr} : K(U) \to \mathbb{Q}$ which assigns to each vector bundle E on U its rank. In addition, $K(X)$ has an involution which takes a vector bundle $[E]$ to its dual $[E^*]$.

Another important property of K-theory is that it is a so-called λ-*ring*. That is, for every non-negative integer i, there is a map $\lambda^i : K(Y) \to K(Y)$ defined by $\lambda^i([E]) := [\bigwedge^i E]$, where $\bigwedge^i E$ is the i-th exterior power of the vector bundle E. In particular, $\lambda^0([E]) = \mathbf{1}$, and $\lambda^i([E]) = 0$ if i is greater than the rank of E.

These maps satisfy the relations

$$\lambda^k(\mathcal{F} \oplus \mathcal{F}') = \bigoplus_{i=0}^{k} \lambda^i(\mathcal{F})\lambda^{k-i}(\mathcal{F}')$$

for all $k = 0, 1, 2, \dots$ and all $\mathcal{F}$, and $\mathcal{F}'$ in $K(Y)$. These relations can be neatly stated in terms of the universal formal power series in t

$$\lambda_t(\mathcal{F}) := \bigoplus_{i=0}^{\infty} \lambda^i(\mathcal{F})t^i \tag{2.3}$$

by demanding that λ_t satisfy the multiplicativity relation

$$\lambda_t(\mathcal{F} \oplus \mathcal{F}') = \lambda_t(\mathcal{F})\lambda_t(\mathcal{F}'). \tag{2.4}$$

If E is a rank-r vector bundle over X, then one can define

$$\lambda_{-1}([E]) := \bigoplus_{i=0}^{r} (-1)^i \lambda^i([E])$$

in $K(X)$, which will play an important role in this paper. In particular, $\lambda_{-1}([E^*])$ is the *K-theoretic Euler class of* E.

Like the Chow ring, the ring $K(X)$ is not quite a Frobenius algebra, because it is generally infinite dimensional, and its pairing may be degenerate; however, if we define the the *trace element* as

$$\tau^K(v) := \chi\big(X, \lambda_{-1}(T^*X) \otimes v\big) \tag{2.5}$$

for all v in $K(X)$, then we have the following proposition.

Proposition 2.3. *Let $K(X)$ be the K-theory of an irreducible, smooth, projective variety X.*

(1) *The tuple $(K(X), \otimes, \mathbf{1}, \eta_K, \tau^K)$ is a pre-Frobenius algebra.*
(2) *If $f : X \to Y$ is any morphism, then the associated pullback morphism $f^* : K(Y) \to K(X)$ is a morphism of commutative, associative algebras.*
(3) *(Projection formula) For any proper morphism $f : X \to Y$, if $\alpha \in K(X)$ and $\beta \in K(Y)$ we have*

$$f_*(\alpha \cup f^*(\beta)) = f_*(\alpha) \cup \beta.$$

2.3. Chern classes, Todd classes, and the Chern character. The *Chern polynomial of $\mathcal{F}$* in $K(X)$ is defined to be the universal formal power series in t

$$c_t(\mathcal{F}) := \sum_{i=0}^{\infty} c_i(\mathcal{F})t^i,$$

where $c_i(\mathcal{F})$, the *i-th Chern class of $\mathcal{F}$*, belongs to $A^i(X)$ for all i, and c_t and the c_i satisfy the following axioms:

(1) If $\mathcal{F} = [\mathcal{O}(D)]$ is a line bundle defined by a divisor D, then

$$c_t(\mathcal{F}) = \mathbf{1} + Dt.$$

(2) The Chern classes commute with pullback, i.e., if $f : X \to Y$ is any morphism, then $c_i(f^*\mathcal{F}) = f^*c_i(\mathcal{F})$ for all $\mathcal{F}$ in $K(X)$ and all i.

(3) If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence, then

$$c_t(\mathcal{F}) = c_t(\mathcal{F}')c_t(\mathcal{F}'').$$

In particular, $c_0(\mathcal{F}) = \mathbf{1}$ for all $\mathcal{F}$.

A fundamental tool is the *splitting principle*, which says that for any vector bundle E on X of rank r, there is a morphism $f : Y \to X$, such that $f^* : A^\bullet(X) \to A^\bullet(Y)$ is injective, and $f^*([E])$ splits (in K-theory) as a sum of line bundles:

$$f^*([E]) = [\mathcal{L}_1] \oplus \cdots \oplus [\mathcal{L}_r]. \tag{2.6}$$

We define the *Chern roots* of $[E]$ to be $a_i := c_1(\mathcal{L}_i)$, and thus by Property (3) of the Chern polynomial, we have

$$c_t([E]) = \prod_{i=1}^{r}(\mathbf{1} + a_i t). \tag{2.7}$$

Of course, the Chern roots depend on the choice of f, but any relations derived in this way among the Chern classes of $[E]$ will hold in $A^\bullet(X)$ regardless of the choice of f.

From the Chern classes, one can construct the *Chern character*

$$\mathbf{ch} : K(X) \to A^\bullet(X)$$

by associating to a rank-r vector bundle E over X the element

$$\mathbf{ch}([E]) := \sum_{i=1}^{r} \exp(a_i) = r + c_1([E]) + \frac{1}{2}\big(c_1^2([E]) - 2c_2([E])\big) + \cdots, \tag{2.8}$$

where $a_1, \ldots, a_r$ are the Chern roots of $[E]$.

For each connected component U of X, the *virtual rank* is the algebra homomorphism $\mathbf{vr} : K(U) \to \mathbb{Q}$, which is the composition of $\mathbf{ch} : K(U) \to A^\bullet(U)$ with the canonical projection $A^\bullet(U) \to A^0(U) \cong \mathbb{Q}$.

Remark 2.4. In general, the Chern character does not commute with push-forward. That is the content of the Grothendieck–Riemann–Roch theorem, which we will review shortly. Since the pairings of both $K(X)$ and $A^\bullet(X)$ are defined by pushforward, this means the Chern character does not respect the pairings.

To state the Grothendieck–Riemann–Roch theorem, we need the *Todd class* $\mathbf{td} : K(X) \to A^\bullet(X)$, which is defined by imposing the multiplicativity condition

$$\mathbf{td}(\mathcal{F} \oplus \mathcal{F}') = \mathbf{td}(\mathcal{F})\mathbf{td}(\mathcal{F}')$$

for all $\mathcal{F}$, $\mathcal{F}'$ in $K(X)$, and by also demanding that if E is a rank r vector bundle on X, then

$$\mathbf{td}([E]) := \prod_{i=1}^{r} \phi(a_i),$$

where $a_i = 1, \dots, r$ are the Chern roots of $[E]$ and

$$\phi(t) := \frac{te^t}{e^t - 1}$$

is regarded as a element in $\mathbb{Q}[\![t]\!]$. Therefore, $\mathbf{td}(\mathcal{F}) = \mathbf{1} + x$, where $x = c_1(\mathcal{F}) + (c_1^2(\mathcal{F}) + c_2(\mathcal{F}))/12 + \cdots$ belongs to $\bigoplus_{i=1}^{D} A^i(X)$.

Theorem 2.5 (Grothendieck–Riemann–Roch). *For any proper morphism $f : X \to Y$ of non-singular varieties and any $\mathcal{F} \in K(X)$, we have*

$$\mathbf{ch}(f_*(\mathcal{F})) \cup \mathbf{td}(TY) = f_*(\mathbf{ch}(\mathcal{F}) \cup \mathbf{td}(TX)), \tag{2.9}$$

where TX and TY are the tangent bundles of X and Y, respectively.

The following useful proposition intertwines many of the structures discussed in this section.

Proposition 2.6 [FL, Prop. I.5.3]. *If E is a vector bundle of rank r over X, then the following identity holds in $A^\bullet(X)$:*

$$\mathbf{td}([E])\mathbf{ch}\big(\lambda_{-1}([E^*])\big) = c_{\mathrm{top}}([E]), \tag{2.10}$$

where $c_{\mathrm{top}}([E])$ is the top Chern class $c_r([E])$.

Notation 2.7. When E is a vector bundle over X, we will often write $c_t(E)$ instead of $c_t([E])$, and similarly for λ_t, $\mathbf{td}$ and $\mathbf{ch}$.

We are now ready to state the key property of the Chern character. As mentioned in Remark 2.4, the Grothendieck–Riemann–Roch theorem implies that the Chern character cannot preserve the pairings, but it does preserve all of the other structures.

Definition 2.8. An *allometric isomorphism* of pre-Frobenius algebras is an isomorphism $\phi : (R, *, \eta, \mathbf{1}, \tau) \to (R', \star, \eta', \mathbf{1}', \tau')$ of unital, associative algebras that does not necessarily preserve the metric but does preserve the trace elements:

$$\phi^* \tau' = \tau.$$

We have the following theorem.

Theorem 2.9. *The Chern character* $\mathbf{ch} : K(X) \to A^\bullet(X)$ *is an allometric isomorphism of pre-Frobenius algebras. Furthermore, if* $f : X \to Y$ *is any morphism, then the following diagram commutes:*

$$\begin{array}{ccc} K(Y) & \xrightarrow{f^*} & K(X) \\ {\scriptstyle \mathbf{ch}}\downarrow & & {\scriptstyle \mathbf{ch}}\downarrow \\ A^\bullet(Y) & \xrightarrow{f^*} & A^\bullet(X). \end{array} \tag{2.11}$$

Proof. The only nonstandard part of this statement is that **ch** preserves the trace elements. This can be seen as follows. For all $\mathcal{F}$ in $K(X)$,

$$\begin{aligned} \tau^K(\mathcal{F}) &= \chi\big(X, \lambda_{-1}(T^*X) \otimes \mathcal{F}\big) \\ &= \int_X \mathbf{td}(TX) \cup \mathbf{ch}\big(\lambda_{-1}(T^*X) \otimes \mathcal{F}\big) \\ &= \int_X \mathbf{td}(TX) \cup \mathbf{ch}\big(\lambda_{-1}(T^*X)\big) \cup \mathbf{ch}(\mathcal{F}) \\ &= \int_X c_{\mathrm{top}}(TX) \cup \mathbf{ch}(\mathcal{F}) \\ &= \tau^A(\mathbf{ch}(\mathcal{F})), \end{aligned}$$

where the second equality holds by the Hirzebruch–Riemann–Roch theorem (a special case of the Grothendieck–Riemann–Roch theorem), the third because **ch** preserves multiplication, and the fourth by (2.10). □

Remark 2.10. Since $K(X)$ is a $\mathbb{Q}$-vector space, we will need to make sense of expressions such as $\mathbf{td}(\frac{1}{n}[E])$, where n is a positive integer and E is a rank r vector bundle over X. Observe that

$$\mathbf{td}([E]) = \mathbf{td}\left(\bigoplus_{i=1}^{n} \frac{1}{n}[E]\right) = \left(\mathbf{td}\left(\frac{1}{n}[E]\right)\right)^n.$$

Consider the formal power series $\Phi(t_1, \ldots, t_r)$ in $\mathbb{Q}[[t_1, \ldots, t_r]]$ defined by

$$\Phi(t_1, \ldots, t_r) := \prod_{i=1}^{r} \phi(t_i).$$

In particular, $\mathbf{td}([E]) = \Phi(a_1, \ldots, a_r)$. Since $\Phi(t_1, \ldots, t_r)$ is equal to $\mathbf{1}$ plus higher order terms, we can define $\Phi^{\frac{1}{n}}(t_1, \ldots, t_r)$ to be the unique formal power series in $\mathbb{Q}[[t_1, \ldots, t_r]]$ equal to $\mathbf{1}$ plus higher order terms such that

$$\left(\Phi^{\frac{1}{r}}(t_1, \ldots, t_r)\right)^r = \Phi(t_1, \ldots, t_r).$$

We define

$$\mathbf{td}^{\frac{1}{r}}([E]) := \Phi^{\frac{1}{r}}(a_1, \ldots, a_r).$$

3. G-graded G-modules and G-(equivariant) Frobenius algebras

In this section we introduce some algebraic structures which we will need throughout the rest of the paper.

Definition 3.1. A G-graded vector space $\mathcal{H} := \bigoplus_{m \in G} \mathcal{H}_m$ endowed with the structure of a G-module by isomorphisms $\rho(\gamma) : \mathcal{H} \xrightarrow{\sim} \mathcal{H}$ for all γ in G is said to be a *G-graded G-module* if $\rho(\gamma)$ takes $\mathcal{H}_m$ to $\mathcal{H}_{\gamma m \gamma^{-1}}$ for all m in G.

G-graded G-modules form a category whose objects are G-graded G-modules and whose morphisms are homomorphisms of G-modules which respect the G-grading. Furthermore, the dual of a G-graded G-module inherits the structure of a G-graded G-module.

Let us adopt the notation that v_m is a vector in $\mathcal{H}_m$ for any $m \in G$.

Definition 3.2. A tuple $((\mathcal{H}, \rho), *, \mathbf{1}, \eta, \tau)$ is said to be a *pre-G-(equivariant) Frobenius algebra* provided that the following properties hold:

(1) (G-graded G-module) $(\mathcal{H}, \rho)$ is a (possibly infinite-dimensional) G-graded G-module.
(2) (Self-invariance) For all γ in G, $\rho(\gamma) : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is the identity map.
(3) (G-graded pairing) η is a symmetric, (possibly degenerate) bilinear form on $\mathcal{H}$ such that $\eta(v_{m_1}, v_{m_2})$ is nonzero only if $m_1 m_2 = 1$.
(4) (G-graded multiplication) The binary product $(v_1, v_2) \mapsto v_1 * v_2$, called the *multiplication* on $\mathcal{H}$, preserves the G-grading (i.e., the multiplication is a map $\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2} \to \mathcal{H}_{m_1 m_2}$) and is distributive over addition.
(5) (Associativity) The multiplication is associative; i.e.,

$$(v_1 * v_2) * v_3 = v_1 * (v_2 * v_3)$$

for all v_1, v_2, and v_3 in $\mathcal{H}$.
(6) (Braided commutativity) The multiplication is invariant with respect to the braiding:

$$v_{m_1} * v_{m_2} = (\rho(m_1) v_{m_2}) * v_{m_1}$$

for all $m_i \in G$ and all $v_{m_i} \in \mathcal{H}_{m_i}$ with $i = 1, 2$.

(7) (G-equivariance of the multiplication)

$$(\rho(\gamma)v_1) * (\rho(\gamma)v_2) = \rho(\gamma)(v_1 * v_2)$$

for all γ in G and all $v_1, v_2 \in \mathcal{H}$.

(8) (G-invariance of the pairing)

$$\eta(\rho(\gamma)v_1, \rho(\gamma)v_2) = \eta(v_1, v_2)$$

for all γ in G and all $v_1, v_2 \in \mathcal{H}$.

(9) (Multiplicative invariance of the pairing)

$$\eta(v_1 * v_2, v_3) = \eta(v_1, v_2 * v_3)$$

for all $v_1, v_2, v_3 \in \mathcal{H}$.

(10) (G-invariant identity) The element $\mathbf{1}$ in $\mathcal{H}_1$ is the identity element under the multiplication, and it satisfies

$$\rho(\gamma)\mathbf{1} = \mathbf{1}$$

for all γ in G.

(11) (G-equivariant trace element) The *trace element* τ is a collection $\{\tau_{a,b}\}_{a,b\in G}$ of components $\tau_{a,b} \in \mathcal{H}^*$, such that $\tau_{a,b}(v_m)$ is nonzero only if $m = [a, b]$, and is G-equivariant, i.e.,

$$\tau_{\gamma a\gamma^{-1},\gamma b\gamma^{-1}} \circ \rho(\gamma) = \tau_{a,b}$$

for all a, b, γ in G.

(12) (Trace axiom) For all a, b in G, the trace element τ satisfies

$$\tau_{a,b} = \tau_{a\ ba^{-1},a^{-1}}.$$

We define the *characteristic element* $\boldsymbol{\tau}$ in $\mathcal{H}^*$ to be

$$\boldsymbol{\tau} := \frac{1}{|G|}\sum_{a,b\in G} \tau_{a,b},$$

and we call the element $\boldsymbol{\tau}(\mathbf{1}) \in \mathbb{Q}$ the *characteristic of the pre-G-Frobenius algebra.*

A *pre-Frobenius algebra* is a pre-G-Frobenius algebra with a trivial group G. In this case, the trace element and characteristic element are equal.

Remark 3.3. The G-equivariance of τ insures that the characteristic element $\boldsymbol{\tau}$ is G-invariant, i.e.,

$$\boldsymbol{\tau} \circ \rho(\gamma) = \boldsymbol{\tau}. \tag{3.1}$$

Remark 3.4. Any pre-G-Frobenius algebra $\mathcal{H}$, has a pre-Frobenius subalgebra $\mathcal{H}_1$ with a G-action which preserves the multiplication, unity, pairing, and trace element.

Remark 3.5. One can readily generalize the above definition to a pre-G-Frobenius superalgebra by introducing an additional $\mathbb{Z}/2\mathbb{Z}$-grading and inserting signs in the usual manner.

Definition 3.6. For a tuple $((\mathcal{H}, \rho), *, \mathbf{1}, \eta)$ satisfying all of the properties of a pre-G-Frobenius algebra which do not involve the trace element and where $\mathcal{H}$ is finite dimensional, we define the *canonical trace* to be

$$\tau_{a,b}(v) := \mathrm{Tr}_{\mathcal{H}_a}(L_v \circ \rho(b)) \tag{3.2}$$

for all a, b in G and v in $\mathcal{H}_{[a,b]}$, where L_v denotes left multiplication by v.

We define a *G-Frobenius algebra* [Kau02,Kau03,Tur] to be a tuple $((\mathcal{H}, \rho), *, \mathbf{1}, \eta)$, such that $\mathcal{H}$ is finite dimensional, the metric η is nondegenerate, and such that the tuple, together with the canonical trace, forms a pre-G-Frobenius algebra.

Remark 3.7. The trace axiom (Axiom (12)) for a G-Frobenius algebra $((\mathcal{H}, \rho), *, \mathbf{1}, \eta)$ with the canonical trace is easily seen to be equivalent to the more familiar form

$$\mathrm{Tr}_{\mathcal{H}_a}(L_v \circ \rho(b)) = \mathrm{Tr}_{\mathcal{H}_b}\left(\rho(a^{-1}) \circ L_v\right) \tag{3.3}$$

for all a, b in G and v in $\mathcal{H}_{[a,b]}$, where L_v denotes left multiplication by v.

A G-Frobenius algebra with trivial group G is nothing more than a Frobenius algebra. Moreover, in this case the canonical trace of the trivial G-Frobenius algebra reduces to the canonical trace of the Frobenius algebra.

Later in the paper we will construct a *stringy Chern character* which maps the pre-G-Frobenius algebra of stringy K-theory to the pre-G-Frobenius algebra of the stringy Chow ring. We will see that, as in the case of the ordinary Chern character, the stringy Chern character preserves all of the structure of a pre-G-Frobenius algebra except the pairing. This inspires the following definition:

Definition 3.8. An *allometric isomorphism* $\phi : ((\mathcal{H}, \rho), *, \eta, \mathbf{1}, \tau) \rightarrow ((\mathcal{H}', \rho'), \star, \eta', \mathbf{1}', \tau')$ of pre-G-Frobenius algebras is a G-equivariant isomorphism of unital algebras that does not necessarily preserve the pairing but does preserve the trace element:

$$\phi(\rho(m)v) = \rho(m)\phi(v)$$

for all $m \in G$ and all $v \in \mathcal{H}$, and

$$\phi^*\tau' = \tau.$$

Definition 3.9. Let $(\mathcal{H}, \rho)$ be a G-graded G-module. Let $\pi_G : \mathcal{H} \to \mathcal{H}$ be the averaging map

$$\pi_G(v) := \frac{1}{|G|} \sum_{\gamma \in G} \rho(\gamma)v$$

for all v in $\mathcal{H}$. Let $\overline{\mathcal{H}}$ be the image of π_G. The vector space $\overline{\mathcal{H}}$ is called the *space of G-coinvariants of $\mathcal{H}$,* and it inherits a grading by the set $\overline{G}$ of conjugacy classes of G:

$$\overline{\mathcal{H}} = \bigoplus_{\overline{\gamma} \in \overline{G}} \overline{\mathcal{H}}_{\overline{\gamma}}.$$

Since the group G is finite, the space $\overline{\mathcal{H}}$ is equal to the space $\mathcal{H}^G$ of G-invariants of $\mathcal{H}$.

For any bilinear form η on $\mathcal{H}$, we define $\overline{\eta}$ to be the restriction of the bilinear form $\frac{1}{|G|}\eta$ to $\overline{\mathcal{H}}$. Finally, define the trace element $\overline{\tau}$ on $\overline{\mathcal{H}}$ to be the restriction of the characteristic element $\boldsymbol{\tau}$ to $\overline{\mathcal{H}}$.

We have the following proposition.

Proposition 3.10. *If the tuple $((\mathcal{H}, \rho), *, \mathbf{1}, \eta, \tau)$ is a pre-G-Frobenius algebra, then its G-coinvariants $(\overline{\mathcal{H}}, *, \mathbf{1}, \overline{\eta}, \overline{\tau})$ form a pre-Frobenius algebra, where $*$ is induced from $\mathcal{H}$. Moreover, if the tuple $((\mathcal{H}, \rho), *, \mathbf{1}, \eta)$ is a G-Frobenius algebra with canonical trace element τ, as defined in (3.2), then its ring of G-coinvariants $(\overline{\mathcal{H}}, *, \mathbf{1}, \overline{\eta})$ is a Frobenius algebra whose induced trace element $\overline{\tau}$ is equal to its canonical trace element, i.e.,*

$$\overline{\tau}(\overline{v}) = \mathrm{Tr}_{\overline{\mathcal{H}}}(L_{\overline{v}}) \tag{3.4}$$

for all $\overline{v}$ in $\overline{\mathcal{H}}$.

Proof. All that must be shown is (3.4). For all m in G, and for all v_m in $\mathcal{H}_m$, we have

$$\begin{aligned}
\mathrm{Tr}_{\overline{\mathcal{H}}}(L_{\pi_G(v_m)}) &= \mathrm{Tr}_{\mathcal{H}}(L_{\pi_G(v_m)} \circ \pi_G) \\
&= \frac{1}{|G|^2} \sum_{\beta,\gamma \in G} \mathrm{Tr}_{\mathcal{H}}(L_{\rho(\gamma)v_m} \circ \rho(\beta)) \\
&= \frac{1}{|G|^2} \sum_{\beta,\gamma \in G} \mathrm{Tr}_{\mathcal{H}}\left(L_{\rho(\gamma)v_m} \circ \rho(\gamma) \circ \rho(\gamma^{-1}) \circ \rho(\beta)\right) \\
&= \frac{1}{|G|^2} \sum_{\beta,\gamma \in G} \mathrm{Tr}_{\mathcal{H}}\left(\rho(\gamma) \circ L_{v_m} \circ \rho(\gamma^{-1}) \circ \rho(\beta)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|G|^2} \sum_{\beta,\gamma \in G} \mathrm{Tr}_{\mathcal{H}} \left(L_{v_m} \circ \rho(\gamma^{-1}) \circ \rho(\beta) \circ \rho(\gamma) \right) \\
&= \frac{1}{|G|^2} \sum_{\beta,\gamma \in G} \mathrm{Tr}_{\mathcal{H}} \left(L_{v_m} \circ \rho(\gamma^{-1}\beta\gamma) \right) \\
&= \frac{1}{|G|^2} \sum_{b,\gamma \in G} \mathrm{Tr}_{\mathcal{H}} (L_{v_m} \circ \rho(b)) \\
&= \frac{1}{|G|} \sum_{b \in G} \mathrm{Tr}_{\mathcal{H}} (L_{v_m} \circ \rho(b)),
\end{aligned}
$$

where the second equality follows from the definition of π_G, the fourth from the G-equivariance of the multiplication, and the fifth from the cyclicity of the trace. Now, for all a and b in G, let $\phi_a : \mathcal{H}_a \to \mathcal{H}_{mba\,b^{-1}}$ be the restriction of $L_{v_m} \circ \rho(b)$ to $\mathcal{H}_a$. The map ϕ_a preserves $\mathcal{H}_a$ if and only if $mbab^{-1} = a$ or, equivalently, if $m = [a, b]$. Furthermore, ϕ_a only contributes to the trace $\mathrm{Tr}_{\mathcal{H}}(L_{v_m} \circ \rho(b))$ when $m = [a, b]$. Therefore,

$$
\frac{1}{|G|} \sum_{b \in G} \mathrm{Tr}_{\mathcal{H}} (L_{v_m} \circ \rho(b)) = \frac{1}{|G|} \sum_{a,b} \mathrm{Tr}_{\mathcal{H}_a} (L_{v_m} \circ \rho(b)) = \frac{1}{|G|} \sum_{a,b} \tau_{a,b}(v_m),
$$

where the last two sums are over all $a, b \in G$ such that $[a, b] = m$. □

4. The stringy Chow ring and stringy K-theory of a variety with G-action

In this section, we discuss the main properties of the stringy Chow ring $\mathcal{A}(X, G)$ and the stringy K-theory $\mathcal{K}(X, G)$ of a smooth, projective variety with an action of a finite group G.

As discussed in the introduction (Sect. 1.3), the vector spaces underlying the stringy Chow ring and stringy K-theory of X are just the usual Chow ring and K-theory, respectively, of the *inertia variety*

$$
I_G(X) := \coprod_{m \in G} X^m \subseteq X \times G,
$$

where $X^m := \{(x, m) | \rho(m)x = x\}$ with its induced G-action. Again, the reader should beware that the inertia *variety* is not the same as the inertia *orbifold*

$$
[I_G(X)/G] = \coprod_{[\![g]\!]} [X^g / Z_G(g)]
$$

of [CR1,AGV], which is the stack quotient of the inertia variety $I_G(X)$ by the action of G.

Recall (see Definition 1.3) that one of the key elements in the construction of both the stringy multiplication and the stringy Chern character is the element $\mathcal{S} \in I_G(X)$, defined as

$$\mathcal{S}_m := \mathcal{S}|_{X^m} := \bigoplus_{k=0}^{r-1} \frac{k}{r} W_{m,k}. \tag{4.1}$$

The G-equivariant involution $\sigma : X^m \to X^{m^{-1}}$ yields a G-equivariant isomorphism $\sigma^* : W_{m^{-1}} \to W_m$ for all m in G. If m acts by multiplication by ζ^k, then m^{-1} acts by ζ^{r-k}, so we have

$$\sigma^* W_{m^{-1},0} = W_{m,0} \tag{4.2}$$

and

$$\sigma^* W_{m^{-1},k} = W_{m,r-k} \tag{4.3}$$

for all $k \in \{1, \ldots, r-1\}$. Consequently, the induced map $\sigma^* : K(X^{m^{-1}}) \to K(X^m)$ satisfies

$$\mathcal{S}_m \oplus \sigma^* \mathcal{S}_{m^{-1}} = N_m, \tag{4.4}$$

since the normal bundle, N_m, of X^m in X satisfies the equation $N_m = W_m \ominus W_{m,0}$.

The virtual rank of $\mathcal{S}_m$ on a connected component U of X^m is the age $a(m, U)$ (see Definition 1.3). Taking the virtual rank of both sides of (4.4) yields the well-known equation

$$a(m, U) + a(m^{-1}, U) = \operatorname{codim}(U \subseteq X). \tag{4.5}$$

This supports the interpretation of $\mathcal{S}_m$ as a K-theoretic version of the age.

Recall that one may use the age to define a rational grading on $\mathcal{A}(X, G)$.

Definition 4.1. For all m in G, all connected components U of X^m, and all elements v_m in $A^p(U) \subseteq \mathcal{A}_m(X)$, for p the usual integral degree in the Chow ring, we define a $\mathbb{Q}$-grading which we call the *stringy grading* $|v_m|_{str}$ on $\mathcal{A}_m(X)$ by

$$|v_m|_{str} := a(m, U) + p. \tag{4.6}$$

Remark 4.2. Sometimes the $\mathbb{Q}$-grading just happens to be integral. For example, if X is n-dimensional and its canonical bundle K_X has a nowhere-vanishing section Ω, then for all m in G, we have

$$\rho(m)^* \Omega = \exp(2\pi i a(m)) \Omega.$$

Thus, if G preserves Ω, then $a(m)$ must be an integer.

A special case is when X is $2n$-dimensional, possessing a (complex algebraic) symplectic form ω in $\bigwedge^2 T^* X$. This can arise if X happens to be a hyper-Kähler manifold. If, in addition, G preserves ω, then G preserves the nowhere vanishing section ω^n of K_X. In this case, for all m in G and for every connected component U of X^m, the associated age [Kal] is the integer

$$a(m, U) = \frac{1}{2}\operatorname{codim}(U \subseteq X).$$

Remark 4.3. Unlike the stringy Chow ring, the ring $\mathcal{K}(X, G)$ lacks a $\mathbb{Q}$-grading. This should not be surprising, as even ordinary K-theory lacks a grading by "dimension." In particular, the virtual rank does not enjoy the same good properties in K-theory that grading by codimension has in the Chow ring.

The multiplication in the string Chow ring and stringy K-theory were already defined in Definitions 1.6 and 1.7, but to see that these form pre-G-Frobenius algebras, we also need to define their trace element.

Definition 4.4. The *trace element* $\tau^{\mathcal{A}}$ *of* $\mathcal{A}(X, G)$ is a collection of *components* $\{\tau_{a,b}\}_{a,b\in G}$, where $\tau_{a,b} \in \mathcal{A}(X, G)^*$ is defined to be

$$\tau_{a,b}(v_m) := \begin{cases} \int_{X^{\langle a,b\rangle}} v_m\big|_{X^{\langle a,b\rangle}} \cup c_{\mathrm{top}}\big(TX^{\langle a,b\rangle} \oplus \mathcal{S}_{[a,b]}\big|_{X^{\langle a,b\rangle}}\big) & \text{if } m = [a,b] \\ 0 & \text{if } m \neq [a,b] \end{cases} \tag{4.7}$$

for all a, b, m in G and v_m in $\mathcal{A}_m(X)$.

We define the trace element of stringy K-theory similarly.

Definition 4.5. The *trace element* $\tau^{\mathcal{K}}$ *of* $\mathcal{K}(X, G)$ is a collection of *components* $\{\tau_{a,b}\}_{a,b\in G}$, where $\tau_{a,b} \in \mathcal{K}(X, G)^*$ is defined to be

$$\tau_{a,b}(\mathcal{F}_m) = \begin{cases} \chi\big(X^{\langle a,b\rangle}, \mathcal{F}_m\big|_{X^{\langle a,b\rangle}} \cup \lambda_{-1}\big(TX^{\langle a,b\rangle} \oplus \mathcal{S}_{[a,b]}\big|_{X^{\langle a,b\rangle}}\big)^*\big) \\ \qquad\qquad \text{if } m = [a,b] \\ 0 \qquad\qquad \text{if } m \neq [a,b] \end{cases} \tag{4.8}$$

for all a, b, m in G, and $\mathcal{F}_m$ in $\mathcal{K}_m(X)$. Here χ denotes the Euler characteristic.

We will show in Sect. 5.2 that the element $TX^{\langle a,b\rangle} \oplus \mathcal{S}_{[a,b]}|_{X^{\langle a,b\rangle}}$ in $K(X^{\langle a,b\rangle})$ can be represented by a vector bundle. Similarly, we will prove in Sect. 8 that $\mathcal{R}(\mathbf{m})$ is represented by a vector bundle. In the meantime, we will use that fact to prove the following results.

Theorem 4.6. *Let X be a smooth, projective variety with an action of a finite group G.*

(1) *The tuple* $((\mathcal{A}(X, G), \rho), *, \mathbf{1}, \eta_{\mathcal{A}}, \tau^{\mathcal{A}})$ *is a pre-G-Frobenius algebra.*
(2) $|\mathbf{1}|_{str} = 0$.

(3) *The multiplication respects the* $\mathbb{Q}$*-grading, i.e., for all homogeneous elements* v_{m_i} *in* $\mathcal{A}_{m_i}(X)$*, for* $i = 1, 2$*, we have*

$$|v_{m_1} * v_{m_2}|_{str} = |v_{m_1}|_{str} + |v_{m_2}|_{str}.$$

(4) *The pairing has a definite* $\mathbb{Q}$*-grading, i.e., for all homogeneous elements* v_{m_1} *in* $\mathcal{A}_{m_1}(X)$ *and* v_{m_2} *in* $\mathcal{A}_{m_2}(X)$ *we have* $\eta_{\mathcal{A}}(v_{m_1}, v_{m_2}) = 0$ *unless* $m_1 m_2 = 1$ *and*

$$|v_{m_1}|_{str} + |v_{m_2}|_{str} = \dim X. \tag{4.9}$$

(5) *The components* $\{\tau_{a,b}\}$ *of* $\tau^{\mathcal{A}}$*, satisfy* $\tau_{a,b}(v_m) = 0$ *unless* $|v_m|_{str} = 0$ *and* $m = [a, b]$*.*

Theorem 4.7. *Let X be a smooth, projective variety with an action of a finite group G. The tuple* $((\mathcal{K}(X, G), \rho), *, \mathbf{1}, \eta_{\mathcal{K}}, \tau^{\mathcal{K}})$ *is a pre-G-Frobenius algebra, where the trace element* $\tau^{\mathcal{K}}$ *is defined (4.8).*

For both Theorems (4.6 and 4.7), the only nontrivial parts are the trace axiom and the associativity of multiplication. These are proved in Lemmas 5.9 and 5.4, respectively.

Example 4.8. Consider the case where $m_i = 1$ for some $i = 1, 2, 3$. In this case the bundle $\mathcal{R}$ on $X^{\mathbf{m}}$ is trivial.

If $m_1 = 1$ and $m_2 m_3 = 1$, then the stringy multiplication is given by the restriction to X^{m_3} of the ordinary multiplication in ordinary K-theory, i.e.,

$$\mathcal{F}_{m_1=1} * \mathcal{F}_{m_2} = \mathcal{F}_{m_1}\big|_{X^{m_3}} \otimes \sigma^* \mathcal{F}_{m_2}. \tag{4.10}$$

A similar result holds if $m_2 = 1$ and $m_1 m_3 = 1$. In particular, this means that stringy multiplication on the untwisted sector $\mathcal{K}_1(X)$ coincides with the ordinary multiplication on $\mathcal{K}_1(X)$.

More interesting is the case where $m_3 = 1$ and $m_1 m_2 = 1$. In this case, we have

$$\mathcal{F}_{m_1} * \mathcal{F}_{m_2} = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}^*_{m_1} \mathcal{F}_{m_1} \otimes \mathbf{e}^*_{m_2} \mathcal{F}_{m_2}\big). \tag{4.11}$$

Here, even though the bundle $\mathcal{R}(\mathbf{m})$ is trivial, the stringy multiplication is nontrivial, since the map $\check{\mathbf{e}}_{m_3}$ will generally be between varieties of different dimensions.

Remark 4.9. If $a = b = 1$ in $\mathcal{A}(X, G)$, then for all v_1 in $\mathcal{A}_1(X)$, we have

$$\tau_{1,1}(v_1) = \int_X v_1 \cup c_{\mathrm{top}}(TX). \tag{4.12}$$

Therefore, the component $\tau_{1,1}$ of the trace element on the untwisted sector $\mathcal{A}_1(X)$ of the stringy Chow ring agrees with the trace element of the ordinary Chow ring $A^\bullet(X)$.

Remark 4.10. The characteristic of $((\mathcal{A}(X, G), \rho), *, \mathbf{1}, \eta_{\mathcal{A}}, \tau^{\mathcal{A}})$ is

$$\boldsymbol{\tau}(\mathbf{1}) = \frac{1}{|G|} \sum_{ab=ba} \chi(X^{\langle a,b\rangle}), \tag{4.13}$$

where the sum is over all commuting pairs a, b in G, and

$$\chi(X^{\langle a,b\rangle}) = \int_{[X^{\langle a,b\rangle}]} c_{\mathrm{top}}(TX^{\langle a,b\rangle})$$

is the usual Euler characteristic. This expression (4.13) coincides with the "stringy Euler characteristic" introduced by physicists [DHVW] (see also [AS]).

5. Associativity and the trace axiom

In this section, we use the fact that the element $\mathcal{R}$ defined in (1.4), is a vector bundle (proved in Corollary 8.4) to give an elementary proof of associativity and the trace axiom for both the stringy Chow ring and stringy K-theory.

5.1. Associativity. Let us recall some excess intersection theory. Consider smooth, projective varieties V, Y_1, Y_2, and Z which form the following Cartesian square

$$\begin{array}{ccc} V & \xrightarrow{i_1} & Y_1 \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ Y_2 & \xrightarrow{i_2} & Z, \end{array} \tag{5.1}$$

where i_1, i_2 are regular embeddings and j_1, j_2 are morphisms of schemes.

Let $E(V, Y_1, Y_2) \to V$ be the *excess normal (vector) bundle*, which is the cokernel of the map $N_{V/Y_1} \to N_{Y_2/Z}|_V$, where N_{V/Y_1} and $N_{Y_2/Z}$ are the normal bundles of V in Y_1 and of Y_2 in Z, respectively. In $K(V)$ one thus obtains the equality

$$[E(Z, Y_1, Y_2)] = TZ|_V \ominus TY_1|_V \ominus TY_2|_V \oplus TV. \tag{5.2}$$

Under these hypotheses, the following theorem holds (see Theorems 1.3 and 1.4 in [FL, Chap. IV.1]).

Theorem 5.1. *For all $\mathcal{F}$ in $K(Y_2)$ and v in $A^\bullet(Y_2)$,*

$$j_1^* i_{2*}\mathcal{F} = i_{1*}\big(\lambda_{-1}\big(E(Z, Y_1, Y_2)^*\big) \otimes j_2^*\mathcal{F}\big) \tag{5.3}$$

and

$$j_1^* i_{2*} v = i_{1*}\big(c_{\mathrm{top}}(E(Z, Y_1, Y_2)) \cup j_2^* v\big). \tag{5.4}$$

The previous theorem gives rise to the following fact about $\mathcal{R}$.

Lemma 5.2. *Let* $\mathbf{m} := (m_1, \dots, m_4)$ *in* G^4 *such that* $m_1m_2m_3m_4 = 1$. *Let* $X^{\mathbf{m}}$ *consist of those points in* X *which are fixed by* m_i *for all* $i \in \{1, \dots, 4\}$. *The following equation holds in* $K(X^{\mathbf{m}})$*:*

$$\begin{aligned}\mathcal{R}\big(m_1, m_2, (m_1m_2)^{-1}\big)\big|_{X^{\mathbf{m}}} &\oplus \mathcal{R}(m_1m_2, m_3, m_4)\big|_{X^{\mathbf{m}}} \oplus E_{m_1,m_2}\\ = \mathcal{R}(m_1, m_2m_3, m_4)\big|_{X^{\mathbf{m}}} &\oplus \mathcal{R}\big(m_2, m_3, (m_2m_3)^{-1}\big)\big|_{X^{\mathbf{m}}} \oplus E_{m_2,m_3},\end{aligned} \tag{5.5}$$

where

$$E_{m_1,m_2} := E(X^{m_1m_2}, X^{\langle m_1,m_2\rangle}, X^{\langle m_1m_2,m_3\rangle}) \tag{5.6}$$

and

$$E_{m_2,m_3} := E(X^{m_2m_3}, X^{\langle m_1,m_2m_3\rangle}, X^{\langle m_2,m_3\rangle}). \tag{5.7}$$

Furthermore, both sides of (5.5) are equal in $K(X^{\mathbf{m}})$ *to*

$$TX^{\mathbf{m}} \ominus TX\big|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{4} \mathcal{S}_{m_i}\big|_{X^{\mathbf{m}}}. \tag{5.8}$$

Proof. Plug in the definitions of the excess normal bundles and the formula for the obstruction bundle $\mathcal{R}$ from (8.3), then apply (4.4) and simplify the result. One discovers that both the right hand and left hand sides of (5.5) are equal in $K(X^{\mathbf{m}})$ to (5.8). □

Remark 5.3. For the reader familiar with the G-stable maps of [JKK], we note that the element $TX^{\mathbf{m}} \ominus TX|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{4} \mathcal{S}_{m_i}|_{X^{\mathbf{m}}}$ in (5.8) may be interpreted as the fiber of the obstruction bundle over $\{q\} \times X^{\mathbf{m}}$ in $\xi_{0,4}(\mathbf{m}) \times X^{\mathbf{m}} = \xi_{0,4}(X, 0, \mathbf{m})$, where q is any point in $\xi_{0,4}(\mathbf{m})$. This can be seen by an argument similar to that in the proof of [JKK, Prop. 6.21].

Lemma 5.4. *Let* X *be a smooth, projective variety with an action of a finite group* G. *The multiplications in stringy K-theory* $((\mathcal{K}(X, G), \rho), *, \mathbf{1}, \eta_{\mathcal{K}})$ *and in the stringy Chow ring* $((\mathcal{A}(X, G), \rho), *, \mathbf{1}, \eta_{\mathcal{A}})$ *are both associative.*

Proof. Consider $\mathbf{m} = (m_1, m_2, m_3, m_4)$ in G^4 such that $m_1m_2m_3m_4 = 1$. If E_{m_1,m_2} and E_{m_2,m_3} are defined as in (5.6) and (5.7), then the following equalities hold:

$$\begin{aligned}c_{\mathrm{top}}\big(\mathcal{R}\big(m_1, m_2, (m_1m_2)^{-1}\big)\big)\big|_{X^{\mathbf{m}}} &\cup c_{\mathrm{top}}(\mathcal{R}(m_1m_2, m_3, m_4))\big|_{X^{\mathbf{m}}}\\ &\cup c_{\mathrm{top}}(E_{m_1,m_2})\\ = c_{\mathrm{top}}(\mathcal{R}(m_1, m_2m_3, m_4))\big|_{X^{\mathbf{m}}} &\cup c_{\mathrm{top}}\big(\mathcal{R}\big(m_2, m_3, (m_2m_3)^{-1}\big)\big)\big|_{X^{\mathbf{m}}}\\ &\cup c_{\mathrm{top}}(E_{m_2,m_3})\end{aligned} \tag{5.9}$$

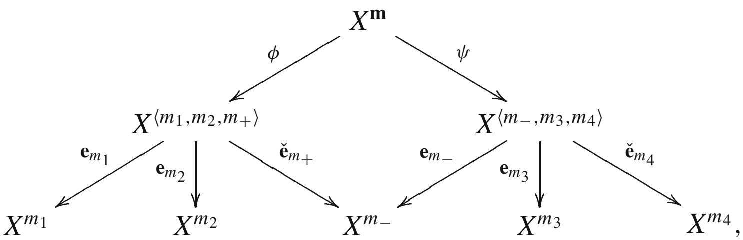
and

$$\begin{aligned}
\lambda_{-1}\big(\mathcal{R}\big(m_1, m_2, (m_1m_2)^{-1}\big)^*\big)\big|_{X^{\mathbf{m}}} &\otimes \lambda_{-1}\big(\mathcal{R}(m_1m_2, m_3, m_4)^*\big)\big|_{X^{\mathbf{m}}} \\
&\otimes \lambda_{-1}\big(E^*_{m_1,m_2}\big) \\
= \lambda_{-1}\big(\mathcal{R}(m_1, m_2m_3, m_4)^*\big)\big|_{X^{\mathbf{m}}} &\otimes \lambda_{-1}\big(\mathcal{R}\big(m_2, m_3, (m_2m_3)^{-1}\big)^*\big)\big|_{X^{\mathbf{m}}} \\
&\otimes \lambda_{-1}\big(E^*_{m_2,m_3}\big). \qquad (5.10)
\end{aligned}$$

Equation (5.9) follows by taking the top Chern class of both sides of (5.5) and then using multiplicativity of c_{top}. Equation (5.10) follows by taking the dual of (5.5), applying λ_{-1}, and then using multiplicativity of λ_{-1}.

Associativity will follow from (5.9) and (5.10) and the definitions of the multiplications as follows.

Let $m_+ = (m_1m_2)^{-1}$ and $m_- = (m_1m_2)$. Consider the following diagram:



where ϕ and ψ are the obvious inclusions. Note that the diamond in the middle is Cartesian and that the usual inclusions $\epsilon_i : X^{\mathbf{m}} \to X^{m_i}$ factor as

$$\epsilon_1 = \mathbf{e}_{m_1} \circ \phi \qquad \epsilon_2 = \mathbf{e}_{m_2} \circ \phi \qquad (5.11)$$

$$\epsilon_3 = \mathbf{e}_{m_3} \circ \psi \qquad \epsilon_4 = \mathbf{e}_{m_4} \circ \psi. \qquad (5.12)$$

Finally, we define

$$\check{\epsilon}_4 = \sigma \circ \epsilon_4 = \check{\mathbf{e}}_{m_4} \circ \psi.$$

For any $\mathcal{F}_1 \in \mathcal{K}_{m_1}$, $\mathcal{F}_2 \in \mathcal{K}_{m_2}$, $\mathcal{F}_3 \in \mathcal{K}_{m_3}$, we have

$$\begin{aligned}
&(\mathcal{F}_1 * \mathcal{F}_2) * \mathcal{F}_3 \\
&\quad = (\check{\mathbf{e}}_{m_4})_*\big(\mathbf{e}^*_{m_-}(\check{\mathbf{e}}_{m_+})_*\big[\mathbf{e}^*_{m_1}\mathcal{F}_1 \otimes \mathbf{e}^*_{m_2}\mathcal{F}_2 \otimes \lambda_{-1}\big(\mathcal{R}(m_1, m_2, m_+)^*\big)\big] \\
&\qquad\qquad \otimes \mathbf{e}^*_{m_3}\mathcal{F}_3 \otimes \lambda_{-1}\big(\mathcal{R}(m_-, m_3, m_4)^*\big)\big) \\
&\quad = (\check{\mathbf{e}}_{m_4})_*\big(\psi_*\big(\phi^*\big[\mathbf{e}^*_{m_1}\mathcal{F}_1 \otimes \mathbf{e}^*_{m_2}\mathcal{F}_2 \otimes \lambda_{-1}\big(\mathcal{R}(m_1, m_2, m_+)^*\big)\big] \\
&\qquad\qquad \otimes \lambda_{-1}\big(E^*_{m_1,m_2}\big)\big) \otimes \mathbf{e}^*_{m_3}\mathcal{F}_3 \otimes \lambda_{-1}\big(\mathcal{R}(m_-, m_3, m_4)^*\big)\big) \\
&\quad = (\check{\mathbf{e}}_{m_4})_*\big(\psi_*\big(\phi^*\mathbf{e}^*_{m_1}\mathcal{F}_1 \otimes \phi^*\mathbf{e}^*_{m_2}\mathcal{F}_2 \otimes \phi^*\big(\lambda_{-1}\big(\mathcal{R}(m_1, m_2, m_+)^*\big)\big) \\
&\qquad\qquad \otimes \lambda_{-1}\big(E^*_{m_1,m_2}\big) \otimes \psi^*\mathbf{e}^*_{m_3}\mathcal{F}_3 \otimes \psi^*\big(\lambda_{-1}\big(\mathcal{R}(m_-, m_3, m_4)^*\big)\big)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&= (\check{\epsilon}_4)_*\big(\epsilon_1^*\mathcal{F}_1 \otimes \epsilon_2^*\mathcal{F}_2 \otimes \phi^*\lambda_{-1}\big(\mathcal{R}(m_1, m_2, m_+)^*\big) \otimes \lambda_{-1}\big(E^*_{m_1,m_2}\big) \\
&\qquad \otimes \epsilon_3^*\mathcal{F}_3 \otimes \psi^*\big(\lambda_{-1}\big(\mathcal{R}(m_-, m_3, m_4)^*\big)\big)\big) \\
&= (\check{\epsilon}_4)_*\big(\epsilon_1^*\mathcal{F}_1 \otimes \epsilon_2^*\mathcal{F}_2 \otimes \epsilon_3^*\mathcal{F}_3 \otimes \lambda_{-1}\big(\mathcal{R}(m_1, m_2, m_+)^*\big)\big|_{X^{\mathbf{m}}} \\
&\qquad \otimes \lambda_{-1}\big(\mathcal{R}(m_-, m_3, m_4)^*\big)\big|_{X^{\mathbf{m}}} \otimes \lambda_{-1}\big(E^*_{m_1,m_2}\big)\big), \qquad (5.13)
\end{aligned}
$$

where the first equality is the definition, the second equality follows from Theorem 5.1, the third equality follows from the projection formula, and the fourth and fifth equalities follow from (5.11) and (5.12) and the definitions of ψ and ϕ.

A similar argument shows that the product $\mathcal{F}_1 * (\mathcal{F}_2 * \mathcal{F}_3)$ is given by

$$
\begin{aligned}
&\mathcal{F}_1 * (\mathcal{F}_2 * \mathcal{F}_3) \\
&\quad = (\check{\epsilon}_4)_*\big(\epsilon_1^*\mathcal{F}_1 \otimes \epsilon_2^*\mathcal{F}_2 \otimes \epsilon_3^*\mathcal{F}_3 \otimes \lambda_{-1}\big(\mathcal{R}(m_1, m_2m_3, m_4)^*\big)\big|_{X^{\mathbf{m}}} \\
&\qquad \otimes \lambda_{-1}\big(\mathcal{R}\big(m_2, m_3, (m_2m_3)^{-1}\big)^*\big)\big|_{X^{\mathbf{m}}} \otimes \lambda_{-1}\big(E^*_{m_2,m_3}\big)\big). \qquad (5.14)
\end{aligned}
$$

By (5.10) these two expressions (5.13) and (5.14) are equal, hence associativity holds. □

5.2. The trace axiom. We now prove the trace axiom in a similar way. Throughout this section, we fix elements a and b in G and let $m_1 := [a, b]$. Let $\mathbf{m}' := ({m'}_1, {m'}_2, {m'}_3) := ([a, b], bab^{-1}, a^{-1})$. Let $H := \langle a, b\rangle$ and let $H' := \langle \mathbf{m}' \rangle \leq H$ be the subgroup generated by the elements of $\mathbf{m}'$. Let $\mathcal{R}(\mathbf{m}')$ denote the element in $K(X^{H'})$ from (1.4).

Consider the commutative diagram

$$
\begin{array}{ccc}
X^H & \xrightarrow{\;j_2'\;} & X^{H'} \\
{\scriptstyle j_1'}\downarrow & & \downarrow{\scriptstyle \Delta_2'} \\
X^a & \xrightarrow{\;\Delta_1'\;} & X^{bab^{-1}} \times X^{a^{-1}}.
\end{array}
\qquad (5.15)
$$

Here j_1' and j_2' are the obvious inclusion morphisms, Δ_2' is the diagonal map, and Δ_1' is the composition of the morphisms

$$X^a \xrightarrow{\Delta} X^a \times X^a \xrightarrow{\rho(b)\times\sigma} X^{bab^{-1}} \times X^{a^{-1}},$$

where Δ is the diagonal map and $\rho(b)$ is the action of b. Let $\mathcal{E}'$ be the excess intersection bundle $E(X^{bab^{-1}} \times X^{a^{-1}}, X^{H'}, X^a)$.

Theorem 5.5. *The following equality holds in $K(X^H)$:*

$$j_2'^*\mathcal{R}(\mathbf{m}') \oplus \mathcal{E}' = TX^H \oplus \mathcal{S}_{m_1}\big|_{X^H}. \qquad (5.16)$$

The previous theorem together with Corollary 8.4 and the fact that $\mathcal{E}'$ is an excess intersection (vector) bundle, yields the following.

Corollary 5.6. *$TX^H \oplus \mathcal{S}_{m_1}|_{X^H}$ can be represented in $K(X^H)$ by a vector bundle.*

Proof of Theorem 5.5. All equalities in this proof are understood to be in $K(X^H)$. Observe that

$$\begin{aligned} j_2'^{*}\Delta_2'^{*}T(X^{bab^{-1}} \times X^{a^{-1}}) &= TX^{bab^{-1}}\big|_{X^H} \oplus TX^{a^{-1}}\big|_{X^H} \\ &= \rho(b)(TX^a)\big|_{X^H} \oplus \sigma^* TX^a\big|_{X^H} \\ &= TX^a\big|_{X^H} \oplus TX^a\big|_{X^H}, \end{aligned}$$

where the third equality follows from the fact that $\rho(b) \times \sigma$ is an isomorphism. Plugging this into the definition of the excess intersection bundle yields

$$\mathcal{E}' = TX^H \oplus TX^a\big|_{X^H} \oplus TX^a\big|_{X^H} \ominus TX^a\big|_{X^H} \ominus TX^{H'}\big|_{X^H},$$

which simplifies to

$$\mathcal{E}' = TX^H \oplus TX^a\big|_{X^H} \ominus TX^{H'}\big|_{X^H}. \tag{5.17}$$

On the other hand, (8.3) yields the equality

$$\mathcal{R}(\mathbf{m}') = TX^{H'}\big|_{X^H} \ominus TX\big|_{X^H} \oplus \mathcal{S}_{m_1}\big|_{X^H} \oplus \mathcal{S}_{bab^{-1}}\big|_{X^H} \oplus \mathcal{S}_{a^{-1}}\big|_{X^H}.$$

Together with the equality

$$\mathcal{S}_{bab^{-1}}\big|_{X^H} = \rho(b)(\mathcal{S}_a)\big|_{X^H}$$

and (4.4), we obtain

$$\mathcal{R}(\mathbf{m}') = TX^{H'}\big|_{X^H} \ominus TX^a\big|_{X^H} \oplus \mathcal{S}_{m_1}\big|_{X^H}. \tag{5.18}$$

Combining (5.17) and (5.18) yields the identity

$$j_2'^{*}\mathcal{R}(\mathbf{m}') \oplus \mathcal{E}' = TX^H \oplus \mathcal{S}_{m_1}\big|_{X^H}. \tag{5.19}$$

□

Remark 5.7. If we make the replacement $(a, b) \mapsto (\tilde{a}, \tilde{b}) := (aba^{-1}, a^{-1})$ everywhere in the Theorem 5.5, then since $\langle a, b\rangle = \langle \tilde{a}, \tilde{b}\rangle$ and $m_1 = [a, b] = [\tilde{a}, \tilde{b}]$, the right hand side of (5.16) stays the same, while the left hand side changes. Hence, one obtains an interesting equality between the corresponding left hand sides of (5.16) before and after making the substitution $(a, b) \mapsto (\tilde{a}, \tilde{b})$. The resulting equality is the analogue of (5.5).

Remark 5.8. For the reader familiar with G-stable maps, we note that the element $TX^H \oplus \mathcal{S}_{m_1}|_{X^H}$ from (5.16) is the restriction of the obstruction bundle over $\xi_{1,1}(m_1, a, b) \times X^H$ to $\{q\} \times X^H$, where q is any point in $\xi_{1,1}(m_1, a, b)$. The details of this are given in [JKK, Prop. 6.21].

Lemma 5.9. *If X is a smooth, projective variety with an action of a finite group G, then stringy K-theory* $((\mathcal{K}(X, G), \rho), *, \mathbf{1}, \eta, \tau^{\mathcal{K}})$ *and the stringy Chow ring* $((\mathcal{A}(X, G), \rho), *, \mathbf{1}, \eta, \tau^{\mathcal{A}})$ *satisfy the trace axiom for pre-G-Frobenius algebras.*

Proof. Consider the case of $\mathcal{K}(X, G)$. Corollary 5.6 implies that the element $\lambda_{-1}(TX^{\langle a,b\rangle} \oplus \mathcal{S}_{[a,b]}|_{X^{\langle a,b\rangle}})^*$ is well-defined, hence the trace element $\tau^{\mathcal{K}}$ given in (4.8) is well-defined. The trace axiom for a pre-G-Frobenius algebra follows immediately from the observation that $m = [aba^{-1}, a^{-1}] = [a, b]$ and $\langle a, b\rangle = \langle aba^{-1}, a^{-1}\rangle$ for all a, b in G.

The case of $\mathcal{A}(X, G)$ is analogous. □

This completes the proof of Theorems 4.6 and 4.7 that stringy Chow and stringy K-theory are pre-G-Frobenius algebras.

6. The stringy Chern character

As mentioned in the introduction, for general G the ordinary Chern character fails to be a ring homomorphism; however, this drawback can be overcome through the introduction of the appropriate correction terms to give what we call the *stringy Chern character* $\mathfrak{Ch} : \mathcal{K}(X, G) \to \mathcal{A}(X, G)$. The main purpose of this section is to prove the key properties of $\mathfrak{Ch}$, and especially to prove that the map $\mathfrak{Ch}$ is an allometric isomorphism for any smooth, projective variety X with an action of a finite group G. When G is the trivial group, $\mathfrak{Ch}$ reduces to the usual Chern character mapping from ordinary K-theory to the ordinary Chow ring of X.

Recall that for any smooth, projective variety X with an action of G, we have defined (see Definition (1.7)) the *stringy Chern character* $\mathfrak{Ch}$: $\mathcal{K}(X, G) \to \mathcal{A}(X, G)$ to be

$$\mathfrak{Ch}(\mathcal{F}_m) := \mathbf{ch}(\mathcal{F}_m) \cup \mathbf{td}^{-1}(\mathcal{S}_m) \tag{6.1}$$

for all m in G and $\mathcal{F}_m$ in $\mathcal{K}_m(X)$, where $\mathcal{S}_m$ is defined in (1.3), **td** is the Todd class, and **ch** is the ordinary Chern character.

The main result of this section is the following theorem.

Theorem 6.1. *The stringy Chern character* $\mathfrak{Ch} : \mathcal{K}(X, G) \to \mathcal{A}(X, G)$ *is an allometric isomorphism. In particular,* $\mathfrak{Ch}$ *is a G-equivariant algebra isomorphism.*

Proof. To see that $\mathfrak{Ch}$ is an isomorphism of G-graded G-modules, note first that **td** is invertible (it is a series starting with **1**), so $\mathfrak{Ch}$ is an isomorphism of G-graded vector spaces. The equivariance under the G-action follows from naturality properties of **td**, **ch**, the cup product, and $\mathcal{S}_m$.

We now prove that $\mathfrak{Ch}$ respects multiplication. We suppress the cup and tensor product symbols to avoid notational clutter. Let $\mathcal{F}_{m_i}$ belong to

$\mathcal{K}_{m_i}(X)$ for $i = 1, 2, 3$, where $m_1 m_2 m_3 = 1$. Let $\mathbf{e}_{m_i}$ denote the inclusion $X^{\mathbf{m}} \to X^{m_i}$ and $\check{\mathbf{e}}_{m_i} := \sigma \circ \mathbf{e}_{m_i} : X^{\mathbf{m}} \to X^{m_i^{-1}}$ for all $i = 1, 2, 3$. We have

$$\begin{aligned}
&\mathcal{C}\mathbf{h}(\mathcal{F}_{m_1} * \mathcal{F}_{m_2}) \\
&\quad = \mathbf{ch}(\mathcal{F}_{m_1} * \mathcal{F}_{m_2})\mathbf{td}(\ominus \mathcal{S}_{m_3^{-1}}) \\
&\quad = \mathbf{ch}\big(\check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}_{m_1}^* \mathcal{F}_{m_1} \mathbf{e}_{m_2}^* \mathcal{F}_{m_2} \lambda_{-1}(\mathcal{R}^*)\big)\big)\mathbf{td}(\ominus \mathcal{S}_{m_3^{-1}}) \\
&\quad = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{ch}\big(\mathbf{e}_{m_1}^* \mathcal{F}_{m_1} \mathbf{e}_{m_2}^* \mathcal{F}_{m_2} \lambda_{-1}(\mathcal{R}^*)\big)\mathbf{td}(T\check{\mathbf{e}}_{m_3})\big)\mathbf{td}(\ominus \mathcal{S}_{m_3^{-1}}) \\
&\quad = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}_{m_1}^* \mathbf{ch}(\mathcal{F}_{m_1}) \mathbf{e}_{m_2}^* \mathbf{ch}(\mathcal{F}_{m_2}) \mathbf{ch}\big(\lambda_{-1}(\mathcal{R}^*)\big)\mathbf{td}(T\check{\mathbf{e}}_{m_3})\big)\mathbf{td}(\ominus \mathcal{S}_{m_3^{-1}}) \\
&\quad = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}_{m_1}^* \mathbf{ch}(\mathcal{F}_{m_1}) \mathbf{e}_{m_2}^* \mathbf{ch}(\mathcal{F}_{m_2}) c_{\mathrm{top}}(\mathcal{R})\mathbf{td}^{-1}(\mathcal{R})\mathbf{td}(T\check{\mathbf{e}}_{m_3})\big)\mathbf{td}(\ominus \mathcal{S}_{m_3^{-1}}) \\
&\quad = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}_{m_1}^* \mathbf{ch}(\mathcal{F}_{m_1}) \mathbf{e}_{m_2}^* \mathbf{ch}(\mathcal{F}_{m_2}) c_{\mathrm{top}}(\mathcal{R})\mathbf{td}(\ominus \mathcal{R} \oplus T\check{\mathbf{e}}_{m_3})\big)\mathbf{td}(\ominus \mathcal{S}_{m_3^{-1}}) \\
&\quad = \mathbf{e}_{m_3 *}\big(\mathbf{e}_{m_1}^* \mathbf{ch}(\mathcal{F}_{m_1}) \mathbf{e}_{m_2}^* \mathbf{ch}(\mathcal{F}_{m_2}) c_{\mathrm{top}}(\mathcal{R})\mathbf{td}(\ominus \mathcal{R} \oplus T\check{\mathbf{e}}_{m_3})\check{\mathbf{e}}_{m_3}^* \mathbf{td}(\ominus \mathcal{S}_{m_3^{-1}})\big) \\
&\quad = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}_{m_1}^* \mathbf{ch}(\mathcal{F}_{m_1}) \mathbf{e}_{m_2}^* \mathbf{ch}(\mathcal{F}_{m_2}) c_{\mathrm{top}}(\mathcal{R})\mathbf{td}\big(\ominus \mathcal{R} \oplus T\check{\mathbf{e}}_{m_3} \ominus \check{\mathbf{e}}_{m_3}^* \mathcal{S}_{m_3^{-1}}\big)\big),
\end{aligned}$$

where the first two equalities follow from the definition of the multiplication and $\mathcal{C}\mathbf{h}$, the third from the Grothendieck–Riemann–Roch theorem, the fourth from the fact that the usual Chern character **ch** commutes with pull back and is a homomorphism with respect to the usual products in the Chow ring, and the fifth from (2.10). The sixth and eighth equalities follow from multiplicativity of **td**, and the seventh follows from the projection formula.

If we let $\mathcal{T} \in K(X^{\mathbf{m}})$ be

$$\mathcal{T} := \ominus \mathcal{R} \oplus TX^{\mathbf{m}} \ominus TX^{m_3^{-1}}\big|_{X^{\mathbf{m}}} \ominus \check{\mathbf{e}}_{m_3}^* \mathcal{S}_{m_3^{-1}},$$

then by plugging in (4.4), we obtain

$$\mathcal{T} = \ominus \mathcal{R} \oplus TX^{\mathbf{m}} \ominus TX|_{X^{\mathbf{m}}} \oplus \mathcal{S}_{m_3}|_{X^{\mathbf{m}}}. \tag{6.2}$$

Therefore, we obtain the equality

$$\mathcal{C}\mathbf{h}(\mathcal{F}_{m_1} * \mathcal{F}_{m_2}) = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}_{m_1}^* \mathbf{ch}(\mathcal{F}_{m_1}) \mathbf{e}_{m_2}^* \mathbf{ch}(\mathcal{F}_{m_2}) c_{\mathrm{top}}(\mathcal{R})\mathbf{td}(\mathcal{T})\big). \tag{6.3}$$

Similarly, we see that

$$\begin{aligned}
&\mathcal{C}\mathbf{h}(\mathcal{F}_{m_1}) * \mathcal{C}\mathbf{h}(\mathcal{F}_{m_2}) \\
&\quad = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}_{m_1}^* \mathcal{C}\mathbf{h}(\mathcal{F}_{m_1}) \mathbf{e}_{m_2}^* \mathcal{C}\mathbf{h}(\mathcal{F}_{m_2}) c_{\mathrm{top}}(\mathcal{R})\big) \\
&\quad = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}_{m_1}^* \big(\mathbf{ch}(\mathcal{F}_{m_1}) e_{m_1}^* \mathbf{td}(\ominus \mathcal{S}_{m_1})\big) \mathbf{e}_{m_2}^* (\mathbf{ch}(\mathcal{F}_{m_2}) \mathbf{td}(\ominus \mathcal{S}_{m_2})) c_{\mathrm{top}}(\mathcal{R})\big) \\
&\quad = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}_{m_1}^* \mathbf{ch}(\mathcal{F}_{m_1}) \mathbf{e}_{m_2}^* \mathbf{ch}(\mathcal{F}_{m_2}) c_{\mathrm{top}}(\mathcal{R})\mathbf{td}\big(\ominus e_{m_1}^* \mathcal{S}_{m_1} \ominus e_{m_2}^* \mathcal{S}_{m_2}\big)\big),
\end{aligned}$$

where the first two equalities are by definition and the third is by multiplicativity of **td**. Thus, if

$$\mathcal{T}' := \ominus \mathcal{S}_{m_1}|_{X^{\mathbf{m}}} \ominus \mathcal{S}_{m_2}|_{X^{\mathbf{m}}}, \tag{6.4}$$

then

$$\mathfrak{Ch}(\mathcal{F}_{m_1}) * \mathfrak{Ch}(\mathcal{F}_{m_2}) = \check{\mathbf{e}}_{m_3 *}\big(\mathbf{e}^*_{m_1}\mathbf{ch}(\mathcal{F}_{m_1})\mathbf{e}^*_{m_2}\mathbf{ch}(\mathcal{F}_{m_2}) c_{\text{top}}(\mathcal{R})\mathbf{td}(\mathcal{T}')\big). \tag{6.5}$$

$\mathfrak{Ch}$ is therefore an algebra homomorphism if and only if the right hand sides of (6.3) and (6.5) are equal. A sufficient condition for this equality to hold is that $\mathcal{T} = \mathcal{T}'$, but this follows immediately from the definition of $\mathcal{R}(\mathbf{m})$ (see 1.4).

We will now prove that $\mathfrak{Ch}$ preserves the trace element. For all a, b in G, for $m = [a, b]$, and for all $\mathcal{F}_m$ in $\mathcal{K}_m(X)$, we have

$$\begin{aligned}
\tau^{\mathcal{K}}_{a,b}(\mathcal{F}_m) &= \chi\big(X^{\langle a,b\rangle}, \mathcal{F}_m\big|_{X^{\langle a,b\rangle}} \otimes \lambda_{-1}\big(TX^{\langle a,b\rangle} \oplus S_m\big|_{X^{\langle a,b\rangle}}\big)^*\big) \\
&= \int_{X^{\langle a,b\rangle}} \mathbf{ch}\big(\mathcal{F}_m\big|_{X^{\langle a,b\rangle}} \otimes \lambda_{-1}\big(TX^{\langle a,b\rangle} \oplus S_m\big|_{X^{\langle a,b\rangle}}\big)^*\big) \cup \mathbf{td}(TX^{\langle a,b\rangle}) \\
&= \int_{X^{\langle a,b\rangle}} \mathbf{ch}(\mathcal{F}_m\big|_{X^{\langle a,b\rangle}}) \cup \mathbf{ch}\big(\lambda_{-1}\big(TX^{\langle a,b\rangle} \oplus S_m\big|_{X^{\langle a,b\rangle}}\big)^*\big) \\
&\qquad \cup \mathbf{td}(TX^{\langle a,b\rangle}) \\
&= \int_{X^{\langle a,b\rangle}} \mathbf{ch}(\mathcal{F}_m\big|_{X^{\langle a,b\rangle}}) \cup c_{\text{top}}\big(TX^{\langle a,b\rangle} \oplus S_m\big|_{X^{\langle a,b\rangle}}\big) \\
&\qquad \cup \mathbf{td}^{-1}\big(TX^{\langle a,b\rangle} \oplus S_m\big|_{X^{\langle a,b\rangle}}\big) \cup \mathbf{td}(TX^{\langle a,b\rangle}) \\
&= \int_{X^{\langle a,b\rangle}} \mathbf{ch}(\mathcal{F}_m\big|_{X^{\langle a,b\rangle}}) \cup c_{\text{top}}\big(TX^{\langle a,b\rangle} \oplus S_m\big|_{X^{\langle a,b\rangle}}\big) \cup \mathbf{td}^{-1}(S_m\big|_{X^{\langle a,b\rangle}}) \\
&= \int_{X^{\langle a,b\rangle}} \mathfrak{Ch}(\mathcal{F}_m)\big|_{X^{\langle a,b\rangle}} \cup c_{\text{top}}\big(TX^{\langle a,b\rangle} \oplus S_m\big|_{X^{\langle a,b\rangle}}\big) \\
&= \tau^{\mathcal{A}}_{a,b}(\mathfrak{Ch}(\mathcal{F}_m)),
\end{aligned}$$

where we have used the Hirzebruch–Riemann–Roch theorem in the second equality, the fact that **ch** preserves the ordinary multiplications in the third, (2.10) in the fourth, the multiplicativity of **td** in the fifth, and the definition of $\mathfrak{Ch}$ (Equation (1.7)) in the sixth. □

Remark 6.2. It is instructive to consider the homomorphism property of $\mathfrak{Ch}$ when the obstruction bundle $\mathcal{R}$ on $X^{\mathbf{m}}$ is trivial. When $m_1 = 1$ and $m_2 m_3 = 1$, it is trivial to verify from (4.11) that

$$\mathfrak{Ch}(\mathcal{F}_{m_1} * \mathcal{F}_{m_2}) = \mathfrak{Ch}(\mathcal{F}_{m_1}) * \mathfrak{Ch}(\mathcal{F}_{m_2}). \tag{6.6}$$

Indeed, (6.6) continues to hold even if $\mathfrak{Ch}$ were replaced by the ordinary Chern character **ch**. A similar result holds if $m_2 = 1$ and $m_1 m_3 = 1$. However, when $m_1 m_2 = 1$ and $m_3 = 1$, then (6.6) would fail to hold if $\mathfrak{Ch}$ were replaced by the ordinary Chern character **ch** because of the presence of the nontrivial pushforward map $\check{\mathbf{e}}_{m_3 *}$ in (4.11). This shows that the stringy corrections to the Chern character are necessary even when the obstruction bundle is trivial.

Finally, $\mathfrak{Ch}$ satisfies the usual functorial properties with respect to equivariant étale morphisms.

Theorem 6.3. *Let $f : X \to Y$ be a G-equivariant, étale morphism between smooth, projective varieties X and Y with G-action. The following properties hold.*

(1) *(Pullback) The pullback maps*

$$f^* : ((\mathcal{A}(Y, G), \rho), *, \mathbf{1}) \to ((\mathcal{A}(X, G), \rho), *, \mathbf{1})$$

and

$$f^* : ((\mathcal{K}(Y, G), \rho), *, \mathbf{1}) \to ((\mathcal{K}(X, G), \rho), *, \mathbf{1})$$

are equivariant morphisms of G-graded associative algebras.

(2) *(Naturality) The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{K}(Y,G) & \xrightarrow{f^*} & \mathcal{K}(X,G) \\ \mathfrak{Ch}\downarrow & & \mathfrak{Ch}\downarrow \\ \mathcal{A}(Y,G) & \xrightarrow{f^*} & \mathcal{A}(X,G) \end{array} \tag{6.7}$$

(3) *(Grothendieck–Riemann–Roch) For all m in G and $\mathcal{F}_m$ in $\mathcal{K}_m(X)$,*

$$f_*\big(\mathfrak{Ch}(\mathcal{F}_m) \cup \mathbf{td}(TX^m)\big) = \mathfrak{Ch}(f_*\mathcal{F}_m) \cup \mathbf{td}(TY^m). \tag{6.8}$$

Proof. The proof of part (1) follows immediately from the fact that since f is G-equivariant and étale, the bundle f^*TY^m is isomorphic to TX^m.

Part (2) follows from the naturality of the ordinary Chern character and the fact that if f is étale, then $f^*\mathcal{S}_m^Y = \mathcal{S}_m^X$, where $\mathcal{S}_m^X$ and $\mathcal{S}_m^Y$ are as defined in (1.3) for X and Y, respectively.

Part (3) follows from these same considerations, since

$$\begin{aligned} f_*\big(\mathfrak{Ch}(\mathcal{F}_m) \cup \mathbf{td}(TX^m)\big) &= f_*\big(\mathbf{ch}(\mathcal{F}_m) \cup \mathbf{td}\big(\ominus \mathcal{S}_m^X\big) \cup \mathbf{td}(TX^m)\big) \\ &= f_*\big(\mathbf{ch}(\mathcal{F}_m) \cup \mathbf{td}(TX^m) \cup \mathbf{td}\big(\ominus \mathcal{S}_m^X\big)\big) \\ &= f_*\big(\mathbf{ch}(\mathcal{F}_m) \cup \mathbf{td}(TX^m) \cup \mathbf{td}\big(\ominus f^*\mathcal{S}_m^Y\big)\big) \\ &= f_*\big(\mathbf{ch}(\mathcal{F}_m) \cup \mathbf{td}(TX^m) \cup f^*\mathbf{td}\big(\ominus \mathcal{S}_m^Y\big)\big) \\ &= f_*\big(\mathbf{ch}(\mathcal{F}_m) \cup \mathbf{td}(TX^m)\big) \cup \mathbf{td}\big(\ominus \mathcal{S}_m^Y\big) \\ &= \mathbf{ch}(f_*\mathcal{F}_m) \cup \mathbf{td}(TY^m) \cup \mathbf{td}\big(\ominus \mathcal{S}_m^Y\big) \\ &= \mathfrak{Ch}(f_*\mathcal{F}_m) \cup \mathbf{td}(TY^m), \end{aligned}$$

where the projection formula was used in the fifth equality and the ordinary Grothendieck–Riemann–Roch theorem was used in the sixth. □

7. Discrete torsion

At this point, we wish to make a short comment about discrete torsion. As discussed in [Kau04], any G-Frobenius algebra can be twisted by a discrete torsion, which is a 2-cocycle $\alpha \in Z^2(G, \mathbb{Q}^*)$, to obtain a G-Frobenius algebra with twisted sectors of the same dimension. Of course, the same is true for any pre-G-Frobenius algebra with trace τ, provided the trace τ is appropriately twisted, as we explain below.

This procedure allows us to "twist" the stringy Chow ring $\mathcal{A}(X, G)$ and the stringy K-theory $\mathcal{K}(X, G)$. If one twists both rings by the same element α, then the stringy Chern character $\mathfrak{Ch}$ again provides an allometric isomorphism.

We briefly recall the main points of the construction of twisting by discrete torsion, omitting the proofs which all follow from rather straightforward computations. A reference for the proofs is [Kau04].

For $\alpha \in Z^2(G, \mathbb{Q}^*)$, let $\mathbb{Q}^\alpha[G]$ be the twisted group ring, i.e., $\mathbb{Q}^\alpha[G] = \bigoplus_{m\in G} \mathbb{Q}e_m$ with the multiplication $e_{m_1} \star e_{m_2} = \alpha(m_1, m_2)e_{m_1 m_2}$.

Set $\epsilon(\gamma, m) := \alpha(\gamma, m)/\alpha(\gamma m\gamma^{-1}, \gamma)$ and define $\rho(\gamma)(e_m) = \epsilon(\gamma, m)e_{\gamma m\gamma^{-1}}$. Define a bi-linear form η by $\eta(e_{m_+}, e_{m_-}) = 0$ unless $m_+m_- = 1$, and $\eta(e_m, e_{m^{-1}}) = \alpha(m, m^{-1})$. Lastly, let $\mathbf{1} = e_1$.

Lemma 7.1. $((\mathbb{Q}^\alpha[G], \rho), \star, \mathbf{1}, \eta)$ *is a G-Frobenius algebra.*

Definition 7.2. We define the tensor product $\hat{\otimes}$ of two pre-G-Frobenius algebras $((\mathcal{H}, \varphi), \star, \mathbf{1}, \eta, \tau)$ and $((\mathcal{H}', \varphi'), \star', \mathbf{1}', \eta', \tau')$ to be the pre-G-Frobenius algebra $\mathcal{H}\hat{\otimes}\mathcal{H}' = \bigoplus_{m\in G}(\mathcal{H}\hat{\otimes}\mathcal{H}')_m$ with $(\mathcal{H}\hat{\otimes}\mathcal{H}')_m := \mathcal{H}_m \otimes_{\mathbb{Q}} \mathcal{H}'_m$, diagonal multiplication $\star \otimes \star'$, diagonal G-action $\rho \otimes \rho'$, the tensor product pairing $\eta \otimes \eta'$, unity $\mathbf{1} \otimes \mathbf{1}'$, and trace $\tau \otimes \tau'$.

Proposition 7.3. *The tensor product of two pre-G-Frobenius algebras is a pre-G-Frobenius algebra. Similarly, the tensor product of two G-Frobenius algebras is a G-Frobenius algebra.*

Definition 7.4. For a pre-G-Frobenius algebra $\mathcal{H}$ and an element $\alpha \in Z^2(G, \mathbb{Q}^*)$, we set

$$\mathcal{H}^\alpha := \mathcal{H}\hat{\otimes}\mathbb{Q}^\alpha[G]. \tag{7.1}$$

Notice that as vector spaces $\mathcal{H}^\alpha_m = \mathcal{H}_m \otimes_{\mathbb{Q}} \mathbb{Q} \simeq \mathcal{H}_m$.

Lemma 7.5. *Using the identification $\mathcal{H}^\alpha_m \cong \mathcal{H}_m$, the G-Frobenius structures for $((\mathcal{H}^\alpha, \rho^\alpha), \star^\alpha, \mathbf{1}^\alpha, \eta^\alpha)$ are*

$$\begin{aligned} v_{m_1} \star^\alpha v_{m_2} &:= \alpha(m_1, m_2)v_{m_1} \star v_{m_2}, \\ \rho^\alpha(\gamma)v_m &:= \epsilon(\gamma, m)\rho(\gamma)v_m, \\ \eta^\alpha(v_m, v_{m^{-1}}) &:= \alpha(m, m^{-1})\eta(v_m, v_{m^{-1}}) \end{aligned}$$

and

$$\tau^{\alpha}_{a,b}(v_{[a,b]}) := \frac{\alpha([a,b], bab^{-1})\alpha(b,a)}{\alpha(bab^{-1}, b)}\tau_{a,b}(v_{[a,b]})$$

for all v_{m_i} *in* $\mathcal{H}^{\alpha}_{m_i}$, v_m *in* $\mathcal{H}^{\alpha}_m$, *and* $v_{m^{-1}}$ *in* $\mathcal{H}^{\alpha}_{m^{-1}}$.

Proposition 7.6. *The pre-G-Frobenius algebras* $\mathcal{H}$ *and* $\mathcal{H}^{\alpha}$ *are isomorphic if and only if* α *is a coboundary; that is,* $[\alpha] = 0 \in H^2(G, \mathbb{Q}^*)$.

Proposition 7.7. *If* $\Phi : \mathcal{H} \to \mathcal{H}'$ *is an isomorphism (or allometric isomorphism) of pre-G-Frobenius algebras, then* $\Phi \otimes id$ *is an isomorphism (respectively allometric isomorphism) between* $\mathcal{H}^{\alpha}$ *and* $\mathcal{H}'^{\alpha}$.

Corollary 7.8. *Let* $\mathfrak{Ch} : \mathcal{K}(X, G) \to \mathcal{A}(X, G)$ *denote the stringy Chern character. For all* $\alpha \in Z^2(G, \mathbb{Q}^*)$, *the map* $\mathfrak{Ch}^{\alpha} = \mathfrak{Ch} \otimes id : \mathcal{K}^{\alpha}(X) \to \mathcal{A}^{\alpha}(X)$ *is an allometric isomorphism.*

8. Relation to Fantechi–Göttsche, Chen–Ruan, and Abramovich–Graber–Vistoli

In [FG] Fantechi and Göttsche describe a ring $\mathcal{H}^{\bullet}(X, G)$, which we call the *stringy cohomology*, associated to every manifold X with an action by a finite group G. The stringy cohomology is also a G-Frobenius algebra [JKK], and the ring of G-invariants of $\mathcal{H}^{\bullet}(X, G)$ is known to be isomorphic to the Chen–Ruan orbifold cohomology $H^{\bullet}_{\text{orb}}([X/G], \mathbb{Q})$ of the quotient stack $[X/G]$. Abramovich, Graber, and Vistoli [AGV] have an algebraic construction, similar to that of Chen and Ruan, of what we would call an orbifold Chow ring.

Remark 8.1. Note that Abramovich, Graber, and Vistoli call their ring the "stringy Chow ring," but we prefer to reserve the word *stringy* for G-Frobenius structures associated to a manifold with a specific group action, and use the word *orbifold* for Frobenius algebras that are associated to orbifolds, and are thus presentation independent. The general philosophy is that a stringy construction associated to a finite group G acting on X should have, as its ring of invariants, an orbifold construction for the quotient stack $[X/G]$. The orbifold construction should also generalize to stacks which are not global quotients by finite groups.

Just as our constructions rely on the special bundle $\mathcal{R}(\mathbf{m})$, the constructions of Fantechi–Göttsche, Chen–Ruan, and Abramovich–Graber–Vistoli all use an obstruction bundle arising in the theory of stable maps – either stable maps into an orbifold or G-stable maps (see [JKK]) into a manifold with G-action. The description of these obstruction bundles is rather technical and is generally difficult to use for computation.

In this section, we prove that the obstruction bundle of Fantechi–Göttsche is equivalent to our bundle $\mathscr{R}(\mathbf{m})$. It is known that their construction agrees (after taking G-invariants) with that of Chen–Ruan and Abramovich–Graber–Vistoli [FG, §2]. In Sect. 9, and especially in Theorem 9.10, we will generalize this to general orbifolds – not just those which are global quotients by finite groups.

For our purposes the most important consequence of the equivalence of our construction with that of Fantechi–Göttsche is that the element $\mathscr{R}(\mathbf{m}) \in K(X^{\mathbf{m}})$ is actually represented by a vector bundle. But another important consequence is that their obstruction bundle may now be described solely in terms of the G-action on the tangent bundle of X, restricted to various fixed-point loci. This greatly simplifies the computation of stringy cohomology, orbifold cohomology, and orbifold Chow. In particular, it allows us to circumvent all of the technical details of those constructions, including stable curves, stable maps, admissible covers, and moduli spaces.

8.1. The obstruction bundle of Fantechi and Göttsche. We briefly review the construction of the obstruction bundle of Fantechi and Göttsche [FG]. For each triple $\mathbf{m} = (m_1, m_2, m_3) \in G^3$ such that $m_1 m_2 m_3 = 1$, let $\langle \mathbf{m} \rangle$ be the subgroup generated by the elements m_1, m_2, and m_3. There is a presentation of the fundamental group $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$ as $\langle c_1, c_2, c_3 | c_1 c_2 c_3 = 1 \rangle$, where c_1, c_2 and c_3 are based loops around $p_1 = 0$, $p_2 = 1$, and $p_3 = \infty$, respectively. We define a natural homomorphism $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \to \langle \mathbf{m} \rangle$, taking c_i to m_i. This defines a principal $\langle \mathbf{m} \rangle$-bundle over $\mathbb{P}^1 - \{0, 1, \infty\}$ which extends to a smooth connected curve E. The curve E has an action of $\langle \mathbf{m} \rangle$ such that the quotient $E/\langle \mathbf{m} \rangle$ has genus zero, and the natural map $E \to E/\langle \mathbf{m} \rangle$ is branched at the three points p_1, p_2, p_3 with monodromy m_1, m_2, m_3, respectively.

Let $\pi : E \times X^{\mathbf{m}} \to X^{\mathbf{m}}$ be the second projection. The obstruction bundle of Fantechi and Göttsche, which we denote by $\mathscr{R}_{FG}(\mathbf{m})$, on $X^{\mathbf{m}}$ is

$$\mathscr{R}_{FG}(\mathbf{m}) := R^1 \pi_*^{\langle \mathbf{m} \rangle} (\mathscr{O}_E \boxtimes TX|_{X^{\mathbf{m}}}). \tag{8.1}$$

One can check that the restriction of the bundle $\mathscr{R}_{FG}(\mathbf{m}) \in K(X^{\mathbf{m}})$ to a connected component U of $X^{\mathbf{m}}$ has rank

$$a(m_1, U) + a(m_2, U) + a(m_3, U) - \operatorname{codim}(U \subseteq X). \tag{8.2}$$

Remark 8.2. For those familiar with quantum cohomology, this obstruction bundle is the analogue of the obstruction bundle for stable maps, but with additional accounting for the structure of the group action on X. That is, $c_{top}(\mathscr{R}_{FG})$ is the virtual fundamental class on (distinguished components of) the moduli space of genus-zero, three-pointed G-stable maps into X. The base space $X^{\mathbf{m}}$ in the definition of the obstruction bundle is the distinguished component $\xi_{0,3}(X, 0, \mathbf{m}) \cong pt \times X^{\mathbf{m}}$ of $\overline{\mathscr{M}}^G_{0,3}(X, 0, \mathbf{m})$. The interested reader may refer to [JKK, §6] for more details.

Theorem 8.3. *Let X be a smooth variety (not necessarily projective, or even proper) with an action of a finite group G. If $\mathbf{m} = (m_1, m_2, m_3) \in G^3$ is such that $m_1 m_2 m_3 = 1$, then on the fixed point locus $X^{\mathbf{m}} := X^{m_1} \cap X^{m_2}$, we have*

$$\mathcal{R}_{FG}(\mathbf{m}) = \mathcal{R}(\mathbf{m}) = TX^{\mathbf{m}} \ominus TX\big|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \mathcal{S}_{m_i}\big|_{X^{\mathbf{m}}}, \tag{8.3}$$

in the K-theory $K(X^{\mathbf{m}})$ of $X^{\mathbf{m}}$.

Corollary 8.4. *For each triple $\mathbf{m} = (m_1, m_2, m_3)$ with $m_1 m_2 m_3 = 1$, the element $\mathcal{R}(\mathbf{m}) \in K(X^{\mathbf{m}})$ is represented by a vector bundle on $X^{\mathbf{m}}$.*

As a first step to proving Theorem 8.3, we prove Lemma 8.5. The basic setup for Lemma 8.5 is as follows. Let E be a smooth algebraic curve of genus $\tilde{g}$, not necessarily connected, with a finite group G acting effectively on E. Assume that the quotient E/G has genus g. Denote the orbits where the action is not free by $p_1, \ldots, p_n \in E/G$. A choice of base point $\widetilde{p} \in E$ induces a homomorphism of groups

$$\varphi_{\widetilde{p}} : \pi_1(E/G - \{p_1, \ldots, p_n\}, p) \to G,$$

where p is the image of $\widetilde{p}$ in E/G (we assume $p \notin \{p_1, \ldots, p_n\}$). Denote by H the image of $\varphi_{\widetilde{p}}$ in G. Note that the number α of connected components of E is the index $[G : H]$. There is a presentation of $\pi_1(E/G - \{p_1, \ldots, p_n\}, p)$ of the form $\langle a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_n | \prod_{i=1}^{n} c_i = \prod_{j=1}^{g} [a_j, b_j] \rangle$, where the c_i are loops around the points p_i. For each $i \in \{1, \ldots, n\}$ we call the image $m_i := \varphi_{\widetilde{p}}(c_i) \in G$ of c_i the *monodromy* around p_i, and we denote the order of m_i in G by r_i. Of course, a different choice of $\widetilde{p}$ will change all of the m_i by simultaneous conjugation with an element of G.

The following lemma describes the G-module structure of the cohomology $H^1(E; \mathcal{O}_E)$. It has recently come to our attention that a related result can be found in [Kan].

Lemma 8.5. *Given the setup described above, and letting $\mathbb{C}[G]$ denote the group ring regarded as a G-module under multiplication, we have the following equality in the representation ring of G,*

$$H^1(E; \mathcal{O}_E) = \mathbb{C}[G/H] \oplus (g-1)\mathbb{C}[G] \oplus \bigoplus_{i=1}^{n} \bigoplus_{k_i=0}^{r_i - 1} \frac{k_i}{r_i} \operatorname{Ind}_{\langle m_i \rangle}^{G} \mathbf{V}_{m_i, k_i}, \tag{8.4}$$

where $\mathbf{V}_{m_i, k_i}$ is the irreducible representation of $\langle m_i \rangle$ such that m_i acts by multiplication by $\exp(-2\pi i k_j / r_j)$, and $\operatorname{Ind}_{\langle m_i \rangle}^{G} \mathbf{V}_{m_i, k_i}$ is the induced representation $\mathbb{C}[G] \otimes_{\mathbb{C}[\langle m_i \rangle]} \mathbf{V}_{m_i, k_i}$.

Proof. It suffices to check that these two virtual representations have the same virtual character. The trace of the action of an element $\gamma \in G$ on the right hand side is

$$\chi_{\mathbb{C}[G/H]}(\gamma) + (g-1)|G|\delta_{\gamma,e} + \sum_{i=1}^{n}\sum_{k_i=0}^{r_i-1}\frac{k_i}{r_i}\chi_{\mathrm{Ind}_{\langle m_i\rangle}^{G}\mathbf{V}_{m_i,k_i}}(\gamma).$$

It is well known (e.g., [FH, Example 3.19]) that, for a representation V of a subgroup $H \leq G$, we have

$$\chi_{\mathrm{Ind}_H^G V}(\gamma) = \frac{|G|}{|H|}\sum_{\sigma\in H\cap[\![\gamma]\!]}\frac{\chi_V(\sigma)}{|[\![\gamma]\!]|},$$

where $[\![\gamma]\!]$ is the conjugacy class of γ in G. In our case, with $H = \langle m_i\rangle$ of order r_i, and $V = \mathbf{V}_{m_i,k_i}$, we have

$$\chi_{\mathrm{Ind}_H^G V}(\gamma) = \frac{|G|}{r_i|[\![\gamma]\!]|}\sum_{m_i^l\in[\![\gamma]\!]}\chi_V(m_i^l) = \frac{|G|}{r_i|[\![\gamma]\!]|}\sum_{m_i^l\in[\![\gamma]\!]}\zeta_i^{lk_i},$$

where $\zeta_j = \exp(-2\pi i/r_j)$ for each $j \in \{1,\ldots,n\}$. Thus the trace of the right hand side of (8.4) becomes

$$\mathrm{Tr}_{RHS}(\gamma) = \chi_{\mathbb{C}[G/H]}(\gamma) + (g-1)|G|\delta_{\gamma,e} + \sum_{i=1}^{n}\sum_{k_i=0}^{r_i-1}\frac{k_i|G|}{r_i^2|[\![\gamma]\!]|}\sum_{m_i^l\in[\![\gamma]\!]}\zeta_i^{lk_i}. \tag{8.5}$$

If $\gamma = e$ is the identity element of G, we have

$$\begin{aligned}\mathrm{Tr}_{RHS}(e) &= \alpha + |G|(g-1) + \sum_{i=1}^{n}\sum_{k_i=0}^{r_i-1}\frac{k_i|G|}{r_i^2}\\ &= \alpha + |G|(g-1) + |G|\sum_{i=1}^{n}\frac{r_i-1}{2r_i}\\ &= \dim_{\mathbb{C}} H^1(E,\mathcal{O}_E),\end{aligned} \tag{8.6}$$

where the last equality follows from the Riemann–Hurwitz formula and the fact that the genus of E/G is g.

If $\gamma \neq e$, then

$$\begin{aligned}\mathrm{Tr}_{RHS}(\gamma) &= \chi_{\mathbb{C}[G/H]}(\gamma) + \sum_{i=1}^{n}\frac{|G|}{r_i^2|[\![\gamma]\!]|}\sum_{m_i^l\in[\![\gamma]\!]}\sum_{k_i=0}^{r_i-1}k_i\zeta_i^{lk_i}\\ &= \chi_{\mathbb{C}[G/H]}(\gamma) + \sum_{i=1}^{n}\frac{|G|}{r_i^2|[\![\gamma]\!]|}\sum_{m_i^l\in[\![\gamma]\!]}r_i\frac{\zeta_i^{-l}}{1-\zeta_i^{-l}}\\ &= \sum_{\substack{\sigma\in G/H\\ \gamma\sigma=\sigma}}1 + \sum_{i=1}^{n}\frac{|G|}{r_i|[\![\gamma]\!]|}\sum_{m_i^l\in[\![\gamma]\!]}\frac{\zeta_i^{-l}}{1-\zeta_i^{-l}},\end{aligned} \tag{8.7}$$

where the last equality follows from standard results on induced representations [FH, 3.18].

This formula is related to fixed points of the action of γ on E as follows. The element γ can only fix points that lie over the p_i, for $i \in \{1, \dots, n\}$. If $\widetilde{p}_i$ is a point over p_i fixed by γ, then $\widetilde{p}_i$ has monodromy conjugate to m_i, and thus γ must be conjugate to m_i^l for some l. Conversely, if γ is conjugate to m_i^l for some l, then γ fixes all points $\widetilde{p}_i$ that lie over p_i, and γ acts on the tangent space $T_{\widetilde{p}_i}E$ by ζ_i^{-l}.

If both $\widetilde{p}_i$ and $\widetilde{p}_i'$ are fixed by γ with action ζ_i^{-l} on the tangent space, then $\widetilde{p}_i' = \widetilde{p}_i\sigma$ for some $\sigma \in G$, such that σ commutes with γ, but if $\sigma \in \langle m_i \rangle$, then $\widetilde{p}_i = \widetilde{p}_i'$. So the number of such points lying over p_i is exactly $\frac{|Z_G(\gamma)|}{|\langle m_i \rangle|} = \frac{|G|}{r_i |[\![\gamma]\!]|}$, where $Z_G(\gamma)$ is the centralizer of γ in G.

The term $\sum_{\substack{\sigma \in G/H \\ \gamma\sigma = \sigma}} 1$ counts connected components of E which are mapped to themselves by γ; that is, it is the trace of γ for the natural representation of G on $H^0(E, \mathcal{O}_E)$. If we now denote by $d\gamma_{\widetilde{p}} = \zeta_i^{-l}$ the action of γ on the tangent space $T_{\widetilde{p}}E$ at a fixed point $\widetilde{p} \in E$, the above argument shows that

$$\mathrm{Tr}_{RHS}(\gamma) = \chi_{H^0(E,\mathcal{O}_E)}(\gamma) + \sum_{\widetilde{p} \text{ fixed by } \gamma} \frac{d\gamma_{\widetilde{p}}}{1 - d\gamma_{\widetilde{p}}}.$$

But the Eichler trace formula says that this is precisely the trace of the action of γ on $H^1(E, \mathcal{O}_E)$; that is, the traces of (8.4) agree (see [FK, §V.2.0] for E connected with $\tilde{g} > 1$, and [Sha, §17] in general). □

Proof of Theorem 8.3. For any $\mathbf{m} = (m_1, m_2, m_3)$ with $m_1 m_2 m_3 = 1$, the curve E in the definition of $\mathcal{R}(\mathbf{m})$ is connected and has an effective action of $G' := \langle m_1, m_2, m_3 \rangle$ with quotient $\mathbb{P}^1 = E/G'$ and three branch points p_1, p_2, p_3.

We have

$$\mathcal{R}_{FG}(\mathbf{m}) = R^1\pi_*^{G'}(\mathcal{O}_E \boxtimes TX|_{X^{\mathbf{m}}}) \cong \left(H^1(E, \mathcal{O}_E) \otimes TX|_{X^{\mathbf{m}}}\right)^{G'},$$

and by Lemma 8.5 this is

$$\begin{aligned} &\left(H^1(E, \mathcal{O}_E) \otimes TX\big|_{X^{\mathbf{m}}}\right)^{G'} \\ &\quad = \left(\left(\mathbb{C} \ominus \mathbb{C}[G'] \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \mathrm{Ind}_{\langle m_i \rangle}^{G'} \mathbf{V}_{m_i,k_i}\right) \otimes TX\big|_{X^{\mathbf{m}}}\right)^{G'} \\ &\quad = TX^{\mathbf{m}} \ominus TX\big|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \left(\mathrm{Ind}_{\langle m_i \rangle}^{G'} \mathbf{V}_{m_i,k_i} \otimes TX\big|_{X^{\mathbf{m}}}\right)^{G'} \\ &\quad = TX^{\mathbf{m}} \ominus TX\big|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \left(TX\big|_{X^{\mathbf{m}}}\right)_{m_i,k_i} \end{aligned}$$

$$
\begin{aligned}
&= TX^{\mathbf{m}} \ominus TX\big|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \mathcal{S}_{m_i}\big|_{X^{\mathbf{m}}} \\
&= \mathcal{R}(\mathbf{m}). \qquad (8.8)
\end{aligned}
$$

□

8.2. The Abelian case. It is instructive to consider the special case where G is an Abelian group. In this case, our analysis of the obstruction bundle $\mathcal{R}_{FG}$ yields, as a simple corollary, a result originally due to Chen and Hu [CH].

Consider the obstruction bundle $\mathcal{R}_{FG}$ over $X^{\mathbf{m}}$, where $\mathbf{m} = (m_1, m_2, m_3)$ in G^3 satisfies $m_1 m_2 m_3 = 1$. Let us assume without loss of generality that $G = \langle \mathbf{m} \rangle$. Since G is Abelian, one can simultaneously diagonalize the actions of m_i, for $i = 1, 2, 3$ on $\mathcal{R}_{FG}$. If $W_{\mathbf{m}}$ denotes the normal bundle of $X^{\mathbf{m}}$ in X, then we have the simultaneous eigenbundle decomposition

$$
W_{\mathbf{m}} = \bigoplus_{\mathbf{k}} W_{\mathbf{m},\mathbf{k}}, \qquad (8.9)
$$

where the sum is over all $\mathbf{k} = (k_1, k_2, k_3)$ such that $k_i = 0, \dots, r_i - 1$, and r_i is the order of m_i for all $i \in \{1, 2, 3\}$. The eigenbundle $W_{\mathbf{m},\mathbf{k}}$ of $W_{\mathbf{m}}$ is the bundle where for all $j \in \{1, 2, 3\}$ each m_j has an eigenvalue $\exp(2\pi i k_j / r_j)$.

The following proposition is an easy corollary of Theorem 8.3.

Proposition 8.6. [CH] *When G is Abelian, under the above assumptions, then*

$$
\mathcal{R}_{FG} = \bigoplus_{\mathbf{k}} W_{\mathbf{m},\mathbf{k}} \qquad (8.10)
$$

in $K(X^{\mathbf{m}})$, where the sum is over triples $\mathbf{k} = (k_1, k_2, k_3)$, for $k_i = 0, \dots,$ $r_i - 1$ and $i = 1, 2, 3$, such that

$$
\frac{k_1}{r_1} + \frac{k_2}{r_2} + \frac{k_3}{r_3} = 2. \qquad (8.11)
$$

Proof. It is a straightforward exercise to verify that the right hand side of (8.3) agrees with (8.10) when G is Abelian. □

Remark 8.7. Chen and Hu use this characterization of the obstruction bundle to give a de Rham model for Chen–Ruan orbifold cohomology when the orbifold arises as the quotient of a variety by an Abelian group. It would be interesting to see how their constructions can be generalized to non-Abelian groups in light of (8.3).

9. The orbifold K-theory of a stack

In this section, we introduce two variants of orbifold K-theory. The first is given by extending the main constructions of stringy K-theory to an orbifold $\mathcal{X}$. We use a special vector bundle $\widetilde{\mathscr{R}}$ on the double inertia stack $\mathfrak{I}\mathfrak{I}_{\mathcal{X}}$, an orbifold analogue to $\widetilde{\mathscr{R}}$ from stringy K-theory, to define a new product $*$ on the K-theory $\mathsf{K}_{\mathrm{orb}}(\mathcal{X}) := K(\mathfrak{I}_{\mathcal{X}})$ of the inertia stack $\mathfrak{I}_{\mathcal{X}}$. We call the resulting algebra the *full orbifold K-theory* of $\mathcal{X}$. The second construction is associated to a global quotient $\mathcal{X} = [X/G]$ of a smooth, projective variety X by a finite group G. The ring $K_{\mathrm{orb}}(\mathcal{X})$, which we call *small orbifold K-theory*, is the algebra of G-invariants of its stringy K-theory. We show that, after tensoring with $\mathbb{C}$, the orbifold K-theory $K_{\mathrm{orb}}(\mathcal{X})$ is a sub-algebra of the pre-Frobenius algebra $\mathsf{K}_{\mathrm{orb}}(\mathcal{X})$.

We also make the analogous constructions for Chow rings, but unlike in the case of K-theory, the (full) orbifold Chow ring is isomorphic, as a pre-Frobenius algebra, to the invariants of the stringy Chow ring.

We also define a ring homomorphism, the *full orbifold Chern character,* $\mathfrak{Ch}_{\mathrm{orb}} : \mathsf{K}_{\mathrm{orb}}(\mathcal{X}) \to A^{\bullet}_{\mathrm{orb}}(\mathcal{X})$. The construction of $\mathsf{K}_{\mathrm{orb}}(\mathcal{X})$ is to Givental and Lee's quantum K-theory [Lee], as Chen–Ruan [CR2] and Abramovich–Graber–Vistoli's [AGV] orbifold quantum cohomology is to quantum cohomology. Furthermore, as a vector space, our construction of $K_{\mathrm{orb}}(\mathcal{X})$ agrees with the construction of Adem and Ruan [AR], but unlike theirs, our orbifold product has the virtue that the associated orbifold Chern character Ch_{orb} is a *ring* isomorphism – not just an additive isomorphism.

The main results of this section are Theorems 9.5 and 9.8 which say, among other things, that $\mathsf{K}_{\mathrm{orb}}(\mathcal{X})$ with the new product is a pre-Frobenius algebra, that $K_{\mathrm{orb}}(\mathcal{X})$ is a sub-pre-Frobenius algebra of $\mathsf{K}_{\mathrm{orb}}(\mathcal{X})$, and that the orbifold Chern character $\mathfrak{Ch}_{\mathrm{orb}} : \mathsf{K}_{\mathrm{orb}}(\mathcal{X}) \to A^{\bullet}_{\mathrm{orb}}(\mathcal{X})$ is a homomorphism of pre-Frobenius algebras which induces the isomorphism $Ch_{\mathrm{orb}} : K_{\mathrm{orb}}(\mathcal{X}) \xrightarrow{\sim} A^{\bullet}_{\mathrm{orb}}(\mathcal{X})$.

9.1. The full orbifold K-theory and the orbifold Chow ring. First, we need to establish some notation. We denote by $Z_G(g)$ the centralizer of an element g in a group G, and we denote by $Z_G(g, g')$ the intersection $Z_G(g, g') := Z_G(g) \cap Z_G(g')$. For any group G, we denote the set of all conjugacy classes in G by $\overline{G}$.

Recall that the inertia stack $\mathfrak{I}_{\mathcal{X}}$ of $\mathcal{X}$ is defined to be the stack whose T-valued points are pairs (x, g), where x is a T-valued point of $\mathcal{X}$ and g is an automorphism of x in $\mathcal{X}(T)$. An equivalent definition is

$$\mathfrak{I}_{\mathcal{X}} := \mathcal{X} \times_{(\mathcal{X}\times\mathcal{X})} \mathcal{X}$$

corresponding to the diagram

$$\begin{array}{ccc} \mathfrak{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}. \end{array}$$

We can write

$$\mathfrak{I}_{\mathcal{X}} = \coprod_{[\![g]\!]} \mathcal{X}_{[\![g]\!]},$$

where the indices run over conjugacy classes $[\![g]\!]$ of local automorphisms and $\mathcal{X}_{[\![g]\!]}$ is the locus of pairs (x, g') with $g' \in [\![g]\!]$. There is an obvious inclusion

$$j : \mathcal{X} \hookrightarrow \mathfrak{I}_{\mathcal{X}},$$

taking x to $(x, \mathbf{1}_x)$. There is also a canonical involution $\sigma : \mathfrak{I}_{\mathcal{X}} \to \mathfrak{I}_{\mathcal{X}}$ given by $\sigma : (x, g) \mapsto (x, g^{-1})$.

The double inertia stack $\mathfrak{II}_{\mathcal{X}}$ is defined to be the stack whose T-valued points are triples (x, g, h), where x is a T-valued point of $\mathcal{X}$ and g and h are both automorphisms of x in $\mathcal{X}(T)$. Again we can decompose $\mathfrak{II}_{\mathcal{X}}$ as

$$\mathfrak{II}_{\mathcal{X}} := \coprod_{[\![g_1,g_2]\!]} \mathcal{X}_{[\![g_1,g_2]\!]},$$

where the indices run over (diagonal) conjugacy classes of pairs of local automorphisms $[\![g_1, g_2]\!]$, and $\mathcal{X}_{[\![g_1,g_2]\!]}$ is the locus of triples (x, g_1', g_2') with $(g_1', g_2') \in [\![g_1, g_2]\!]$.

There are three "evaluation" morphisms

$$\mathfrak{II}_{\mathcal{X}} \xrightarrow{ev_i} \mathfrak{I}_{\mathcal{X}}$$

defined by

$$\begin{aligned} ev_1 &: (x, g_1, g_2) \mapsto (x, g_1) \\ ev_2 &: (x, g_1, g_2) \mapsto (x, g_2) \\ ev_3 &: (x, g_1, g_2) \mapsto (x, (g_1 g_2)^{-1}). \end{aligned}$$

More generally, given any elements $a, b, m \in G$, such that m is in the subgroup $\langle a, b\rangle$ generated by a and b, there is a morphism

$$\varepsilon_{a,b,m} : \mathcal{X}_{[\![a,b]\!]} \to \mathcal{X}_{[\![m]\!]}.$$

We define

$$\check{ev}_i = \sigma \circ ev_i,$$

where $\sigma : \mathfrak{I}_{\mathcal{X}} \to \mathfrak{I}_{\mathcal{X}}$ is the canonical involution.

Also, forgetting automorphisms entirely gives morphisms

$$i : \mathfrak{I}_{\mathcal{X}} \to \mathcal{X}$$

and

$$J : \mathfrak{II}_{\mathcal{X}} \to \mathcal{X}.$$

We have

$$J = i \circ ev_1 = i \circ ev_2 = i \circ ev_3,$$

and

$$i \circ j = \mathbf{1}_{\mathcal{X}}.$$

For most of our constructions we will need to impose an additional condition on the stacks.

Definition 9.1. We say that a stack $\mathcal{X}$ *satisfies the KG-condition* if the Grothendieck group $K^{\mathrm{naive}}(\mathcal{X}, \mathbb{Z})$ of (orbi-)vector bundles is isomorphic to the Grothendieck group $K^0(\mathcal{X}, \mathbb{Z})$ of perfect complexes and to the Grothendieck group $G_0(\mathcal{X}, \mathbb{Z})$ of coherent sheaves on $\mathcal{X}$.

If $\mathcal{X}$ satisfies the KG-condition, we will simply write $K(\mathcal{X})$ to denote this group with rational coefficients:

$$K(\mathcal{X}) := K^0(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{Q} \cong K^{\mathrm{naive}}(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{Q} \cong G_0(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{Q}.$$

If the stack $\mathcal{X}$, its inertia stack $\mathfrak{I}_{\mathcal{X}}$, and its double inertia stack $\mathfrak{II}_{\mathcal{X}}$ all satisfy the KG-condition, then we say that $\mathcal{X}$ *satisfies condition* $(\star)$.

If a stack $\mathcal{X}$ is smooth, then it is always true that $K(\mathcal{X}) \cong G(\mathcal{X})$ [Jos02, §2]. Moreover, if $\mathcal{X}$ has the *resolution property* that every coherent sheaf is a quotient by a vector bundle, then $K^{naive}(\mathcal{X}) \cong K(\mathcal{X})$ [Jos05, Prop. 2.3]. Smooth Deligne–Mumford stacks with a finite stabilizer group which is generically trivial and with a coarse moduli space which is a separated scheme satisfy the resolution property [Tot, Thm. 1.2], thus they satisfy the KG-property. In particular, condition $(\star)$ holds for the quotient $\mathcal{X} = [X/G]$ of a smooth projective variety X by a finite group G.

As in the stringy case, for each conjugacy class $[\![g]\!]$, with g of order r, the element g acts by r-th roots of unity on $W_{[\![g]\!]} := T\mathcal{X}|_{\mathcal{X}_{[\![g]\!]}}$. We define $W_{[\![g]\!],k}$ to be the eigenbundle of $W_{[\![g]\!]}$, where g acts by multiplication by $\zeta^k = \exp(2\pi i k/r)$. Note that this eigenbundle is determined only by the conjugacy class $[\![g]\!]$ rather than by the particular representative g. Finally, we define

$$\tilde{\mathscr{S}}_{[\![g]\!]} := \bigoplus_{k=0}^{r-1} \frac{k}{r} W_{[\![g]\!],k} \in K(\mathcal{X}_{[\![g]\!]}). \tag{9.1}$$

This allows us to define $\tilde{\mathscr{S}} \in K(\mathfrak{I}_{\mathcal{X}})$ as

$$\tilde{\mathscr{S}} = \bigoplus_{[\![g]\!]} \tilde{\mathscr{S}}_{[\![g]\!]}. \tag{9.2}$$

Definition 9.2. The *orbifold obstruction bundle* $\widetilde{\mathscr{R}} \in K(\mathfrak{II}_{\mathcal{X}})$ is

$$\widetilde{\mathscr{R}} := T\mathfrak{II}_{\mathcal{X}} \ominus J^* T\mathcal{X} \oplus \bigoplus_{i=1}^{3} ev_i^* \tilde{\mathscr{S}}. \tag{9.3}$$

We can now define $\mathsf{K}_{\mathrm{orb}}(\mathcal{X})$ as follows.

Definition 9.3. As a vector space, the *full orbifold K-theory* $\mathsf{K}_{\mathrm{orb}}(\mathcal{X})$ is $K(\mathfrak{I}_{\mathcal{X}})$. The orbifold product of two bundles $\mathcal{F}$ and $\mathcal{F}'$ in $\mathsf{K}_{\mathrm{orb}}(\mathcal{X})$ is defined to be

$$\mathcal{F} * \mathcal{F}' := (\check{ev}_3)_*\big(ev_1^*(\mathcal{F}) \otimes ev_2^*(\mathcal{F}') \otimes \lambda_{-1}(\widetilde{\mathcal{R}}^*)\big). \tag{9.4}$$

The trace element $\tau^{\mathsf{K}_{\mathrm{orb}}} \in \mathsf{K}_{\mathrm{orb}}(\mathcal{X})^*$ is defined by

$$\tau^{\mathsf{K}_{\mathrm{orb}}}(\mathcal{F}_{[\![m]\!]}) := \sum_{\substack{[\![a,b]\!] \\ [a,b]\in[\![m]\!]}} \chi\Big(\mathcal{X}_{[\![a,b]\!]}, \varepsilon^*_{a,b,m}(\mathcal{F}_{[\![m]\!]}) \otimes \lambda_{-1}\Big(\big(T\mathcal{X}_{[\![a,b]\!]} \oplus \varepsilon^*_{a,b,m}(\mathcal{S}_{[\![m]\!]})\big)^*\Big)\Big), \tag{9.5}$$

where the sum runs over all (diagonal) conjugacy classes of pairs $[\![a,b]\!]$ of local automorphisms such that $[a,b] \in [\![m]\!]$.

Finally, the symmetric bilinear form $\eta_{\mathsf{K}_{\mathrm{orb}}}$ on $\mathsf{K}_{\mathrm{orb}}(\mathcal{X})$ is

$$\eta_{\mathsf{K}_{\mathrm{orb}}}(\mathcal{F}, \mathcal{G}) := \chi\big(\mathfrak{I}_{\mathcal{X}}, \mathcal{F} \otimes \sigma^*(\mathcal{G})\big).$$

We make a similar definition for the Chow ring.

Definition 9.4. As a vector space, we let $A^\bullet_{\mathrm{orb}}(\mathcal{X}) := A^\bullet(\mathfrak{I}_{\mathcal{X}})$. The orbifold product of $\mathcal{F}$ and $\mathcal{F}'$ in $A^\bullet_{\mathrm{orb}}(\mathcal{X})$ is defined to be

$$\mathcal{F} * \mathcal{F}' := (\check{ev}_3)_*\big(ev_1^*(\mathcal{F}) \otimes ev_2^*(\mathcal{F}') \otimes c_{top}(\widetilde{\mathcal{R}})\big). \tag{9.6}$$

The trace element, denoted by $\tau^{A^\bullet_{\mathrm{orb}}}$, where $\tau^{A^\bullet_{\mathrm{orb}}} \in A^\bullet_{\mathrm{orb}}(\mathcal{X})^*$ is given by

$$\tau^{A^\bullet_{\mathrm{orb}}}(\mathcal{F}_{[\![m]\!]}) := \sum_{\substack{[\![a,b]\!] \\ [a,b]\in[\![m]\!]}} \int_{\mathcal{X}_{[\![a,b]\!]}} \varepsilon^*_{a,b,m}(\mathcal{F}_{[\![m]\!]}) \cup c_{\mathrm{top}}\big(T\mathcal{X}_{[\![a,b]\!]} \oplus \varepsilon^*_{a,b,m}(\mathcal{S}_{[\![m]\!]})\big), \tag{9.7}$$

and the pairing, denoted by $\eta_{A^\bullet_{\mathrm{orb}}}$, on $A^\bullet_{\mathrm{orb}}(\mathcal{X})$ as

$$\eta_{A^\bullet_{\mathrm{orb}}}(\mathcal{F}, \mathcal{G}) := \int_{\mathfrak{I}_{\mathcal{X}}} \mathcal{F} \cup \sigma^*(\mathcal{G}).$$

A related construction has now appeared in [ARZ] in the topological category although they not describe a Frobenius structure or a Chern character. It would be interesting to explore the connections between their construction and ours.

Theorem 9.5. *Let $\mathcal{X}$ be a smooth Deligne–Mumford stack.*

(1) $(A^\bullet_{\mathrm{orb}}(\mathcal{X}), *, \eta_{A^\bullet_{\mathrm{orb}}}, \tau^{A^\bullet_{\mathrm{orb}}})$ *is a pre-Frobenius algebra called* the orbifold Chow ring of $\mathcal{X}$. *Moreover, $(A^\bullet_{\mathrm{orb}}(\mathcal{X}), *)$ is isomorphic to the "orbifold Chow ring" of* [AGV].

(2) *If* $\mathscr{X}$ *satisfies condition* $(\star)$*, then* $(\mathsf{K}_{\mathrm{orb}}(\mathscr{X}), *, \eta_{\mathsf{K}_{\mathrm{orb}}}, \tau^{\mathsf{K}_{\mathrm{orb}}})$ *is a pre-Frobenius algebra called* the full orbifold K-theory of $\mathscr{X}$.

(3) *If* $\mathscr{X}$ *satisfies condition* $(\star)$*, then the* full orbifold Chern character $\mathfrak{Ch}_{\mathrm{orb}} : \mathsf{K}_{\mathrm{orb}}(\mathscr{X}) \to A^{\bullet}_{\mathrm{orb}}(\mathscr{X})$ *is an allometric homomorphism of pre-Frobenius algebras, where* $\mathfrak{Ch}_{\mathrm{orb}} := \mathbf{ch} \cup \mathbf{td}(\ominus \tilde{\mathscr{S}})$*, and* $\tilde{\mathscr{S}}$ *is given in (9.2).*

Proof. Parts (1) and (2) will be proved in Sect. 9.3.

Part (3) follows from the definition of $\widetilde{\mathscr{R}}$, of $\mathscr{S}$, and of $\mathfrak{Ch}_{\mathrm{orb}}$, in a manner similar to the proof of Theorem 6.1. □

We conclude this section with a full orbifold version of Theorem 6.3.

Theorem 9.6. *Let* $f : \mathscr{X} \to \mathscr{Y}$ *be an étale morphism of smooth, Deligne–Mumford stacks such that* $\mathscr{X}$ *and* $\mathscr{Y}$ *both satisfy condition* $(\star)$*. The following properties hold.*

(1) *(Pullback) The pullback maps* $f^* : (A^{\bullet}_{\mathrm{orb}}(\mathscr{Y}), *, \mathbf{1}_{\mathscr{Y}}) \to (A^{\bullet}_{\mathrm{orb}}(\mathscr{X}), *, \mathbf{1}_{\mathscr{X}})$ *and* $f^* : (\mathsf{K}_{\mathrm{orb}}(\mathscr{Y}), *, \mathbf{1}_{\mathscr{Y}}) \to (\mathsf{K}_{\mathrm{orb}}(\mathscr{X}), *, \mathbf{1}_{\mathscr{X}})$ *are ring homomorphisms.*

(2) *(Naturality) The following diagram commutes.*

$$\begin{array}{ccc} \mathsf{K}_{\mathrm{orb}}(\mathscr{Y}) & \xrightarrow{f^*} & \mathsf{K}_{\mathrm{orb}}(\mathscr{X}) \\ {\scriptstyle \mathfrak{Ch}_{\mathrm{orb}}}\downarrow & & {\scriptstyle \mathfrak{Ch}_{\mathrm{orb}}}\downarrow \\ A^{\bullet}_{\mathrm{orb}}(\mathscr{Y}) & \xrightarrow{f^*} & A^{\bullet}_{\mathrm{orb}}(\mathscr{X}) \end{array} \tag{9.8}$$

(3) *(Grothendieck–Riemann–Roch) For all* $\mathscr{F}$ *in* $\mathsf{K}_{\mathrm{orb}}(\mathscr{X})$*,*

$$f_*(\mathfrak{Ch}_{\mathrm{orb}}(\mathscr{F}) \cup \mathbf{td}(T\mathscr{X})) = \mathfrak{Ch}_{\mathrm{orb}}(f_*\mathscr{F}) \cup \mathbf{td}(T\mathscr{Y}). \tag{9.9}$$

The proof of this theorem is a straightforward adaptation of the proof of its stringy counterpart, Theorem 6.3.

9.2. Small orbifold K-theory. We now introduce the algebra called the *small orbifold K-theory* $K_{\mathrm{orb}}(\mathscr{X})$ when $\mathscr{X} = [X/G]$ is the global quotient of a smooth, projective variety X by a finite group G. We will then explain its relationship to $\mathsf{K}_{\mathrm{orb}}(\mathscr{X})$.

For the rest of this section, assume that $\mathscr{X}$ is a global quotient $[X/G]$ of a smooth, projective variety X by the action of a finite group G.

In this case, the inertia stack $\mathfrak{I}_{\mathscr{X}}$ and double inertia stack $\mathfrak{II}_{\mathscr{X}}$ can also be written as global quotients

$$\mathfrak{I}_{\mathscr{X}} = [I_G(X)/G] = \Big[\Big(\coprod_{g\in G} X^g\Big)/G\Big],$$

and

$$\mathfrak{II}_{\mathscr{X}} = [\mathbb{I}_G(X)/G] := \Big[\Big(\coprod_{g,h\in G} X^{g,h}\Big)/G\Big],$$

where each X^g and $X^{g,h}$ is also smooth. As usual, there are morphisms $ev_i : \mathbb{I}_G(X) \to I_G(X)$ and $\check{ev}_i : \mathbb{I}_G(X) \to I_G(X)$ for $i \in \{1, 2, 3\}$.

Flat descent [SGA6, VIII§1] shows that

$$K(\mathcal{X}) \cong K_G(X).$$

The projection

$$\pi : I_G(X) \to \mathfrak{I}_{\mathcal{X}}$$

induces a ring homomorphism (with respect to the usual product $\otimes$)

$$\pi^* : K(\mathfrak{I}_{\mathcal{X}}) = K_G(I_G(X)) \to K(I_G(X))^G.$$

Similarly, we also have a ring *isomorphism* (with respect to the usual product $\cup$)

$$\pi^* : A^\bullet(\mathfrak{I}_{\mathcal{X}}) = A^\bullet_G(I_G(X)) \to A^\bullet(I_G(X))^G$$

because G is a finite group.

It is straightforward to see that π^* commutes with both pullback and pushforward along the morphisms ev_i and $\check{ev}_i$ for all $i \in \{1, 2, 3\}$.

Definition 9.7. For any stack $\mathcal{X}$ which is a global quotient $[X/G]$ of a smooth, projective variety X by a finite group G, the *small orbifold K-theory* $K_{\mathrm{orb}}(\mathcal{X})$ of $\mathcal{X}$ is the pre-Frobenius algebra $\overline{\mathcal{K}}(X, G) = \mathcal{K}(X, G)^G$ of coinvariants of stringy K-theory:

$$K_{\mathrm{orb}}(\mathcal{X}) := \overline{\mathcal{K}}(X, G).$$

This is linearly isomorphic to $K(\mathcal{X})$ and to $K(I_G(X))^G$, but with the stringy product instead of the tensor product.

Theorem 9.8. *Let X be a smooth, projective variety with the action of a finite group G.*

(1) π^* *is a ring homomorphism from* $(\mathsf{K}_{\mathrm{orb}}(\mathcal{X}), *)$ *to* $(K_{\mathrm{orb}}(\mathcal{X}), *)$.
(2) π^* *is an isomorphism of pre-Frobenius algebras from* $(A^\bullet_{\mathrm{orb}}(\mathcal{X}), *)$ *to* $(\mathcal{A}^\bullet(X, G)^G, *) = \overline{\mathcal{A}}(X, G)$.
(3) *The stringy Chern character* $\mathfrak{C}\mathbf{h} : \mathcal{K}(X, G) \to \mathcal{A}^\bullet(X, G)$ *induces an allometric isomorphism on the invariants*

$$Ch_{\mathrm{orb}} : K_{\mathrm{orb}}(\mathcal{X}) = \overline{\mathcal{K}}(X, G) \to \overline{\mathcal{A}}(X, G) \cong A^\bullet_{\mathrm{orb}}(\mathcal{X}),$$

which we call the small orbifold Chern character.
(4) *The following diagram of ring homomorphisms (with respect to the orbifold, or stringy product $*$) commutes:*

$$\begin{array}{ccccccc}
\mathsf{K}_{\mathrm{orb}}(\mathcal{X}) & = & K_G(I_G X) & \xrightarrow{\pi^*} & K(I_G X)^G & = & K_{\mathrm{orb}}(\mathcal{X}) \\
\Big\downarrow \scriptstyle{\mathfrak{Ch}_{\mathrm{orb}}} & & & & & & \Big\downarrow \scriptstyle{Ch_{\mathrm{orb}}} \\
A^\bullet_{\mathrm{orb}}(\mathcal{X}) & = & A^\bullet_G(I_G X) & \overset{\pi^*}{=} & \mathcal{A}^\bullet(X, G)^G & = & \overline{\mathcal{A}}(X, G).
\end{array}$$

(5) *The ring* $K_{\mathrm{orb}}(\mathcal{X})$ *is independent of the choice of presentation of the stack* $\mathcal{X}$ *as a global quotient* $[X/G]$.
(6) *There is an embedding of pre-Frobenius algebras*

$$\iota : K_{\mathrm{orb}}([X/G]) \otimes \mathbb{C} \to \mathsf{K}_{\mathrm{orb}}([X/G]) \otimes \mathbb{C},$$

such that $\pi^* \circ \iota = \mathbf{1}_{K_{\mathrm{orb}}([X/G])}$.

Lemma 9.9. *Let* $\tilde{\mathcal{S}} \in K(\mathfrak{I}_{\mathcal{X}})$ *be the (virtual) sheaf given in (9.2), and let* $\mathcal{S} \in \overline{\mathcal{K}}(X, G)$ *be the K-theoretic age sheaf given in Definition 1.3. Similarly, let* $\widetilde{\mathcal{R}}$ *be the obstruction bundle in* $K(\mathfrak{I}\mathfrak{I}_{\mathcal{X}}) = K_G(\mathbb{I}_G(X))$*, and let* $\mathcal{R}$ *be the obstruction bundle in* $K(\mathbb{I}_G(X))$ *arising in the stringy K-theory* $\mathcal{K}(X, G)$*. We have*

$$\mathcal{S} = \pi^* \tilde{\mathcal{S}} \tag{9.10}$$

and

$$\mathcal{R} = \pi^* \widetilde{\mathcal{R}}. \tag{9.11}$$

Proof. Equation (9.10) follows immediately from the definitions of π^*, $\mathcal{S}$, and $\tilde{\mathcal{S}}$. Equation (9.11) follows from the fact that the $\mathcal{S}_m$, $\tilde{\mathcal{S}}_m$, and the normal bundles in the definition of $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ match term by term. □

Proof of Theorem 9.8. Parts (1) and (2) follow immediately from Lemma 9.9 and the fact that π^* commutes with pullback and pushforward along the maps ev_i and $\check{ev}_i$ for all $i \in \{1, 2, 3\}$.

Part (3) follows from (taking invariants of) Theorem 6.1.

Part (4) follows from naturality of the classical Chern character and Lemma 9.9.

Part (5) follows from the fact that $A^\bullet_{\mathrm{orb}}(\mathcal{X})$ is presentation independent and Ch_{orb} is an allometric isomorphism.

Finally, Part (6) follows from the fact that if Y is a smooth, projective variety with the action of a finite group G, then there is a canonical isomorphism [AS] (see also [Vis, §1], [VV, §7] and [EG, §3.4]) of algebras (with the ordinary multiplication $\otimes$):

$$K_G(Y) \otimes \mathbb{C} \cong (K(I_G Y) \otimes \mathbb{C})^G \cong \bigoplus_{[\![m]\!] \in \overline{G}} (K(Y^m) \otimes \mathbb{C})^{Z_G(m)}. \tag{9.12}$$

The isomorphism takes $\mathcal{F} \in K_G(Y)$ to an eigenbundle decomposition of $\mathcal{F}|_{Y^m}$ in the $[\![m]\!]$-sector $(K(Y^m) \otimes \mathbb{C})^{Z_G(m)}$.

Setting $Y = I_G X$ then we obtain

$$\begin{aligned} \mathsf{K}_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C} &:= K([I_G(X)/G]) \otimes \mathbb{C} \\ &\cong (K(I_G(I_G(X))) \otimes \mathbb{C})^G \\ &= \bigoplus_{[\![g,m]\!]} (K(X^{\langle g,m \rangle}) \otimes \mathbb{C})^{Z_G(g,m)}, \end{aligned}$$

where the indices run over diagonal conjugacy classes of commuting pairs (g, m) in G^2. We denote the sector $(K(X^{\langle g,m\rangle}) \otimes \mathbb{C})^{Z_G(g,m)}$ by $K^{\mathfrak{I}_{\mathcal{X}}}_{[\![g,m]\!]}$. Similarly, setting $Y = \mathbb{I}_G(X)$ we obtain

$$\begin{aligned} K(\mathfrak{II}_{\mathcal{X}}) \otimes \mathbb{C} &= K([\mathbb{I}_G(X)/G]) \otimes \mathbb{C} \\ &\cong (K(I_G(\mathbb{I}_G(X))) \otimes \mathbb{C})^G \\ &= \bigoplus_{[\![g,h,m]\!]} (K(X^{\langle g,h,m\rangle}) \otimes \mathbb{C})^{Z_G(g,h,m)} \\ &=: \bigoplus_{[\![g,h,m]\!]} K^{\mathfrak{II}_{\mathcal{X}}}_{[\![g,h,m]\!]}, \end{aligned}$$

where the indices run over diagonal conjugacy classes of triples (g, h, m) in G^3 such that m commutes with both g and h. It is easy to see that pullback along the morphism $ev_1 : \mathfrak{II}_{\mathcal{X}} \to \mathfrak{I}_{\mathcal{X}}$ takes the sector $K^{\mathfrak{I}_{\mathcal{X}}}_{[\![a,m]\!]}$ to the sum of sectors $\bigoplus_{[\![a,h,m]\!]} K^{\mathfrak{II}_{\mathcal{X}}}_{[\![a,h,m]\!]}$ where the indices run over conjugacy classes $[\![a, h, m]\!]$ with fixed pair $[\![a, m]\!]$. Similarly, pullback along the morphism $ev_2 : \mathfrak{II}_{\mathcal{X}} \to \mathfrak{I}_{\mathcal{X}}$ takes the sector $K^{\mathfrak{I}_{\mathcal{X}}}_{[\![a,m]\!]}$ to the sum of sectors $\bigoplus_{[\![g,a,m]\!]} K^{\mathfrak{II}_{\mathcal{X}}}_{[\![g,a,m]\!]}$ where the indices run over conjugacy classes $[\![g, a, m]\!]$ with fixed pair $[\![a, m]\!]$. Pushforward along $\check{ev}_3$ maps sectors of the form $K^{\mathfrak{II}_{\mathcal{X}}}_{[\![a,b,m]\!]}$ to the sector $K^{\mathfrak{I}_{\mathcal{X}}}_{[\![ab,m]\!]}$.

Define the map $\iota : K_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C} \to \mathsf{K}_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C}$ to send the sector $(K(X^g) \otimes \mathbb{C})^{Z_G(g)} \subseteq K_{\mathrm{orb}}(\mathcal{X})$ identically to the "untwisted" sector $K^{\mathfrak{I}_{\mathcal{X}}}_{[\![g,1]\!]} \subseteq K(\mathfrak{I}_{\mathcal{X}}) \otimes \mathbb{C} = \mathsf{K}_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C}$. Similarly, define a map $\iota' : K(\mathbb{I}_G(X))^G = \bigoplus_{[\![g,h]\!]} (K(X^{\langle g,h\rangle}) \otimes \mathbb{C})^{Z_G(g,h)} \otimes \mathbb{C} \to K(\mathfrak{II}_{\mathcal{X}}) \otimes \mathbb{C}$ by taking the $[\![g, h]\!]$-sector identically to the $[\![g, h, 1]\!]$-sector.

The map $\pi^* : \mathsf{K}_{\mathrm{orb}}(\mathcal{X}) \to K_{\mathrm{orb}}(\mathcal{X})$ sends a sector of the form $K^{\mathfrak{I}_{\mathcal{X}}}_{[\![g,m]\!]}$ to $(K(X^g) \otimes \mathbb{C})^{Z_G(g)}$, and on sectors of the form $K^{\mathfrak{I}_{\mathcal{X}}}_{[\![g,1]\!]}$ the map π^* sends $K^{\mathfrak{I}_{\mathcal{X}}}_{[\![g,1]\!]}$ identically to the sector $(K(X^g) \otimes \mathbb{C})^{Z_G(g)}$. From this it is clear that $\pi^* \circ \iota = \mathbf{1}$ and that ι is injective.

Clearly, ι and ι' commute with ev_i^* and $(\check{ev}_i)_*$, and λ_{-1} preserves the decomposition. Moreover, the tensor product of two homogeneous elements will vanish unless both elements lie in the same sector. Thus, although $\iota'(\mathcal{R})$ is *not* equal to $\widetilde{\mathcal{R}}$ in $K(\mathfrak{II}_{\mathcal{X}})$, it is true that for any $\mathcal{F}$ and $\mathcal{F}'$ in $K_{\mathrm{orb}}(\mathcal{X})$ we have

$$\begin{aligned} \iota(\mathcal{F}) * \iota(\mathcal{F}') &= (\check{ev}_3)_*\big(ev_1^*(\iota(\mathcal{F})) \otimes ev_2^*(\iota(\mathcal{F}')) \otimes \lambda_{-1}(\widetilde{\mathcal{R}}^*)\big) \\ &= (\check{ev}_3)_*\big(ev_1^*(\iota(\mathcal{F})) \otimes ev_2^*(\iota(\mathcal{F}')) \otimes \iota'(\lambda_{-1}(\mathcal{R}^*))\big) \\ &= (\check{ev}_3)_*\big(\iota'(ev_1^*(\mathcal{F})) \otimes \iota'(ev_2^*(\mathcal{F}')) \otimes \iota'(\lambda_{-1}(\mathcal{R}^*))\big) \\ &= \iota\big((\check{ev}_3)_*\big(ev_1^*(\mathcal{F}) \otimes ev_2^*(\mathcal{F}') \otimes (\lambda_{-1}(\mathcal{R}^*))\big)\big) \\ &= \iota(\mathcal{F} * \mathcal{F}'). \end{aligned}$$

This shows that ι is a ring homomorphism. Similar arguments show that ι preserves the trace and pairing. □

9.3. Proof of Theorem 9.5. In this subsection we will prove Theorem 9.5.

The key step is showing that, as in the stringy case, the element $\widetilde{\mathscr{R}} \in K(\mathfrak{I}\mathcal{X})$ is represented by a vector bundle. Associativity of the multiplication follows from this fact. To prove that $\widetilde{\mathscr{R}}$ is a vector bundle, we will show that it is equal to the obstruction bundle for genus-zero, three-pointed orbifold stable maps into $\mathcal{X}$.

It is not hard to see that the stack $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$ of degree-zero, genus-zero, 3-pointed orbifold stable maps into $\mathcal{X}$ is naturally isomorphic to the double inertia stack $\mathfrak{I}\mathcal{X}$. We will equate the two from now on. We denote the universal curve over it by $\varpi : \mathcal{C} \to \overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$, and the universal stable map by $\bar{f} : \mathcal{C} \to \mathcal{X}$. The evaluation maps from $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$ to $\mathscr{I}\mathcal{X}$ are given by $ev_i([\bar{f} : \mathcal{C} \to \mathcal{X}]) = (f(p_i), [\![g_{p_i}]\!]) \in \mathcal{X}_{[\![g_{p_i}]\!]}$, where p_i is the i-th marked (gerbe) point of $\mathcal{C}$, and g_{p_i} is the image of the canonical generator of $\mathrm{stab}(p_i)$ in $\mathrm{stab}(f(p_i))$. Of course, this image is only defined up to conjugacy, since if $\mathcal{X}$ is locally presented as $[X/G]$ near a point $p_i \in \mathcal{X}$, then a representative $\widetilde{p}_i \in X$ of p_i can be replaced by another representative $\gamma\widetilde{p}_i$ for any $\gamma \in G$, which replaces g_{p_i} by $\gamma g_{p_i}\gamma^{-1}$. Because of this, the i-th evaluation map $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0) \to \mathscr{I}\mathcal{X}$ agrees with the ev_i described above for $\mathfrak{I}\mathcal{X} \to \mathscr{I}\mathcal{X}$ for all $i \in \{1, 2, 3\}$.

Theorem 9.10. *In the K-theory of $\mathfrak{I}\mathcal{X} = \overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$, the following relation holds for the bundle $\widetilde{\mathscr{R}}$:*

$$\widetilde{\mathscr{R}} \cong R^1\varpi_*(\bar{f}^*T\mathcal{X}).$$

Proof. The idea of the proof is to use distinguished components of the stack of pointed admissible covers $\xi_{0,3}$ [JKK, §2.5.1] to produce an étale cover of the moduli stack $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$. On this cover, we can easily produce an isomorphism of equivariant bundles, but it is not unique – it is only determined up to conjugacy. However, the bundles we really want are the invariant sub-bundles of these equivariant bundles, and on these subbundles the induced isomorphism is independent of conjugation. Thus, étale descent applies, and we obtain the desired isomorphism.

We first recall the definitions from [JKK, §2.5.1 and §6] of $\xi^G_{0,3}(\mathbf{m})$ and $\xi^G_{0,3}(X, \mathbf{m})$. As briefly described in Sect. 8, to each triple $\mathbf{m} = (m_1, m_2, m_3)$ with $m_1m_2m_3 = 1$, there is a canonical choice of pointed admissible $\langle m\rangle$-cover $(E, \widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3) \to (\mathbb{P}^1, 0, 1, \infty)$, ramified only over the points 0, 1, and ∞, and with monodromy m_1, m_2, and m_3, respectively, at those points. There is also a canonical choice of pointed admissible G-cover $(\tilde{E}, \widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3) \to (\mathbb{P}^1, 0, 1, \infty)$ with $\tilde{E} = E \times_{\langle m\rangle} G$. For a complete discussion of these constructions, see [JKK, §2.5.1].

Definition 9.11. We define $\xi_{0,3}^{\langle \mathbf{m} \rangle}$ to be the connected component of the stack of 3-pointed admissible $\langle \mathbf{m} \rangle$-covers of genus zero that contains the canonical admissible cover $(E, \widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3)$. Similarly, we define $\xi_{0,3}^G$ to be the connected component of the stack of three-pointed admissible $\langle \mathbf{m} \rangle$-covers of genus zero that contains the admissible cover $(\tilde{E}, \widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3)$.

If X is any variety with a G-action, a degree-zero, 3-pointed G-stable map of genus zero is a G-equivariant morphism $\tilde{E} \to X$ from a 3-pointed admissible G-cover to X, such that the induced morphism $\tilde{E}/G \to X/G$ is a 3-pointed stable map of genus zero. We define $\xi_{0,3}^G(X, 0, \mathbf{m})$ to be the component of the stack of pointed G-stable maps whose underlying 3-pointed admissible G-covers $\tilde{E}$ correspond to points of $\xi_{0,3}^G(\mathbf{m})$.

It is easy to see that there is a canonical isomorphism $\xi_{0,3}^{\langle \mathbf{m} \rangle}(\mathbf{m}) \xrightarrow{\sim} \xi_{0,3}^G(\mathbf{m})$, and that $\xi_{0,3}^G(\mathbf{m})$ is the stack quotient $\mathcal{B}H_{\mathbf{m}} = [pt/H_{\mathbf{m}}]$ of a point modulo the group $H_{\mathbf{m}} := \langle m_1 \rangle \cap \langle m_2 \rangle \cap \langle m_3 \rangle$ (see [JKK, Prop. 2.20]). Moreover, in [JKK, Lemma 6.7] it is shown that $\xi_{0,3}^G(X, 0, \mathbf{m})$ is canonically isomorphic to $\xi_{0,3}^G \times X^{\mathbf{m}}$. Finally, we have a morphism $q : \xi_{0,3}(X, 0, \mathbf{m}) \to \overline{\mathcal{M}}_{0,3}([X/G], 0)$ given by sending a pointed G-stable map $[f : E \to X]$ to the induced map of quotient stacks $[\bar{f} : [E/G] \to [X/G]]$. This morphism is easily seen to be étale.

Now we may begin the proof. First consider any étale cover U of $\mathcal{X}$, consisting of a disjoint union of smooth varieties X_α with finite groups G_α acting to make $q_\alpha : X_\alpha \to \mathcal{X}$ induce an isomorphism $[X_\alpha/G_\alpha]$ to a neighborhood in $\mathcal{X}$ (that is, $\{(X_\alpha, G_\alpha, q_\alpha)\}$ form a uniformizing system). We may construct an étale cover

$$p : \coprod_{\alpha,\mathbf{m}} X_\alpha^{\mathbf{m}} \to \coprod_{\alpha,\mathbf{m}} \xi_{0,3}^{G_\alpha} \times X_\alpha^{\mathbf{m}} = \coprod_{\alpha,\mathbf{m}} \xi_{0,3}^{G_\alpha}(X_\alpha, 0, \mathbf{m}) \to \overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0),$$

where for each α, the $\mathbf{m}$ runs through all triples in G_α whose product is 1, and the first morphism is induced by the obvious (étale) map $pt \times X_\alpha^{\mathbf{m}} \to [pt/H_{\mathbf{m}}] \times X_\alpha^{\mathbf{m}} = \xi_{0,3}^{G_\alpha} \times X_\alpha^{\mathbf{m}}$.

For each α and $\mathbf{m}$, the pullback $p^*\widetilde{\mathcal{R}}$ is easily seen to be the usual obstruction bundle $\mathcal{R}(\mathbf{m})$ on $X_\alpha^{\mathbf{m}}$, and the pullback $p^*\big(T\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0) \ominus J^*T\mathcal{X} \oplus \bigoplus_{i=1}^3 ev_i^* \mathcal{S}\big)$ is clearly equal to $TX_\alpha^{\mathbf{m}} \ominus TX_\alpha|_{X_\alpha^{\mathbf{m}}} \oplus \bigoplus_{i=1}^3 \mathcal{S}_{m_i}|_{X_\alpha^{\mathbf{m}}}$. But to prove the theorem, we will need to provide a canonical isomorphism between these bundles.

The fibered product $X_\alpha^{\mathbf{m}} \times_{\overline{\mathcal{M}}_{0,3}(\mathcal{X},0)} X_\beta^{\mathbf{m}'}$ is non-empty if and only if the stack $\xi_{0,3}^{G_\alpha}(X_\alpha, 0, \mathbf{m}) \times_{\overline{\mathcal{M}}_{0,3}(\mathcal{X},0)} \xi_{0,3}^{G_\beta}(X_\beta, 0, \mathbf{m}')$ is non-empty; and it is straightforward to see that this occurs only if there is a group G with injective homomorphisms $G \hookrightarrow G_\alpha$ and $G \hookrightarrow G_\beta$, such that the triple $\mathbf{m}'$ is diagonally (i.e., all three terms simultaneously) conjugate to $\mathbf{m}$ in G^3. Moreover, for each connected component of $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$, there is a well-defined diagonal conjugacy class of such triples.

For each such conjugacy class, choose a representative $\mathbf{m}$ and let $K = \langle m_1, m_2, m_3\rangle$ be the group generated by the triple. As described above, this triple determines a well-defined distinguished component $\xi^K_{0,3}(\mathbf{m})$ of the stack of three-pointed, admissible K-covers of genus zero.

Choose, once and for all, an isomorphism $\Phi_{\mathbf{m}}$ of K-representations giving the (virtual) equality of (8.4) in Lemma 8.5, but where G is replaced by K. For any other triple $\mathbf{m}'$ in the same conjugacy class, there is a canonical isomorphism of groups $K' = \langle m'_1, m'_2, m'_3\rangle \xrightarrow{\sim} K$ taking $\mathbf{m}'$ to $\mathbf{m}$, and a canonical (equivariant) isomorphism of representations $H^1(E'; \mathcal{O}_{E'}) \cong H^1(E; \mathcal{O}_E)$, where $E \to C \to \xi^K_{0,3}(\mathbf{m})$ is the three-pointed admissible K-cover with holonomy $\mathbf{m}$, and $E' \to C' \to \xi^{K'}_{0,3}(\mathbf{m}')$ is the three-pointed admissible K'-cover with holonomy $\mathbf{m}'$. Similarly, we have canonical (equivariant) isomorphisms of the representations

$$\mathbb{C} \ominus \mathbb{C}[K] \oplus \bigoplus_{i=1}^{n} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \operatorname{Ind}^K_{\langle m_i\rangle} \mathbf{V}_{m_i,k_i}$$
$$\cong \mathbb{C} \ominus \mathbb{C}[K'] \oplus \bigoplus_{i=1}^{n} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \operatorname{Ind}^{K'}_{\langle m_i\rangle} \mathbf{V}_{m_i,k_i}. \tag{9.13}$$

Thus $\Phi_{\mathbf{m}}$ induces an isomorphism $\Phi_{\mathbf{m}'}$ for each triple $\mathbf{m}'$ which is conjugate to $\mathbf{m}$.

If G is any group containing both K and K', with K' a conjugate (say by $\gamma \in G$) of K, then letting $\tilde{E} \to C \to \xi^G_{0,3}(\mathbf{m})$ denote the distinguished three-pointed G-cover with holonomy $\mathbf{m}$, and $\tilde{E}' \to C \to \xi^G_{0,3}(\mathbf{m}')$ denote the distinguished universal three-pointed G-cover with holonomy $\mathbf{m}'$, the group action $\rho(\gamma) : \xi^G_{0,3}(\mathbf{m}) \xrightarrow{\sim} \xi^G_{0,3}(\mathbf{m}')$ identifies the base (γ acts on E and E'). Furthermore, we have canonical isomorphisms of G-representations

$$H^1\big(\tilde{E}; \mathcal{O}_{\tilde{E}}\big) \cong \operatorname{Ind}^G_K \big(H^1(E; \mathcal{O}_E)\big) \tag{9.14}$$

and

$$H^1\big(\tilde{E}'; \mathcal{O}_{\tilde{E}'}\big) \cong \operatorname{Ind}^G_{K'} \big(H^1(E'; \mathcal{O}_{E'})\big). \tag{9.15}$$

As G-representations, $H^1(\tilde{E}; \mathcal{O}_{\tilde{E}})$ and $\rho(\gamma)^* H^1(\tilde{E}'; \mathcal{O}_{\tilde{E}'})$ are not identical, but rather are conjugate; that is, $H^1(\tilde{E}'; \mathcal{O}_{\tilde{E}'})$ is the representation of G arising from conjugating the action of G on $H^1(\tilde{E}; \mathcal{O}_{\tilde{E}})$ by γ. The same holds for the induced representations

$$\mathbb{C}[G/K] \ominus \mathbb{C}[G] \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \operatorname{Ind}^G_{\langle m_i\rangle} \mathbf{V}_{m_i,k_i}$$
$$\cong \operatorname{Ind}^G_K \left(\mathbb{C} \ominus \mathbb{C}[K] \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \operatorname{Ind}^K_{\langle m_i\rangle} \mathbf{V}_{m_i,k_i}\right), \tag{9.16}$$

and

$$\mathbb{C}[G/K'] \ominus \mathbb{C}[G] \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \operatorname{Ind}_{\langle m_i' \rangle}^{G} \mathbf{V}_{m_i',k_i}$$
$$\cong \operatorname{Ind}_{K'}^{G} \left(\mathbb{C} \ominus \mathbb{C}[K'] \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \operatorname{Ind}_{\langle m_i' \rangle}^{K'} \mathbf{V}_{m_i',k_i} \right). \tag{9.17}$$

Finally, for an open subset V of any X_α with G acting on V, pulling back by the action

$$\rho(\gamma) : V^{\mathbf{m}} \xrightarrow{\sim} V^{\mathbf{m}'}$$

makes the G-bundle $\rho(\gamma)^* f^* TV = \mathcal{O}_{\tilde{E}'} \boxtimes \operatorname{Ind}_{K'}^{G} TV|_{V^{\mathbf{m}'}}$ on $V^{\mathbf{m}}$ isomorphic to the conjugate by γ of the G-bundle $f^*TV = \mathcal{O}_{\tilde{E}} \boxtimes \operatorname{Ind}_{K}^{G} TV|_{V^{\mathbf{m}}}$. Thus the isomorphisms $\Phi_{\mathbf{m}'}$ and the induced isomorphisms

$$\tilde{\Phi}_{\mathbf{m}} : R^1\pi_*(f^*TV) = H^1(\tilde{E}; \mathcal{O}_{\tilde{E}}) \otimes \operatorname{Ind}_{K}^{G} TV|_{V^{\mathbf{m}}}$$
$$\xrightarrow{\sim} \left(\mathbb{C}[G/K] \ominus \mathbb{C}[G] \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \operatorname{Ind}_{\langle m_i \rangle}^{G} \mathbf{V}_{m_i,k_i} \right) \otimes \operatorname{Ind}_{K}^{G} TV\Big|_{V^{\mathbf{m}}} \tag{9.18}$$

on $\xi_{0,3}^{G}(\mathbf{m}) \times V^{\mathbf{m}}$ are determined up to conjugacy by an element in G.

However, a representation and any conjugate of that representation have canonically identified invariants, so the isomorphisms $\tilde{\Phi}_{\mathbf{m}}$ induce isomorphisms of the invariant bundles

$$\overline{\Phi}_{\mathbf{m}} : R^1\pi_*^{G}(f^*TV)$$
$$\xrightarrow{\sim} \left(\left(\mathbb{C}[G/K] \ominus \mathbb{C}[G] \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \operatorname{Ind}_{\langle m_i \rangle}^{G} \mathbf{V}_{m_i,k_i} \right) \otimes TV\Big|_{V^{\mathbf{m}}} \right)^{G}$$
$$= TV^{\mathbf{m}} \ominus TV\Big|_{V^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \mathcal{S}_{m_i}\Big|_{V^{\mathbf{m}}}, \tag{9.19}$$

which are independent of conjugation.

In summary, we have chosen an explicit isomorphism

$$\overline{\Phi} : p^* \widetilde{\mathcal{R}} \xrightarrow{\sim} p^* \left(T\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0) \ominus J^*T\mathcal{X} \oplus \bigoplus_{i=1}^{3} ev^* \mathcal{S} \right)$$

on the étale cover $\coprod_{\alpha,\mathbf{m}} X_\alpha^{\mathbf{m}} \xrightarrow{p} \overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$, with the particular property that on the product

$$\coprod X_\alpha^{\mathbf{m}} \times_{\overline{\mathcal{M}}_{0,3}(\mathcal{X},0)} \coprod X_\alpha^{\mathbf{m}} \underset{t}{\overset{s}{\rightrightarrows}} \coprod X_\alpha^{\mathbf{m}}$$

we have $s^*\overline{\Phi} = t^*\overline{\Phi}$. Thus by étale descent the isomorphism $\overline{\Phi}$ descends from the cover $\coprod X_\alpha^{\mathbf{m}}$ to the stack $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$. □

The proof of associativity given in Lemma 5.4 is now easily adapted to give a proof of associativity for the orbifold product in $A^\bullet_{\text{orb}}(\mathcal{X})$ and $\mathsf{K}_{\text{orb}}(\mathcal{X})$. The rest of the properties of a pre-Frobenius algebra are straightforward to check.

The fact that $A^\bullet_{\text{orb}}(\mathcal{X})$ is isomorphic to the construction of [AGV] also follows from Theorem 9.10 and from the equality $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0) = \mathfrak{I}\mathcal{X}$, since the definition of the product in [AGV] is the usual quantum product with the obstruction bundle $R^1\varpi_*(\bar{f}^*T\mathcal{X})$.

10. Stringy topological K-theory and stringy cohomology

All of the results in the previous sections have their counterparts in the topological category.

10.1. Ordinary topological K-theory and cohomology. Throughout this section, unless otherwise stated, G is a finite group acting on a compact, almost complex manifold X, preserving the almost complex structure.

Furthermore, let $H^\bullet(X)$ be the rational cohomology of X. It is a Frobenius superalgebra: a Frobenius algebra with a multiplication that is graded commutative.

Topological K-theory $K_{\text{top}}(X) := K_{\text{top}}(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is also a Frobenius superalgebra with the $\mathbb{Z}/2\mathbb{Z}$-grading:

$$K_{\text{top}}(X) = K^0_{\text{top}}(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus K^1_{\text{top}}(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Here $K^0_{\text{top}}(X; \mathbb{Z})$ is defined exactly as $K(X; \mathbb{Z})$ but in the topological category. That is, $K^0_{\text{top}}(X; \mathbb{Z})$ is additively generated by isomorphism classes of complex topological vector bundles over X modulo the relation of (2.1) whenever (2.2) holds. The odd part $K^1_{\text{top}}(X; \mathbb{Z})$ is defined to be $K^0_{\text{top}}(X \times \mathbb{R}; \mathbb{Z})$. Equivalently, we may take $K^1_{\text{top}}(X; \mathbb{Z})$ to be the kernel of the restriction map $i^* : K^0_{\text{top}}(X \times S^1) \to K^0_{\text{top}}(X \times pt)$ induced in K-theory from the inclusion of a point $i : X \times pt \to X \times S^1$.

Associated to a differentiable proper map of almost complex manifolds $f : X \to Y$, there is the induced pushforward morphism $f_* : K_{\text{top}}(X) \to K_{\text{top}}(Y)$ (see [Kar, IV 5.24] and [AH, Sect. 4]). In particular, if Y is a point and $f : X \to Y$ is the obvious map, we again define the Euler characteristic as

$$\chi(X, \mathcal{F}) := f_*\mathcal{F}.$$

Associated to any continuous $f : X \to Y$, there is a pullback homomorphism $f^* : K_{\text{top}}(Y) \to K_{\text{top}}(X)$ [Kar, II.1.12].

For any compact, almost complex manifolds X and Y, there are natural morphisms

$$\nu : K^n_{\mathrm{top}}(X) \otimes K^m_{\mathrm{top}}(Y) \to K^0_{\mathrm{top}}(X \times Y \times \mathbb{R}^{n+m}).$$

Bott periodicity says that if Y is a point, there is an isomorphism

$$\beta : K^0_{\mathrm{top}}(X) \xrightarrow{\sim} K^0_{\mathrm{top}}(X \times \mathbb{R}^2)$$

[Kar, III.1.3], which is natural with respect to both pullback and pushforward. Therefore, for any compact, almost complex manifold X, composition of ν with pullback along the diagonal map $\Delta : X \to X \times X$ gives a multiplication

$$\mu : K^n_{\mathrm{top}}(X) \otimes K^m_{\mathrm{top}}(X) \to K^0_{\mathrm{top}}(X \times \mathbb{R}^{n+m}) \subseteq K_{\mathrm{top}}(X)$$

if $n + m \leq 1$, and

$$\mu : K^1_{\mathrm{top}}(X) \otimes K^1_{\mathrm{top}}(X) \to K^0_{\mathrm{top}}(X \times \mathbb{R}^2) \xrightarrow{\beta^{-1}} K_{\mathrm{top}}(X),$$

if $n = m = 1$. Here β^{-1} is the inverse of the Bott isomorphism. We will write $\mathcal{F}_1 \otimes \mathcal{F}_2$ to denote $\mu(\mathcal{F}_1, \mathcal{F}_2)$. This product makes $K_{\mathrm{top}}(X)$ into a commutative, associative superalgebra [Kar, II.5.1 and II.5.27].

One can define a metric on $K_{\mathrm{top}}(X)$ by

$$\eta_{K_{\mathrm{top}}}(\mathcal{F}_1, \mathcal{F}_2) := \chi(X, \mathcal{F}_1 \otimes \mathcal{F}_2),$$

and we define $\mathbf{1} := \mathcal{O}_X$. It is straightforward to check that $(K_{\mathrm{top}}(X), \otimes, \mathbf{1}, \eta_{K_{\mathrm{top}}})$ is a Frobenius superalgebra. Moreover, the projection formula holds for proper, differentiable maps with a compact target [Kar, IV.5.24].

The Frobenius superalgebra of topological K-theory satisfies the usual naturality properties with respect to pullback, is also a λ-ring [Kar, §7.2], and satisfies the splitting principle [Kar, Thm. IV.2.15].

For all i, the i-th Chern class $c_i(\mathcal{F})$ associated to any $\mathcal{F}$ in $K^0_{\mathrm{top}}(X)$ belongs to $H^{2i}(X)$, and so $H^{2p}(X)$ may be regarded as the analogue of the Chow group $A^p(X)$. The associated Chern polynomial c_t satisfies the usual multiplicativity and naturality properties, and the Chern character $\mathbf{ch} : K_{\mathrm{top}}(X) \to H^\bullet(X)$, defined by (2.8), is an isomorphism of commutative, associative superalgebras [Kar, Thm. V.3.25]. The Todd classes are defined from the ordinary Chern classes as before. In addition, Proposition (2.6) holds in topological K-theory since it follows from the splitting principle, the Chern character isomorphism, and the λ-ring properties [FH, Prop. I.5.3].

Finally, the Grothendieck–Riemann–Roch formula (see [Kar, Cor. V.4.18] or [AH, Thm. 4.1]) and the excess intersection formula (Theorem 5.1) hold (see [Qui, Prop. 3.3], which is written for cobordism, but the proof works as well for topological K-theory).

Remark 10.1. Let X be a compact G-manifold with a smoothly varying one-parameter family of G-equivariant, almost complex structures $J_t : TX \to TX$ for all t, say, in the interval [0, 1]. Because of the homotopy invariance of characteristic classes, the resulting G-Frobenius algebras $\mathcal{H}(X; G)$ and $\mathcal{K}(X; G)$, and the stringy Chern character are all independent of t. Therefore, these stringy algebraic structures depend only upon the homotopy class of the G-equivariant almost complex structure on the G-manifold X.

In particular, when X is a compact symplectic manifold with an action of G preserving the symplectic structure, since up to homotopy there exists a unique, G-equivariant, almost complex structure compatible with the symplectic form [GGK, Ex. D.12], these stringy algebraic structures are invariants of the symplectic manifold with G-action.

Remark 10.2. While we are primarily interested in G-equivariant almost complex manifolds in this section, our constructions generalize in a straightforward way to the case where X is a compact manifold with an oriented, G-equivariant stable complex structure (see [GGK, App. D]). The key point [GHK] is that for any subgroup $H \leq G$, a G-equivariant stable complex structure induces an almost complex structure on the normal bundle to the submanifold X^H (the locus of points fixed by H). Furthermore, both $\mathcal{S}_m$ (see Remark 1.4) and the right hand side of (8.3) only depend upon such normal bundles.

10.2. Stringy topological K-theory and stringy cohomology. Let X be a compact, almost complex manifold with an action of a finite group $\rho : G \to \mathrm{Aut}(X)$ preserving the almost complex structure.

Fantechi and Göttsche's [FG] stringy cohomology $\mathcal{H}(X, G)$ of X is given by

$$\mathcal{H}(X, G) := \bigoplus_{m \in G} \mathcal{H}_m(X),$$

where $\mathcal{H}_m(X) := H^\bullet(X^m)$, and the definition of the multiplication is still given by (1.5), the trace element τ by (4.7), and similarly for the metric and unity. However, the $\mathbb{Q}$-grading here is not quite that defined by (4.6), but is defined instead by the equation

$$|v_m|_{str} := 2a(v_m) + |v_m|, \tag{10.1}$$

where $|v_m| := p$ when v_m belongs to $H^p(X^m)$ and $a(v_m) := a(m, U)$.

Furthermore, Theorem (4.6) holds, provided that $\mathcal{A}(X, G)$ is everywhere replaced by $\mathcal{H}(X, G)$, $\dim X$ is understood to be the dimension of X as a real manifold, and "pre-G-Frobenius algebra" is replaced by "G-Frobenius superalgebra." However, we need to establish the following Proposition to complete the proof.

Proposition 10.3. *Let X be a compact, almost complex manifold with the action of a finite group G with stringy cohomology $\mathcal{H}(X, G)$. If the trace element τ is given by (4.7), then (3.2) holds, where $\mathcal{H} := \mathcal{H}(X, G)$. Consequently, the trace axiom (3.3) is satisfied, and $\mathcal{H}(X, G)$ is a G-Frobenius superalgebra. The characteristic $\boldsymbol{\tau}(\mathbf{1})$ of $\mathcal{H}(X, G)$ also satisfies (4.13) and is an integer.*

Proof. To avoid distracting signs, we assume that $\mathcal{H}(X, G)$ has only even dimensional cohomology classes.

We first prove (3.2). We henceforth adopt the notation from Sect. (5.2). Let $\{\nu_{\alpha[a]}\}$ be a homogeneous basis for $\mathcal{H}_a(X)$ with $\alpha[a] \in \{1, \dots, d_a\}$, where d_a is the dimension of $\mathcal{H}_a(X)$. Similarly, let $\{\mu_{\beta[a^{-1}]}\}$ with $\beta[a] \in \{1, \dots, d_a\}$ be a homogeneous basis for $\mathcal{H}_{a^{-1}}(X)$. Let $\eta_{\alpha[a],\beta[a^{-1}]}$ be the matrix of the metric pairing $\mathcal{H}_a(X) \times \mathcal{H}_{a^{-1}}(X) \to \mathbb{Q}$, with respect to these bases, and let $\eta^{\alpha[a]\beta[a^{-1}]}$ be the inverse of $\eta_{\alpha[a],\beta[a^{-1}]}$. We observe, from the Künneth theorem, that

$$\Delta'^{*}_{2}\Delta'_{1*}\mathbf{1}_{X^a} = \eta^{\alpha[a]\beta[a^{-1}]}(\rho(b)\nu_{\alpha[a]})\big|_{X^{H'}} \otimes \nu_{\beta[a^{-1}]}\big|_{X^{H'}}. \tag{10.2}$$

Thus, we have for $m_1 = [a, b]$,

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_a(X)}(L_{v_{m_1}} \circ \rho(b)) \\
&= \int_{X^{H'}} c_{\mathrm{top}}(\mathcal{R}(\mathbf{m}')) \cup \mathbf{e}^{*}_{m_1} v_{m_1} \cup (\rho(b)\nu_{\alpha[a]})\big|_{X^{H'}} \cup \nu_{\alpha[a^{-1}]}\big|_{X^{H'}} \eta^{\alpha[a]\alpha[a^{-1}]} \\
&= \int_{X^{H'}} c_{\mathrm{top}}(\mathcal{R}(\mathbf{m}')) \cup \mathbf{e}^{*}_{m_1} v_{m_1} \cup \Delta'^{*}_{2}\Delta'_{1*}\mathbf{1}_{X^a} \\
&= \int_{X^{H'}} c_{\mathrm{top}}(\mathcal{R}(\mathbf{m}')) \cup \mathbf{e}^{*}_{m_1} v_{m_1} \cup j'_{2*} c_{\mathrm{top}}(\mathcal{E}') \\
&= \int_{X^{H'}} j'_{2*}\big(j'^{*}_{2}\mathbf{e}^{*}_{m_1} v_{m_1}\big) \cup j'^{*}_{2} c_{\mathrm{top}}(\mathcal{R}(\mathbf{m}')) \cup c_{\mathrm{top}}(\mathcal{E}') \\
&= \int_{X^{H'}} j'_{2*}\big(j'^{*}_{2}\mathbf{e}^{*}_{m_1} v_{m_1}\big) \cup c_{\mathrm{top}}\big(j'^{*}_{2}\mathcal{R}(\mathbf{m}') \oplus \mathcal{E}'\big) \\
&= \int_{X^{H}} v_{m_1}\big|_{X^H} \cup c_{\mathrm{top}}\big(j'^{*}_{2}\mathcal{R}(\mathbf{m}') \oplus \mathcal{E}'\big) \\
&= \int_{X^{H}} v_{m_1}\big|_{X^H} \cup c_{\mathrm{top}}\big(TX^H \oplus \mathcal{S}_{m_1}\big|_{X^H}\big) \\
&= \tau_{a,b}(v_{m_1}),
\end{aligned}$$

where the first equality holds by definition of the trace, the second by (10.2), the third by Theorem (5.1), the fourth by the projection formula, the sixth by properties of the top Chern class, and the seventh by Theorem (5.5).

The trace axiom (3.3) for the G-Frobenius algebra $\mathcal{H}(X, G)$ follows from the trace axiom for a pre-G-Frobenius algebra together with (3.2).

The integrality of the characteristic follows from (3.4) by plugging in $\overline{v} = \mathbf{1}$ for the Frobenius superalgebra of G-coinvariants $\overline{\mathcal{H}} = \mathcal{H}(X, G)^G$. □

Stringy topological K-theory $\mathcal{K}^{\text{top}}(X, G) := \bigoplus_{m \in G} \mathcal{K}_m^{\text{top}}(X)$ is defined additively by $\mathcal{K}_m^{\text{top}}(X) := K_{\text{top}}(X^m)$ for all m in G. The stringy multiplication, metric, the trace element τ and unity are defined just as in the case of $\mathcal{K}(X, G)$. This is compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading because $\mathcal{R}$ is an element of $K^0_{\text{top}}(X^{\mathbf{m}})$.

Since the Eichler trace formula holds for all compact Riemann surfaces, our formula (8.3) for the obstruction bundle, and indeed the entire analysis in Sects. 5 and 8, holds in topological K-theory. The K-theoretic version of Proposition 10.3 holds as the arguments are purely functorial. Now an argument essentially identical to that for stringy cohomology shows that $((\mathcal{K}^{\text{top}}(X, G), \rho), *, \mathbf{1}, \eta)$ is a G-Frobenius superalgebra. We state this formally in the following proposition.

Proposition 10.4. *Let X be a compact, almost complex manifold with the action of a finite group G with stringy topological K-theory $\mathcal{K}^{\text{top}}(X, G)$. If the trace element $\tau^{\mathcal{K}}$ is given by (4.8), then (3.2) holds, where $\mathcal{H} := \mathcal{K}^{\text{top}}(X, G)$. Consequently, the trace axiom (3.3) is satisfied, and $\mathcal{K}^{\text{top}}(X, G)$ is a G-Frobenius superalgebra.*

Furthermore, the stringy Chern character $\mathfrak{Ch} : \mathcal{K}^{\text{top}}(X, G) \to \mathcal{H}(X, G)$ is still defined by (1.7). The rest of the analysis in Sect. 6 holds, provided that $\mathcal{A}(X, G)$ is everywhere replaced by $\mathcal{H}(X, G)$ and K-theory is everywhere replaced by topological K-theory. Therefore, $\mathfrak{Ch} : \mathcal{K}^{\text{top}}(X, G) \to \mathcal{H}(X, G)$ is an allometric isomorphism.

Finally, the analysis in Sect. 9 holds after replacing Chow groups by cohomology everywhere. In particular, since $\overline{\mathcal{H}}(X, G)$ is isomorphic [FG] to the stringy (or Chen–Ruan orbifold) cohomology $H^\bullet_{\text{orb}}([X/G])$, the stringy Chern character $\overline{\mathfrak{Ch}} : \overline{\mathcal{K}}_{\text{top}}(X, G) \to \overline{\mathcal{H}}(X, G)$ gives a ring isomorphism $Ch_{\text{orb}} : K^{\text{top}}_{\text{orb}}([X/G]) \to H^\bullet_{\text{orb}}([X/G])$, where $K^{\text{top}}_{\text{orb}}([X/G])$ is the topological small orbifold K-theory of $[X/G]$.

10.3. The symmetric product and crepant resolutions. One of the most interesting examples of stringy K-theory and cohomology is the symmetric product. Let $X := Y^n$, where Y is an almost complex manifold of complex dimension d with the symmetric group S_n acting on Y^n by permuting its factors. In this case, for any $m \in S_n$ it is easy to see that the age $a(m)$ is related to the length of the permutation $l(m)$:

$$a(m) = l(m)d/2.$$

Consequently, by (10.1), the $\mathbb{Q}$-grading on $\mathcal{H}(X, G)$ is, in fact, a grading by (possibly odd) integers.

Consider stringy topological K-theory $\mathcal{K}^{\text{top}}(Y^n, S_n)$ of the S_n-variety Y^n. Choose the 2-cocycle (discrete torsion) α in $Z^2(S_n, \mathbb{Q}^*)$

$$\alpha(m_1, m_2) := (-1)^{\varepsilon(m_1, m_2)},$$

where ε is defined by

$$\varepsilon(m_1, m_2) := \frac{1}{2}(l(m_1) + l(m_2) - l(m_1 m_2)).$$

It is straightforward to verify that $\varepsilon(m_1, m_2)$ is an integer. Now, twist the S_n-Frobenius algebra $\mathcal{K}^{\text{top}}(Y^n, S_n)$ by α, as in Sect. 7, to yield a new S_n-Frobenius algebra $((\mathcal{K}^{\text{top}}(Y^n, S_n), \rho), \star, \mathbf{1}, \eta^\alpha)$, which we will denote by $\mathbf{K}^{\text{top}}(Y^n, S_n)$. Notice that the G-action is unchanged by the twist, but the twisted multiplication $\mathcal{K}^{\text{top}}(Y^n, S_n)$ is given by the formula

$$v_{m_1} \star v_{m_2} := v_{m_1} *^\alpha v_{m_2} = \alpha(m_1, m_2) v_{m_1} * v_{m_2}, \tag{10.3}$$

where $*$ denotes the stringy multiplication in $\mathcal{K}^{\text{top}}(Y^n, S_n)$.

Twisting the multiplication on the stringy cohomology of Y^n in the same fashion, we obtain the S_n-Frobenius algebra $((\mathcal{H}(Y^n, S_n), \rho), \star, \mathbf{1}, \eta^\alpha)$, which we will denote by $\mathbf{H}(Y^n, S_n)$. By the obvious topological analogue of Corollary 7.8, the stringy Chern character $\mathfrak{Ch} : \mathbf{K}^{\text{top}}(Y^n, S_n) \to \mathbf{H}(Y^n, S_n)$ is an isomorphism of S_n-commutative algebras. After taking S_n-coinvariants, we obtain a ring isomorphism

$$Ch_{\text{orb}} : \mathbf{K}^{\text{top}}_{\text{orb}}\big([Y^n/S_n]\big) \to \mathbf{H}^\bullet_{\text{orb}}\big([Y^n/S_n]\big),$$

where the ring $\mathbf{K}^{\text{top}}_{\text{orb}}([Y^n/S_n])$ is the topological small orbifold K-theory $K^{\text{top}}_{\text{orb}}([Y^n/G])$, but with the α-twisted multiplication, and similarly for $\mathbf{H}^\bullet_{\text{orb}}([Y^n/S_n])$.

What makes these particular twisted rings interesting is the following theorem.

Theorem 10.5. *Let Y be a complex, projective surface such that $c_1(Y) = 0$. Consider Y^n with S_n acting by permutation of its factors. If $Y^{[n]}$ denotes the Hilbert scheme of n points in Y, then $\mathbf{K}^{\text{top}}_{\text{orb}}([Y^n/S_n])$ is isomorphic as a Frobenius superalgebra to $K_{\text{top}}(Y^{[n]})$.*

Proof. We define ψ so that the following diagram commutes

$$\begin{array}{ccc} \mathbf{K}^{\text{top}}_{\text{orb}}\big([Y^n/S_n]\big) & \xrightarrow{Ch_{\text{orb}}} & \mathbf{H}^\bullet_{\text{orb}}\big([Y^n/S_n]\big) \\ \psi \big\downarrow & & \psi' \big\downarrow \\ K_{\text{top}}(Y^{[n]}) & \xrightarrow{\mathbf{ch}} & H^\bullet(Y^{[n]}), \end{array} \tag{10.4}$$

where ψ' is the ring isomorphism Ψ^{-1} in [FG, Thm. 3.10]. This uniquely defines ψ, since $\mathbf{ch}$ and Ch_{orb} are ring isomorphisms.

The homomorphism ψ also preserves the metrics because of the Hirzebruch–Riemann–Roch theorem and the fact that ψ' preserves the metrics. $\square$

Remark 10.6. The rings $\mathbf{K}_{\mathrm{orb}}^{\mathrm{top}}([Y^n/S_n]) \otimes_{\mathbb{Q}} \mathbb{C}$ and $K_{\mathrm{orb}}^{\mathrm{top}}([Y^n/S_n]) \otimes_{\mathbb{Q}} \mathbb{C}$ are isomorphic (see [Rua]). Since $Y^{[n]} \to Y^n/S_n$ in the previous theorem is a crepant (and hyper-Kähler) resolution, this is an example of our K-theoretic version of Conjecture 1.2. Our result is nontrivial precisely because of the nontrivial definition of multiplication on $K_{\mathrm{orb}}^{\mathrm{top}}([Y^n/S_n])$ and the stringy Chern character.

References

[AGV] Abramovich, D., Graber, T., Vistoli, A.: Algebraic orbifold quantum products. In: Adem, A., Morava, J., Ruan, Y. (eds.) Orbifolds in Mathematics and Physics. Contemp. Math., vol. 310, pp. 1–25. Am. Math. Soc., Providence, RI (2002). `math.AG/0112004`

[AR] Adem, A., Ruan, Y.: Twisted orbifold K-theory. Commun. Math. Phys. **273**(3), 533–56 (2003). `math.AT/0107168`

[ARZ] Adem, A., Ruan, Y., Zhan, B.: A stringy product on twisted orbifold K-theory. Preprint. `math.AT/0605534`

[AH] Atiyah, M.F., Hirzebruch, F.: The Riemann–Roch theorem for analytic embeddings. Topology **1**, 151–166 (1962)

[AS] Atiyah, M.F., Segal, G.: On equivariant Euler characteristics. J. Geom. Phys. **6**, 671–677 (1989)

[CR1] Chen, W., Ruan, Y.: A new cohomology theory for orbifold. Commun. Math. Phys. **248**(1), 1–31 (2004). `math.AG/0004129`

[CR2] Chen, W., Ruan, Y.: Orbifold Gromov–Witten theory. In: Adem, A., Morava, J., Ruan, Y., (eds.) Orbifolds in Mathematics and Physics. Contemp. Math., vol. 310, pp. 25–85. Am. Math. Soc., Providence, RI (2002). `math.AG/0103156`

[CH] Chen, B., Hu, S.: A deRham model for Chen–Ruan cohomology ring of Abelian orbifolds. Math. Ann. **336**(1), 51–71 (2006). `math.SG/0408265`

[DHVW] Dixon, L., Harvey, J.A., Vafa, C., Witten, E.: Strings on Orbifolds. Nucl. Phys. **B261**, 678 (1985)

[DE] Dolgushev, V., Etingof, P.: Hochschild cohomology of quantized symplectic orbifolds and the Chen–Ruan cohomology. Int. Math. Res. Not. **2005**(27), 1657–1688 (2005). `math.QA/0410562`

[EG] Edidin, D., Graham, W.: NonAbelian localization in equivariant K-theory and Riemann–Roch for quotients. Adv. Math. **198**(2), 547–582 (2005). `math.AG/0411213`

[FG] Fantechi, B., Göttsche, L.: Orbifold cohomology for global quotients. Duke Math. J. **117**(2), 197–227 (2003). `math.AG/0104207`

[FK] Farkas, H., Kra, I.: Riemann Surfaces, 2nd edn. Springer, New York (1991)

[FS] Frenkel, E., Szczesny, M.: Chiral de Rham complex and orbifolds. Preprint. `math.AG/0307181`

[Ful] Fulton, W.: Intersection Theory. Springer, New York (1998)

[FH] Fulton, W., Harris, J.: Representation Theory: a First Course. Springer, New York (1991)

[FL] Fulton, W., Lang, S.: Riemann–Roch Algebra. Springer, New York (1985)

[GHK] Goldin, R., Holm, T.S., Knutson, A.: Orbifold cohomology of torus quotients. Preprint. `math.SG/0502429`

[GGK] Ginzburg, V., Guillemin, V., Karshon, Y.: Moment maps, cobordisms, and Hamiltonian group actions. Am. Math. Soc., Providence, RI (2002)

[Giv] Givental, A.: On the WDVV-equation in quantum K-theory. Mich. Math. J. **48**, 295–304 (2000). math.AG/0003158

[JKK] Jarvis, T., Kaufmann, R., Kimura, T.: Pointed admissible G-covers and G-equivariant cohomological Field Theories. Compos. Math. **141**(4), 926–978 (2005). math.AG/0302316

[Jos02] Joshua, R.: Higher Intersection theory for algebraic stacks: I. K-Theory, **27**(2), 134–195 (2002)

[Jos05] Joshua, R.: K-Theory and absolute cohomology for algebraic stacks. K-theory archive #0732. Preprint (2005)

[Kal] Kaledin, D.: Multiplicative McKay correspondence in the symplectic case. Preprint. math.AG/0311409

[Kan] Kani, E.: The Galois-module structure of the space of holomorphic differentials of a curve. J. Reine Angew. Math. **367**, 187–206 (1986)

[Kar] Karoubi, M.: K-Theory, An Introduction. Springer, Berlin–New York (1978)

[Kau02] Kaufmann, R.: Orbifold Frobenius algebras, cobordisms, and monodromies. In: Adem, A., Morava, J., Ruan, Y. (eds.) Orbifolds in Mathematics and Physics. Contemp. Math., vol. 310, pp. 135–162 (2002)

[Kau03] Kaufmann, R.: Orbifolding Frobenius algebras. Int. J. Math. **14**(6), 573–617 (2003). math.AG/0107163

[Kau04] Kaufmann, R.: The algebra of discrete torsion. J. Algebra **282**(1), 232–259 (2004). math.AG/0208081

[Kau05] Kaufmann, R.: Discrete torsion, symmetric products and the Hilbert scheme. In: Hertling, C., Marcolli, M. (eds.) Frobenius Manifolds, Quantum Cohomology and Singularities. Aspects Math., vol. E36. Vieweg, Wiesbaden (2004)

[Kle] Kleiman, S.: Algebraic Cycles and the Weil Conjectures. In: Grothendieck, A., Kuiper, N. (eds.) Dix exposés sur la cohomologie des schémas. North-Holland, Amsterdam (1968)

[Lee] Lee, Y.P.: Quantum K-theory I: Foundations. Duke Math. J. **121**(3), 389–424 (2004). math.AG/0105014

[Qui] Quillen, D.: Elementary proofs of some results of cobordism theory using Steenrod operations. Adv. Math. **7**, 29–56 (1971)

[Rua] Ruan, Y.: Stringy orbifolds. In: Adem, A., Morava, J., Ruan, Y. (eds.) Orbifolds in Mathematics and Physics. Contemp. Math., vol. 310, pp. 259–299 (2002)

[SGA6] Berthelot, P., Grothendieck, A., Illusie, L.: Théorie des intersections et théorème de Riemann–Roch. Lect. Notes Math., vol. 225. Springer, Berlin (1971)

[Sha] Shanahan, P.: The Atiyah–Singer Index Theorem. Springer, New York (1978)

[Toe] Toen, B.: Théorèmes de Riemann–Roch pour les champs de Deligne–Mumford. K-Theory **18**(1), 33–76 (1999)

[Tot] Totaro, B.: The resolution property for schemes and stacks. J. Reine Angew. Math. **577**, 1–22 (2004) math.AG/0207210

[Tur] Turaev, V.: Homotopy field theory in dimension 2 and group-algebras. Preprint. math.QA/9910010

[Uri] Uribe, B.: Orbifold cohomology of the symmetric product. Commun. Anal. Geom. **13**(1), 113–128 (2005). math.AT/0109125

[Vis] Vistoli, A.: Higher equivariant K-theory for finite group actions. Duke Math. J. **63**(2), 399–419 (1991)

[VV] Vezzosi, G., Vistoli, A.: Higher algebraic K-theory for actions of diagonalizable groups. Invent. Math. **153**(1), 1–44 (2003). math.AG/0107174