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THE DRINFEL'D DOUBLE AND TWISTING IN STRINGY ORBIFOLD THEORY

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This paper exposes the fundamental role that the Drinfel'd double $D(k[G])$ of the group ring of a finite group G and its twists $D^{\beta}(k[G])$, $\beta \in Z^3(G, k^*)$ as defined by Dijkgraaf–Pasquier–Roche play in stringy orbifold theories and their twistings. The results pertain to three different aspects of the theory. First, we show that G-Frobenius algebras arising in global orbifold cohomology or K-theory are most naturally defined as elements in the braided category of $D(k[G])$-modules. Secondly, we obtain a geometric realization of the Drinfel'd double as the global orbifold K-theory of global quotient given by the inertia variety of a point with a G action on the one hand and more stunningly a geometric realization of its representation ring in the braided category sense as the full K-theory of the stack $[\mathrm{pt}/G]$. Finally, we show how one can use the co-cycles β above to twist the global orbifold K-theory of the inertia of a global quotient and more importantly, the stacky K-theory of a global quotient $[X/G]$. This corresponds to twistings with a special type of two-gerbe.

Keywords: Quantum cohomology; Hopf algebras; orbifolds; stringy K-theory; twisted K-theory; gerbes; fusion ring.

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1. Introduction

The Drinfel'd double $D(k[G])$ of a group ring of a finite group G and in particular its twisted version $D^{\beta}(k[G])$ where $\beta \in Z^3(G, k^*)$ were introduced and studied by Dijkgraaf, Pasquier and Roche [17] (see [2] for a very nice brief summary). Their aim was to understand the constructions of [13] concerning orbifold conformal field theory on the one hand and the constructions of [14] pertaining to orbifold Chern–Simons theory on the other. We will realize these algebraic constructions geometrically using the orbifold K-theory of [25] and newly defined twists.

Mathematically, the appearance of the Drinfel'd double $D(k[G])$ as a main character in orbifold theory has its roots in [22, 18] where a $(2+1)$-dimensional theory was considered. See also [20, 21] for related material on equivariant K-theory of a compact group G. The importance and algebraic relevance of $D(k[G])$ in the theory of G-Frobenius algebras was made precise in [30] where we showed that any G-Frobenius algebra is a $D(k[G])$-module and in particular also a $k[G]$-module algebra and $k[G]$ co-module algebra. G-Frobenius algebras arise in the $(1+1)$-dimensional theory [29] such as orbifold Gromov–Witten theory [11] and hence in orbifold cohomology [11, 19] in particular. In this paper, we go one step further and give a definition of a G-Frobenius algebra and more generally a G-Frobenius object in terms of the braided tensor category $D(k[G])$-Mod of $D(k[G])$-modules. The rather lengthy original definition of a G-Frobenius algebra [28, 29] then can be replaced by the statement that a G-Frobenius object is a Frobenius object in $D(k[G])$-Mod which satisfies two additional axioms (S) and (T) of which the former is the famous trace axiom. This is the content of Theorem 3.16.

Another upshot of the categorical treatment is that these objects give the right algebraic structure to encode the trace axiom in infinite dimensional situations. We recall that in [25], we introduced pre-Frobenius algebras with trace elements to be able to write the trace axiom. This was necessary since the Chow ring of a smooth projective variety may not be a Frobenius algebra as it can be infinite dimensional. Here, by a Frobenius algebra we mean a unital associative commutative algebra with a non-degenerate even symmetric invariant bi-linear pairing. Nevertheless, there are traces one can define using the trace elements and for these, the trace axiom holds. In the categorical context, any Frobenius object defines a trace for any endomorphism which we call Frobenius trace or F-trace for short. In particular, the trace elements of [25] can be recovered as the F-traces of the relevant endomorphisms. This fact holds true in all the known constructions involving the string and global versions of the functors $\mathcal{F} \in \{H^*, K^*, A^*, K_0\}$ [19, 3, 4, 11, 25] which is shown in Theorem 4.3. Thus, $D(k[G])$ is at the bottom of the very definition of the algebras associated to global orbifolds. Analogous statements are true for singularities with symmetries [28, 29, 31].

The Drinfel'd double makes its appearance in two more guises. First, we show that in the case of an Abelian symmetry group G, the global K-theory as defined in [25]. See also Sec. 3 for a review, of the inertia variety of a point with the trivial G action satisfying $K^*_{\mathrm{global}}(I(\mathrm{pt}, G), G) = D(k[G])$ as an algebra. In the non-Abelian case, the resulting algebra together with its G-action is Morita equivalent to $D(k[G])$ as a groupoid. See Corollary 4.12.

The most stunning appearance of $D(k[G])$ is the one of Theorem 4.13 where we prove that $K_{\mathrm{full}}([\mathrm{pt}/G]) \cong \mathrm{Rep}(D(k[G]))$. Here, the non-commutativity in the ring structure is now given by the natural braiding of the moniodal category of representations.

Armed with these results, we define twistings by co-cycles in $Z^i(G, k^*)$ where $i = 1, 2, 3$ for the various theories-associated to a global quotient (X, G). In other words, twist by 0-,1-, and 2-gerbes that are pulled back from $[\mathrm{pt}/G]$ or gerbes on X that are trivial but not equivariantly trivial. See [34, 23] for this point of view of gerbes.

The 0-twists are performed on $K_{\mathrm{global}}(X, G)$ or any of the other stringy functors $\mathcal{F}$. They correspond to the Ramond twist defined in [28, 29]. The twists by 1-gerbes are identified as the twist of discrete torsion that were algebraically defined in [30]. Finally, the most interesting twists come from 2-gerbes. There are basically two types. First, we can transgress the 2-gerbe to the inertia variety $I(X, G)$ considered together with its G-action and then consider twists on $K_{\text{global}}((I(X, G), G))$. Here, the twist will just be a special type of discrete torsion. However, we do recover the algebra structure of $D^\beta(k[G])$ for the β-twisted $K^\beta((I(\text{pt}, G), G)$. The more intriguing twist is on $K_{\text{full}}[X/G]$. Twists of orbifold K-theory of the inerita have been independently studied in [6]. In our case, we remain on $K_{\text{full}}(\mathfrak{X})$ and our twist yields the natural generalization of the results above. Namely $K^\beta([\text{pt}/G]) \cong \text{Rep}(D^\beta(k[G]))$. See Theorem 5.8. This result is striking in several aspects. The most prominent feature being that the representation ring of $D^\beta(k[G])$ is understood in the braided monoidal setting with a non-trivial associator. This tells us that this twist twists outside the associative world. *A posteriori* this is however not totally unexpected, since we know from the work of Moore and Seiberg [33] that the fusion ring is not associative in general, but only associative in the braided monoidal category sense. We can of course get an associative algebra by restricting to the dimensions of the intertwiners and defining a Verlinde algebra. See Sec. 4 and also [20, 21] for related material.

The paper is organized as follows:

In Sec. 2, we review all the necessary definitions for the twisted Drinfel'd double including DPR induction and the relevant background from braided monoidal categories. Section 3 contains the first set of results that pertain to the definition of G-Frobenius algebra objects. The third section starts with a brief review of the constructions of [25] and introduces all the variants of stringy K-theory we will consider. Section 4 terminates with the second and third appearance of the Drinfel'd double: (i) as the global K-theory of the inertia of (pt, G) and (ii) in the theorem that $K_{\text{full}}([\text{pt}/G]) \cong \text{Rep}(D(k[G]))$. The various twistings are contained in Sec. 5. Here, we consider twists of 0-, 1-, and 2-gerbes on global quotients that are trivial but not equivariantly trivial.

2. The Twisted Drinfel'd Double

In this section, we collect the basic definitions and constructions of the twisted Drinfel'd double for the readers' convenience.

2.1. *Basic definitions*

Definition 2.1. For a finite group G and an element $\beta \in Z^3(G, k^*)$, the *twisted Drinfel'd double* $D^\beta(k[G])$ is the quasi-triangular quasi-Hopf algebra whose

(1) underlying vector space has the basis $g\underset{x}{\llcorner}$ with $x, g \in G$

$$D^\beta(k[G]) = \bigoplus k\; g\underset{x}{\llcorner}, \tag{2.1}$$

(2) algebra structure is given by

$$\underset{x}{g\llcorner}\,\underset{y}{h\llcorner} = \delta_{g,xhx^{-1}}\theta_g(x,y)\,\underset{xy}{g\llcorner}, \tag{2.2}$$

where

$$\theta_g(x,y) = \frac{\beta(g,x,y)\beta(x,y,(xy)^{-1}g(xy))}{\beta(x,x^{-1}gx,y)}, \tag{2.3}$$

(3) co-algebra structure is given by

$$\Delta(\underset{x}{g\llcorner}) = \sum_{g_1g_2=g} \gamma_x(g_1,g_2)\,\underset{x}{g_1\llcorner} \otimes \underset{x}{g_2\llcorner}, \tag{2.4}$$

where

$$\gamma_x(g_1,g_2) = \frac{\beta(g_1,g_2,x)\beta(x,x^{-1}g_1x,x^{-1}g_2x)}{\beta(g_1,x,x^{-1}g_2x)}. \tag{2.5}$$

(4) The Drinfel'd associator Φ is given by

$$\Phi = \sum_{g,h,k\in G} \beta(g,h,k)^{-1}\,\underset{e}{g\llcorner} \otimes \underset{e}{h\llcorner} \otimes \underset{e}{k\llcorner}. \tag{2.6}$$

(5) The R matrix is given by

$$R = \sum_{g\in G} \underset{e}{g\llcorner} \otimes \underset{g}{\mathbf{1}\llcorner}, \quad \text{where } \underset{g}{\mathbf{1}\llcorner} = \sum_{h\in G} \underset{g}{h\llcorner}. \tag{2.7}$$

(5) The antipode S is given by

$$S(\underset{x}{g\llcorner}) = \frac{1}{\theta_{g^{-1}}(x,x^{-1})\gamma_x(g,g^{-1})}\ \underset{x^{-1}}{x^{-1}g^{-1}x\llcorner}. \tag{2.8}$$

Remark 2.2. There are several things which we would like to point out:

(1) In case $\beta \equiv 1$, that is, β is trivial, we obtain the a braided Hopf algebra $D(k[G])$ which is the Drinfel'd double of the group ring.
(2) The algebra is associative and the unit of this algebra is $\underset{e}{\mathbf{1}\llcorner}$.
(3) There is an injection of algebras $k[G]^* \to D^\beta(k[G])$ given by $\delta_g \mapsto \underset{e}{g\llcorner}$, where $\delta_g(h) := \delta_{g,h}$, since

$$\underset{e}{g\llcorner}\,\underset{e}{h\llcorner} = \delta_{g,h}\,\underset{e}{g\llcorner}. \tag{2.9}$$

(4) There is a special element v^{-1} which is central. It is given by

$$v^{-1} = \sum_{g\in G} \underset{g}{g\llcorner}. \tag{2.10}$$

In case $\beta \equiv 1$, this is the element which gives the inner operation of S^2 of the braided Hopf-algebra $D(k[G])$ [27].
(5) The various θ_g are almost co-cycles for G,

$$\theta_g(x,y)\theta_g(xy,z) = \theta_g(xy,z)\theta_{x^{-1}gx}(y,z). \tag{2.11}$$

It follows that when θ_g is restricted to $Z(g)\times Z(g)$, it is a two-co-cycle for $Z(g)$.

2.2. *The braided monoidal category $D^\beta(k[G])$-Mod*

Since $D^\beta(k[G])$ is a quasi-triangular quasi-Hopf algebra, there is a natural braided monoidal structure on the category of its modules. We recall that if U and V are modules over $D^\beta(k[G])$ or in general any quasi-triangular quasi-Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S, \Phi, R)$, then $U \otimes V$ has the structure of an H module via $\Delta : H \to H \otimes H$.

Recall (see e.g. [27]) that in general for three representation U, V, W and elements $u \in U, v \in V, w \in W$, the associator is given by

$$\begin{aligned} a_{U,V,W} &: (U \otimes V) \otimes W \to U \otimes (V \otimes W), \\ a_{U,V,W}((u \otimes v) \otimes w) &= \Phi(u \otimes (v \otimes w)), \end{aligned} \tag{2.12}$$

and likewise for two representations U, V and elements $u \in U, v \in V$, the braiding is given by

$$\begin{aligned} c_{U,V} &: U \otimes V \to V \otimes U, \\ c_{U,V}(u \otimes v) &= \tau_{U,V}(R(u \otimes v)), \end{aligned} \tag{2.13}$$

where $\tau_{U,V}(u \otimes v) = v \otimes u$.

In particular, let U, V, W be $D^\beta(k[G])$-modules and let $u_g \in U_g$, $v_h \in V_h$, $w_k \in W_k$ be homogeneous elements with respect to the grading by G, then

$$a_{U,V,W}((u_g \otimes v_h) \otimes w_k) = \beta^{-1}(g, h, k) u_g \otimes (v_h \otimes w_k), \tag{2.14}$$

and

$$c_{U,V}(u_g \otimes v_h) = \rho(\underset{g}{ghg^{-1}\llcorner})(v_h) \otimes u_g = \phi(g)(v_h) \otimes u_g. \tag{2.15}$$

Moreover on $U \otimes V$, the $D^\beta(k[G])$-module structure is given by

$$\rho(\underset{x}{g\llcorner})(u_h \otimes v_k) = \delta_{xhkx^{-1},g} \gamma_x(xhx^{-1}, xkx^{-1}) \rho(\underset{x}{xhx^{-1}\llcorner})(u_h) \otimes \rho(\underset{v_k}{xkx^{-1}\llcorner}). \tag{2.16}$$

Remark 2.3. It is well-known [33] that the pentagon relation for associativity constraint is equivalent to the fact that β as a function on G^3 is an element of $Z^3(G, k^*)$.

Proposition 2.4. *Any left* $D^\beta(k[G])$-module (ρ, A) *is* G *graded* $A = \bigoplus_{g \in G} A_g$ *and if* π_g *denotes the projection of* A *onto* A_g, *then*

(1) $\rho(\underset{e}{g\llcorner}) = \pi_g$.

(2) $\rho(\underset{x}{g\llcorner}) = \pi_g \circ \rho(\underset{x}{g\llcorner}) \circ \pi_{x^{-1}gx}$ *and* $\rho(\underset{x}{g\llcorner}) : A_{x^{-1}gx} \xrightarrow{\sim} A_g$ *by isomorphisms.*
In particular, $\rho(\underset{x}{g\llcorner})(a_h) = \delta_{x^{-1}gx,h} \rho(\underset{x}{g\llcorner})(a_h)$.

Proof. Equation (2.9) means that the $\rho(\underset{e}{g\llcorner})$ act as projectors and since $\rho(\underset{e}{\mathbf{1}\llcorner}) = \mathrm{id}_A$, the first claim follows from Eq. (2.9) by setting

$$A_g := \rho(\underset{e}{g\llcorner})(A). \tag{2.17}$$

For the first part of the second claim, we notice that

$$\underset{x}{g\llcorner} = \underset{e}{g\llcorner}\,\underset{x}{g\llcorner}\,\underset{e}{(x^{-1}gx)\llcorner}, \tag{2.18}$$

which implies the statement in conjunction with (1). For the second part, we calculate that $\underset{x^{-1}}{x^{-1}gx\llcorner}\,\underset{x}{g\llcorner} = \theta_{x^{-1}gx}(x^{-1},x)\,\underset{e}{x^{-1}gx\llcorner}$ and since $\theta_{xgx^{-1}}(x^{-1},x) \neq 0$ and $\rho(\underset{e}{x^{-1}gx\llcorner})|_{A_{x^{-1}gx}} = \pi_{x^{-1}gx}|_{A_{x^{-1}gx}} = \mathrm{id}$, the claim follows. □

Notation 2.5. It will be convenient to denote $\rho(\underset{g}{\mathbf{1}\llcorner})$ by $\phi(g)$. For any $D(k[G])$ module A, we let $A_g := \mathrm{Im}(\rho(\underset{e}{g\llcorner}))$ and denote the projection by π_g. Notice that then

$$\rho(\underset{x}{g\llcorner}) = \phi(x) \circ \pi_{x^{-1}gx} = \phi(x)|_{A_{x^{-1}gx}} : A_{x^{-1}gx} \to A_g. \tag{2.19}$$

Remark 2.6. If $\beta \equiv 1$ then ϕ yields a $k[G]$ module structure on A while the grading corresponds to the $k[G]$ co-module structure given by $a_g \mapsto a_g \otimes g$. Moreover, one can check that these two structures are compatible so as to form a crossed $D(k[G])$ module in the sense of [27], as is well-known.

2.3. *DPR induction*

A very useful tool in the theory of the twisted Drinfel'd double is the Dijkgraaf–Pasquier–Roche (DPR) induction [17].

For any $\alpha \in Z^2(G,k^*)$, let $R^\alpha(G)$ be the group of α twisted representations, that is, maps $\rho : G \to GL(V)$ with $\rho(g)\rho(h) = \alpha(g,h)\rho(g,h)$. We write $C(G)$ for the set of conjugacy classes of G. With this notation, DPR induction allows one to constructively prove the following result.

Theorem 2.7 [17]. $\mathrm{Rep}(D^\beta(k[G]))$ *is equivalent to* $\bigoplus_{[g]\in C(G)} R^{\theta_g}(Z(g))$.

Remark 2.8. One can also view the theorem above as following from the Morita equivalence of the loop version of the inertia groupoid and the fiber product/stack version of the inertia groupoid.

A very nice compilation of the results is given in [2]. We also review the DPR induction process below.

Remark 2.9. We wish to point our several facts:

(1) Notice that the individual $R^\alpha(G)$ do not form rings. The product induced by the tensor product on the underlying modules is rather from $R^\alpha(G) \otimes R^\beta(G) \to R^{\alpha\beta}(G)$. The direct sum over the θ_g is in a certain sense "closed" under this operation, whence the product structure. We refer to [17] for the details, but also see Sec. 2.3 below.
(2) The product in $\mathrm{Rep}(D^\beta(k[G]))$ is not associative for general β, but only braided associative, with the braiding given by the Drinfel'd associator Φ. See Sec. 2.2.

(3) We write $k^\alpha[G]$ for the twisted group ring that is $\bigoplus_{g\in G} k1_g$ with multiplication $1_g 1_h = \alpha(g,h)1_{gh}$. It is worth remarking that G acts by conjugation $\rho(g)1_h = \epsilon(g,h)1_{ghg^{-1}}$ with $\epsilon(g,h) = \frac{\alpha(g,h)}{\alpha(ghg^{-1},g)}$. With this action (see e.g. [26]):

$$(k^\alpha[G])^G \otimes \mathbb{C} = R^\alpha(G) \otimes \mathbb{C}. \tag{2.20}$$

Also, a module over $k^\alpha[G]$ is the same as an α twisted representation.

Here and everywhere, the superscipt G denotes the G-invariants.

Definition 2.10 [17]. Fix β and $g \in G$. Given (V, λ) a left θ_g twisted representation of $Z(g)$ the *DPR induced* representation is $\mathrm{Ind}^{\mathrm{DPR}}(\mathrm{V}) := \mathrm{k[G]}\otimes_{\mathrm{k}^{\theta_\mathrm{g}}[\mathrm{Z(g)}]}\mathrm{V}$ where for the tensor product k^{θ_g} acts on the right on $k[G]$ via $x\rho(h) = \theta_{xgx^{-1}}(x,h)xh$ with the action of $D^\beta(k[G])$ given by

$$\underset{x}{h\llcorner}(r\otimes v) := \delta_{h,xrg(xr)^{-1}}\theta_h(x,r)xr\otimes v. \tag{2.21}$$

Remark 2.11. Notice that if one chooses representatives x_i for $G/Z(g)$, then the action amounts to

$$\begin{aligned} \underset{x}{h\llcorner}(x_i\otimes v) &= \delta_{h,xrg(xr)^{-1}}\theta_h(x,x_i)xx_i\otimes v \\ &= \delta_{h,xrg(xr)^{-1}}\theta_h(x,x_i)x_k z\otimes v \\ &= \delta_{h,xrg(xr)^{-1}}\frac{\theta_h(x,x_i)}{\theta_{x_k g x_k^{-1}}(x_k,z)}x_k\rho(z)\otimes v \\ &= \delta_{h,x_k g x_k^{-1}}\frac{\theta_h(x,x_i)}{\theta_h(x_k,z)}x_k\otimes\lambda(z)(v), \end{aligned} \tag{2.22}$$

which is the formula one can find for instance in [2].

2.4. *An exterior tensor product*

Recall [28, 29] that for G-graded spaces $A = \bigoplus_{g\in G}$ and $B = \bigoplus_{g\in G} B_g$, there is another natural tensor product, which is given by

$$A\widehat{\otimes}B := \bigoplus_{g\in G} A_g\otimes B_g. \tag{2.23}$$

Proposition 2.12. *If A is a $D^\beta(k[G])$-module and B is a $D^{\beta'}(k[G])$-module, then $A\widehat{\otimes}B$ is a $D^{\beta\beta'}(k[G])$-module via the diagonal action $\widehat{\Delta}(\underset{x}{g\llcorner}) = \underset{x}{g\llcorner}\otimes\underset{x}{g\llcorner}$.*

Proof. First notice that indeed A and B are G-graded by Proposition 2.4.

We need to check that

$$\widehat{\Delta}(\underset{x}{g\llcorner}\underset{y}{h\llcorner})(a_k\otimes b_k) = \widehat{\Delta}(\underset{x}{g\llcorner})(\widehat{\Delta}(\underset{y}{h\llcorner})(a_k\otimes b_k)). \tag{2.24}$$

For this to be non-zero, we need $h = yky^{-1}$ and $g = xyk(xy)^{-1}$, so fix these values, then $\underset{x}{g\llcorner}\underset{y}{h\llcorner} = \theta^\omega_g(x,y)\underset{xy}{g\llcorner}$ in any $D^\omega(k[G])$. Also, notice that by a simple substitution into the definitions $\theta_g^{\beta\beta'}(x,y) = \theta_g^{\beta}(x,y)\theta_g^{\beta'}(x,y)$, with which the claim follows. Finally, $\widehat{\Delta}(g_{\underset{e}{\llcorner}}) = g_{\underset{e}{\llcorner}} \otimes g_{\underset{e}{\llcorner}}$ which means that indeed the degree g part of $A\widehat{\otimes}B$ is given by $A_g \otimes B_g$. □

2.5. *A second exterior tensor product*

Notice that $D(k[G])$ as a vector space is actually bi-graded by $G \times G$ and for bi-graded modules, there is again a tensor product.

Now, given any bi-graded $A = \bigoplus_{(g,x)\in G\times G} A_{g,x}$ and $B = \bigoplus_{(g,x)\in G\times G} B_{g,x}$, we define

$$A\widehat{\widehat{\otimes}}B := \bigoplus_{(g,x)\in G\times G} A_{g,x} \otimes B_{g,x}. \tag{2.25}$$

Of course, this is just $\widehat{\otimes}$ for the group $H = G \times G$, but since we consider the group G to be fixed, this notation will be very useful.

Lemma 2.13. *When using the diagonal product,* $D^{\beta}(k[G])\widehat{\widehat{\otimes}}D^{\beta'}(k[G]) = D^{\beta\beta'}(k[G])$.

Proof. Straightforward calculation. □

3. *G*-Frobenius Algebras

3.1. *Frobenius algebras*

We wish to recall that there are two notions of Frobenius algebra. The first goes back to Frobenius and is given as follows:

Definition 3.1. A *Frobenius algebra* is a finite-dimensional commutative associative unital algebra A together with a non-degenerate symmetric pairing η that is invariant, that is,

$$\eta(a, bc) = \eta(ab, c). \tag{3.1}$$

A *possibly degenerate Frobenius algebra* is the same data as above, only that we do note require that η is non-degenerate.

In the categorical setting, there is the notion of a Frobenius algebra object in a monoidal category.

Definition 3.2. A *non-unital Frobenius algebra object* or Frobenius object for short in a monoidal category $\mathscr{C}$ is an associative commutative algebra object, which is also a co-associative co-commutative object given by a datum $(A, \mu : A \otimes A \to A, \Delta : A \to A \otimes A)$ that additionally satisfies

$$\Delta \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = (\mathrm{id} \otimes \mu) \circ (\Delta \otimes \mathrm{id}). \tag{3.2}$$

A *Frobenius algebra object* is the data above together with a unit $\upsilon : \mathbb{I}_{\mathcal{C}} \to A$ and a co-unit $\epsilon : A \to \mathbb{I}_{\mathcal{C}}$, where $\mathbb{I}_{\mathcal{C}}$ is the unit object of $\mathscr{C}$.

Remark 3.3. Notice that a Frobenius algebra always gives a Frobenius algebra object in the monoidal category (k-Vect, $\otimes$), by letting Δ be the adjoint of μ with respect to the pairing. The co-unit is given by pairing with the unit of the algebra.

Vice-versa if A is a Frobenius algebra object in (k-Vect, $\otimes$) then A with its unit, multiplication and $\eta(a,b) := \epsilon(\mu(a \otimes b))$ is a possibly degenerate Frobenius algebra.

3.2. *F-traces and trace elements*

One main difference between the finite-dimensional and the non-finite dimensional case is the existence of traces. In the finite-dimensional case, for any operator $\phi \in \mathrm{Aut}(A)$ we can consider $\mathrm{Tr}(\phi)$. The trace actually has an analog in the Frobenius object case. For this, we need an expression in terms of the morphisms.

Proposition 3.4. *For a Frobenius algebra, let 1_k be the unit in k, then*

$$\mathrm{Tr}(\phi) = \epsilon(\mu(\phi \otimes \mathrm{id})\Delta(\upsilon(1_k))). \tag{3.3}$$

Proof. Let $1_A = \upsilon(1_k)$ be the unit of A and let Δ_i be a basis of A. If $g_{ij} = \eta(\Delta_i, \Delta_j)$ is the metric and g^{ij} is its inverse, then $\Delta(\upsilon(1_k)) = \Delta(1_A) = \sum_{ij} g^{ij}\Delta_i \otimes \Delta_j$, since

$$\eta \otimes \eta(\Delta_k \otimes \Delta_l, \Delta(1_A)) := \eta(\Delta_k\Delta_l, 1_a) = \eta(\Delta_k, \Delta_l) = g_{kl},$$

and

$$\eta \otimes \eta\left(\Delta_k \otimes \Delta_l, \sum_{ij} g^{ij}\Delta_i \otimes \Delta_j\right) = \sum_{ij} g_{ki}g^{ij}g_{jl} = \sum_i g_{ki}\delta_{i,l} = \delta_{k,l},$$

and hence, if $\check{\Delta}_i := \sum_j g^{ij}\Delta_j$ is the inverse basis,

$$\begin{aligned}\epsilon(\mu(\phi \otimes \mathrm{id})\Delta(\upsilon(1_k))) &= \epsilon\left(\sum_{ij} g^{ij}\phi(\Delta_i)\Delta_j\right)\\ &= \sum_{i,j}\eta\left(\sum_{ij} g^{ij}\phi(\Delta_i), \Delta_j\right) = \sum_i \check{\Delta}_i(\phi(\Delta_i)) = \mathrm{Tr}(\phi). \qquad \square\end{aligned}$$

Definition 3.5. Given a Frobenius algebra object A and $\phi \in \mathrm{Aut}(A)$, we define the F-Trace $\tau(\phi) : \mathbb{I}_{\mathcal{C}} \to \mathbb{I}_{\mathcal{C}}$ of ϕ via

$$\tau(\phi) := \epsilon \circ \mu \circ (\phi \otimes \mathrm{id}) \circ \Delta \circ \eta. \tag{3.4}$$

Remark 3.6. If $\mathbb{I}_{\mathcal{C}} = k$ and all morphisms are k-linear, the map $\tau(\phi)$ is of course given by its value on 1_k. In this case, we will not distinguish between the map and this value.

Proposition 3.7. *Let $\mathcal{F}$ be a monoidal functor with values in vector spaces for a category with products given by the monoidal structure. Also assume that $\mathcal{F}$ has* *pull-backs, push-forwards and satisfies the projection formula for the diagonal morphisms. Then for any object V, it gives rise to a Frobenius algebra object and hence F-traces.*

Proof. We let μ be given by the pull-back along the diagonal $\Delta_V : V \to V \times V$ where the co-multiplication is given by push-forward along the diagonal: $\mu = \Delta_V^*, \Delta = \Delta_{V*}$.

Equation (3.2) is guaranteed by the projection formula. On one hand,

$$\Delta_{V*}(\Delta_V^*(\mathcal{F}_1 \otimes \mathcal{F}_2)) = (\mathcal{F}_1 \otimes \mathcal{F}_2)\Delta_{V*}(1). \tag{3.5}$$

On the other hand,

$$\begin{aligned}(\Delta_V^* \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_{V*})(\mathcal{F}_1 \otimes \mathcal{F}_2) &= (\Delta_V^* \otimes \mathrm{id})(\mathcal{F}_1 \otimes \Delta_{V*}(\Delta_V^*(1 \otimes \mathcal{F}_2))) \\ &= (\Delta_V^* \otimes \mathrm{id})(\mathcal{F}_1 \otimes (1 \otimes \mathcal{F}_2)\Delta_{V*}(1)) \\ &= \sum \mathcal{F}_1\Delta^{(1)} \otimes \mathcal{F}_2\Delta^{(2)} \\ &= (\mathcal{F}_1 \otimes \mathcal{F}_2)\Delta_{V*}(1),\end{aligned} \tag{3.6}$$

where we used Sweedler's notation $\Delta_{V*}(1) = \sum \Delta^{(1)} \otimes \Delta^{(2)}$ and analogously for the third equation.

The co-unit is furnished by the push-forward to the unit of the monoidal category which is a final object, and the unit of the Frobenius algebra object by the pull-back from it. In our cases of interest, this will be a point or the one-dimensional vector space of the ground field. □

Corollary 3.8. *In the situation above, we also obtain pre-Frobenius algebras in the sense of* [25], *where the trace element is the morphism given by $\forall a \in A : a \mapsto \tau(\lambda_a)$, that is, the F-trace of the morphism of left-multiplication by a; $\lambda_a(b) := ab$.*

Example 3.9. Notice that this gives the canonical trace elements considered in [25] for the pre-Frobenius algebras $A^*(V)$ and $K_0(V)$, which are prime examples of Frobenius algebra objects, that give rise to possibly degenerate Frobenius algebras, as they might be infinite-dimensional. Here $\epsilon = \int$ or χ respectively, which are the push-forwards to a point. For example in A^*, we can calculate the F-trace λ_v which is the operation of left-multiplication by v to be given by

$$\begin{aligned}\tau(\lambda_v) &= \int_V \Delta_V^*[(v \otimes 1_V) \cup (\Delta_{V_*}(1))] \\ &= \int_V (v \cup \Delta^{(1)}) \cup \Delta^{(2)} \\ &= \int_V v \cup \Delta_V^* \Delta_{V_*}(1) \\ &= \int_V v \cup c_{\mathrm{top}}(TV),\end{aligned} \tag{3.7}$$

where we used the notation of the last proposition for the co-product. This is exactly the expression appearing in [25]. The analogous statement of course holds for K-theory.

3.3. *Twisted Frobenius objects*

In general, there is a twisted version of Frobenius algebra objects. This appears in the definition of G-Frobenius algebras and is necessary for considerations concerning singularities with symmetries, see e.g. [28, 29, 31]. We again fix a monoidal category $\mathscr{C}$.

Definition 3.10. Let $\mathbb{I}_\chi$ be an even invertible element in $\mathscr{C}$.

A $\mathbb{I}_\chi$-twisted Frobenius algebra object is the datum $(A, \mu : A \to A \otimes A, \Delta : A \to A \otimes A \otimes \mathbb{I}_\chi^{\otimes 2}, \upsilon : \mathbb{I}_\mathcal{C} \to A, \epsilon : A \to \mathbb{I}_\chi^{\otimes 2})$ such that (3.2) is satisfied, where μ is associative commutative, ϵ is co-associative, co-commutative, υ is a unit, and ϵ is a co-unit using the isomorphism $m : \mathbb{I}_\chi \otimes \mathbb{I}_\chi^{-1} \cong \mathbb{I}_\mathcal{C}$. More precisely,

$$\begin{tikzcd}
\mathbb{I}_\chi^{\otimes -2} \otimes A \otimes \mathbb{I}_\chi^{\otimes 2} \arrow[d, "m \circ m^{\otimes 2}"'] & A \otimes A \otimes \mathbb{I}_\chi^{\otimes 2} \arrow[l, "\epsilon\otimes\mathrm{id}\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\epsilon"] & A \otimes \mathbb{I}_\chi^{\otimes -2} \otimes \mathbb{I}_\chi^{\otimes 2} \arrow[d, "m \circ m^{\otimes 2}"] \\
\mathbb{I}_\mathcal{C} \otimes A \arrow[r] & A \arrow[u, "\Delta"] & A \otimes \mathbb{I}_\mathcal{C} \arrow[l]
\end{tikzcd}$$

where on the left $m^{\otimes 2}$ is m applied to the first and fourth and the second and fifth component and then to the two copies of $\mathbb{I}_\mathcal{C}$ and on the right to the second and fourth and to the third and fifth and then again to the two copies of $\mathbb{I}_\mathcal{C}$.

Remark 3.11. One could of course twist $A \rightsquigarrow \bar{A} := A \otimes \mathbb{I}_\chi^{-1}$ and obtain similar operations and axioms. In the language of [28, 29], this is the Ramond twist or Ramond sector.

3.4. *G-Frobenius algebras*

First, we recall the main definition. See [28, 29].

Definition 3.12. A *G-Frobenius algebra* or GFA for short, over a field k of characteristic 0 is $\langle G, A, \circ, 1, \eta, \varphi, \chi \rangle$, where

G: finite group,
A: finite dim G-graded k-vector space,
  $A = \oplus_{g \in G} A_g$,
  A_e is called the untwisted sector and,
  the A_g for $g \neq e$ are called the twisted sectors,
$\circ$: a multiplication on A which respects the grading:
  $\circ : A_g \otimes A_h \to A_{gh}$,
1: a fixed element in A_e-the unit,

η: non-degenerate bilinear form which respects grading, i.e. $g|_{A_g \otimes A_h} = 0$ unless $gh = e$,

φ: an action of G on A (which will be by algebra automorphisms), $\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, s.t. $\varphi_g(A_h) \subset A_{ghg^{-1}}$,

χ: a character $\chi \in \mathrm{Hom}(G, k^*)$,

satisfying the following axioms:

Notation. We use a subscript on an element of A to signify that it has homogeneous group degree — e.g. a_g means $a_g \in A_g$ — and we write $\varphi_g := \varphi(g)$ and $\chi_g := \chi(g)$.

(a) Associativity

$$(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k),$$

(b) Twisted commutativity

$$a_g \circ a_h = \varphi_g(a_h) \circ a_g,$$

(c) G Invariant Unit:

$$1 \circ a_g = a_g \circ 1 = a_g \quad \text{and} \quad \varphi_g(1) = 1,$$

(d) Invariance of the metric:

$$\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k),$$

(i) Projective self-invariance of the twisted sectors

$$\varphi_g | A_g = \chi_g^{-1} \mathrm{id},$$

(ii) G-Invariance of the multiplication

$$\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h),$$

(iii) Projective G-invariance of the metric

$$\varphi_g^*(\eta) = \chi_g^2 \eta,$$

(iv) Projective trace axiom

$$\begin{aligned} &\forall c \in A_{[g,h]} \text{ and } l_c \text{ left-multiplication by } c : \chi_h \mathrm{Tr}(l_c \varphi_h|_{A_g}) \\ &= \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h}). \end{aligned}$$

We call a G-Frobenius algebra *strict*, if $\chi \equiv 1$.

Remark 3.13. It was shown in [30] that a GFA is a module over $D(k[G])$ and moreover proved that it is a $k[G]$ module algebra and a $k[G]$ co-module algebra. The first part also follows from Remark 2.6.

Example 3.14. Important examples are furnished by the twisted group rings $k^\alpha[G]$ with $\alpha \in Z^2(G, k^*)$. This group actually acts on the set (category) of G-Frobenius algebras through $\widehat{\otimes}$ and gives rise to the action of discrete torsion. See [30] for full details.

Proposition 3.15. *A G-Frobenius algebra with character χ is a unital, associative, commutative algebra object in the category $D(k[G])$-Mod. It moreover defines a k_χ*

twisted Frobenius algebra object, where k_χ is the one-dimensional $D(k[G])$-module concentrated in group degree e with G action on k given by the character χ.

Proof. This follows in a straightforward fashion, by reinterpreting the pertinent diagrams using the braided monoidal structure.

Since $\beta \equiv 1$ associativity in the category $D(k[G])$-Mod is just the ordinary associativity (a).

Let μ denote the multiplication in A. In view of Eq. (2.16) the G-invariance of the multiplication (ii) is equivalent to $\mu : A \otimes A \to A$ being a morphism in the category $D(k[G])$-Mod.

Using Eq. (2.15) we see that the condition that the following diagram commutes — which is the commutativity in $D(k[G])$-Mod — is equivalent to the condition (b) of twisted commutativity.

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu"] \arrow[d, "c_{A,A}"'] & A \arrow[d, "\mathrm{id}"] \\ A \otimes A \arrow[r] & A \end{tikzcd}$$

The fact that the unit is invariant is equivalent to the diagram

$$\begin{tikzcd} k \otimes A \arrow[r, "v\otimes\mathrm{id}"] & A \otimes A \arrow[d, "\mu"'] & A \otimes k \arrow[l, "\mathrm{id}\otimes v"'] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

being a diagram of $D(k[G])$ modules where k has the structure of a trivial $D(k[G])$ module.

We define the co-unit via $\epsilon(a) := \eta(a, 1_k)$. Then the projective G-invarince of the metric (iii) becomes the condition on the co-unit in a twisted Frobenius algebra.

We set $\Delta := \mu^\dagger$, that is, the adjoint of the multiplication under the non-degenerate metric η. Then, the invariace of the metric (d) together with the projective G-invariance (iii) yields the Frobenius equation (3.2). □

Theorem 3.16. *A G-Frobenius algebra with character χ is precisely a $\mathbb{I}_\chi$-twisted Frobenius algebra object* $D(k[G])$-Mod *with the following additional restrictions*:

(1) *The associated pairing $\eta = \epsilon \circ \mu$ is non-degenerate.*
(2) *Denoting the $D(k[G])$ action induced by the G action φ by ρ the following two axioms hold,*
  (T) $\rho(v^{-1}) = \chi^{-1}$ *for a character* $\chi \in \mathrm{Hom}(G, k^*)$
  (S) *Using Notation* 2.5 *let l_c denote the left-multiplication by c:$c \in A$*:

$$\chi_h \tau(l_c \circ \rho(\,hgh^{-1}\underset{h}{\llcorner})) = \chi_{g^{-1}} \tau(\rho(\,h\underset{g^{-1}}{\llcorner}) \circ l_c), \tag{3.8}$$

where τ is the F-trace.

Proof. Given a GFA, it is a unital, associative, commutative algebra object in $D(k[G])$-Mod by the above proposition and it also satisfies the additional axioms. By the Proposition 2.4, we see that any $D(k[G])$-Mod is G-graded and has an action of G by automorphisms of G given by ϕ of Notation 2.5, which act in the prescribed way. Now by the proof of the proposition above, we have that a unital associative commutative algebra object satisfies the axioms (a), (b), (c) and (ii). What remains to be shown is that the multiplication preserves the G-grading, but this follows from the fact that $\Delta(\underset{e}{k_\llcorner}) = \sum_{gh=k} \underset{e}{g_\llcorner} \otimes \underset{e}{h_\llcorner}$ so that if the multiplication is a morphism, the multiplication is graded since the $\underset{e}{g_\llcorner}$ act as projectors. Explicitly,

$$\rho(\underset{e}{k_\llcorner})(a_g b_h) = \mu \circ (\rho \otimes \rho)(\Delta(\underset{e}{k_\llcorner}))(a_g \otimes a_h) = \delta_{k,gh} a_g b_h.$$

It is clear that $\eta = \epsilon \circ \mu$ defines a pairing given a Frobenius algebra object and as above vice-versa defines ϵ in the presence of a unit. The invariance of the metric (d) follows from the Frobenius equation and the structure of the co-unit. The latter is also equivalent to the projective G-invariance of the metric (iii).

For the equivalence of the projective trace axiom with (S), we recall that the elements $\underset{x}{g_\llcorner}$ act as explained in Notation 2.5. Notice that if $c \notin A_{[gh]}$ then both sides are zero. In the same notation with the definition of v^{-1} given, Eq. (2.10) condition (T) is just condition (i). $\square$

Here (S) and (T) stand for the generators of $SL(2,\mathbb{Z})$ and are a reminder that these axioms correspond to the invariance of the conformal blocks.[a]

Dropping the condition (1) we come to the main definition of the paragraph.

Definition 3.17. We define a *G-Frobenius algebra object* to be a Frobenius algebra object in the category $D(k[G])$-Mod which satisfies the axioms (S) and (T).

Remark 3.18. Going beyond the aesthetics and the practicality of the above definition, it is a necessary generalization if we are to deal with the stringy Chow ring or Grothendieck K-theory of a global quotient stack as in [25], where the natural metric may be degenerate. See the next paragraph for details.

3.5. *The Drinfel'd double as a G-Frobenius algebra*

We have seen that any GFA is actually a $D(k[G])$-module. Now as it happens, $D(k[G])$ is itself a $D(k[G])$ module, but not quite a G-Frobenius algebra for general G. This is because the G-degree of $\underset{x}{g_\llcorner}$ is g and the multiplication is not multiplicative in g but rather in x.

Notice that the elements $\underset{x}{g_\llcorner}$ with $[g,x] = e$ form a subalgebra $D^\beta(k[G])^{\mathrm{comm}}$ of $D^\beta(k[G])$ which is actually additively isomorphic to $\bigoplus_{g\in G} k^{\theta_g}(Z(g))$. In the case that G is Abelian, of course $D^\beta(k[G])^{\mathrm{comm}} = D^\beta(k[G])$.

[a]As someone suggested, (S) could of course also stand for Spur.

Proposition 3.19. *$D^\beta(k[G])^{\mathrm{comm}}$ is a GFA for the $D(k[G])$ action given by*

$$\rho(\underset{x}{g\llcorner})(\underset{y}{h\llcorner}) = \frac{\theta_{xhx^{-1}}(x,y)}{\theta_{xhx^{-1}}(xyx^{-1},x)}\delta_{g,xyx^{-1}}\, \underset{xyx^{-1}}{xhx^{-1}\llcorner}, \tag{3.9}$$

which means that the G-degree of $\underset{y}{h\llcorner}$ is y.

Proof. This follows from the fact that the each $k^{\theta_g}[Z(g)]$ is actually a $Z(g)$-FA. This means, for instance, that it satisfies all the axioms for the $Z(g)$ action pertaining to the $Z(g)$ alone. The other axioms then follow from the G-equivariance of the θ_g or are straightforward. For $\beta = 1$, the statement also follows from Proposition 4.10. $\square$

Remark 3.20. In the case of $D(k[G])$, if one uses the grading that the G degree of $\underset{x}{g\llcorner}$ is x so that the multiplication is indeed G-graded, then twisted commutativity dictates that $\rho(\underset{h}{\mathbf{1}\llcorner})(\underset{x}{g\llcorner}) = \underset{hxh^{-1}}{hxgx^{-1}h^{-1}\llcorner}$. In turn, postulating the compatibility of this G action with the multiplication requires that $[g,x] = e$.

Definition 3.21. We call a GFA a *free GFA* if it is of the form $A = k^{\theta_g}[G] \otimes A_e$ for a Frobenius algebra A_e that is a G-module, with the multiplication given by the diagonal multiplication, the G-degree of $g \otimes a$ being g and the G-action given by the conjugation action on the left factor and the postulated G action on A_e.

Remark 3.22. Notice that in this case, we have a second $k^{\theta_g}[G]$ action, given by multiplication from the left on the factor $k^{\theta_g}[G]$. This action sends $\lambda_h : A_g \to A_{hg}$. This is similar to the quantum symmetry considered in [30].

Definition 3.23. Given $\beta \in Z^3(G, k^*)$, if $A = \bigoplus_{g\in G} B_g$ is the direct sum of free $Z(g)$-Frobenius algebras $B_g = k^{\theta_g}[Z(g)] \otimes B_e$, then we define the *DPR induced free* algebra $\mathrm{Ind}^{\mathrm{DPR}}(\mathrm{A}) := \bigoplus \mathrm{k[G]} \otimes_{\mathrm{k}^{\theta_g}[\mathrm{Z(g)}]} \mathrm{B_g} \cong \mathrm{D}^\beta(\mathrm{k[G]}) \otimes \mathrm{B_e}$, where the action is analogous to Definition 2.10 and the algebra structure is the diagonal algebra structure.

Remark 3.24. At the moment, we do not see how to induce this algebra in the non-free case. Geometrically, this amounts to the fact that on the inertia, the automorphisms have to commute so that the double twisted sectors for non-commuting elements are not accessible. Also in the general case, the double twisted sectors $A_{x^{-1}gx,x}$ for $x \in G$ are not equidimensional. This is, however, an interesting detail which should be studied further, but is unfortunately beyond the scope of the present considerations.

4. Orbifold Cohomology and K-Theory

In this section, we recall the various stringy functors introduced in [25] and re-express them in the current framework. First, we recall from [25] that we have the following stringy functors for a global quotient (X, G), $\mathcal{F} \in \{A^*, H^*, K_0, K^{\text{top}}\}$ as well as isomorphisms $\mathfrak{Ch} : K_0(X, G) \to A^*(K, G)$ and $\mathfrak{Ch} : K^{\text{top}}(X, G) \to H^*(X, G)$. Then, we also recall the stack versions of these functors and maps for a suitably nice stack $\mathfrak{X}$. In order to simplify things, we will work over $\mathbb{Q}$ or extensions of it. See Remark 4.1.

4.1. *General setup — global quotient case*

We recall the setup as in the global part of [25]. We simultaneously treat two flavors of geometry: algebraic and differential. For the latter, we consider a stably almost complex manifold X with the action of a finite group G such that the stably almost complex bundle is G equivariant. While for the former, X is taken to be a smooth projective variety with a G-action.

In both situations for $m \in G$, we denote the fixed point set of m by X^m and let

$$I(X) = \amalg_{m \in G} X^m \tag{4.1}$$

be the inertia variety.

We let $\mathcal{F}$ be any of the functors $H^*, K_0, A^*, K^{\text{top}}$, that is cohomology, Grothendieck K_0, Chow ring or topological K-theory with $\mathbb{Q}$ coefficients, and define

$$\mathcal{F}_{\text{stringy}}(X, G) := \mathcal{F}(I(X)) = \bigoplus_{m \in G} \mathcal{F}(X^m) \tag{4.2}$$

additively.

We furthermore set

$$\mathrm{Eu}_{\mathcal{F}}(E) = \begin{cases} c_{\text{top}}(E) & \text{if } \mathcal{F} = H^* \text{ or } A^* \text{ and } E \text{ is a bundle} \\ \lambda_{-1}(E^*) & \text{if } \mathcal{F} = K \text{ or } K^{\text{top}}. \end{cases} \tag{4.3}$$

Notice that on bundles Eu is multiplicative. For general, K-theory elements we set

$$\mathrm{Eu}_{\mathcal{F},t}(E) = \begin{cases} c_t(E) & \text{if } \mathcal{F} = H^* \text{ or } A^* \\ \lambda_t(E^*) & \text{if } \mathcal{F} = K \text{ or } K^{\text{top}}. \end{cases} \tag{4.4}$$

4.2. *The stringy product*

For $m \in G$, we let X^m be the fixed point set of m and for a triple $\mathbf{m} = (m_1, m_2, m_3)$ (or more generally an n-tuple) such that $\prod m_i = \mathbf{1}$ (where $\mathbf{1}$ is the identity of G). We let $X^{\mathbf{m}}$ be the common fixed point set, that is, the set fixed under the subgroup generated by them.

In this situation, recall the following definitions. Fix $m \in G$, let $r = \text{ord}(m)$ be its order. Furthermore, let $W_{m,k}$ be the sub-bundle of $TX|_{X^m}$ on which m acts with character $\exp(2\pi i \frac{k}{r})$, then

$$S_m = \bigoplus_k \frac{k}{r} W_{m,k}. \tag{4.5}$$

Notice this formula is invariant under stabilization.

We also wish to point out that using the identification $X^m = X^{m^{-1}}$

$$S_m \oplus (S_{m^{-1}}) = N_{X^m/X}, \tag{4.6}$$

where for an embedding $X \to Y$ we will use the notation $N_{X/Y}$ for the normal bundle.

Recall from [25] that in such a situation there is a product on $\mathcal{F}(X, G)$ which is given by

$$v_{m_1} * v_{m_2} := \check{e}_{m_3 *}(e_1^*(v_{m_1}) e_2^*(v_{m_2}) \mathrm{Eu}(\mathcal{R}(\mathbf{m}))), \tag{4.7}$$

where the obstruction bundle $\mathcal{R}(\mathbf{m})$ is defined by

$$\mathcal{R}(\mathbf{m}) = S_{m_1} \oplus S_{m_2} \oplus S_{m_3} \ominus N_{X^{\mathbf{m}}/X}, \tag{4.8}$$

and the $e_i : X^{m_i} \to X$ and $\check{e}_3 : X^{m_3^{-1}} \to X$ are the inclusions. Notice, that as it is written, $\mathcal{R}(\mathbf{m})$ only has to be an element of K-theory with rational coefficients, but is actually indeed represented by a bundle [25].

Remark 4.1. This bundle and hence the multiplication below are actually defined over $\mathbb{Z}$. The point is that in [25], we identified $\mathcal{R}(\mathbf{m})$ as a bundle and true representation in the representation ring. Since there is no torsion in this ring, the bundle is identified over $\mathbb{Z}$.

Remark 4.2. The first appearance of a push-pull formula was given in [11] in terms of a moduli space of maps. The product was for the G invariants, that is, for the H^* of the inertia orbifold and is known as Chen–Ruan cohomology. In [19], the obstruction bundle was given using Galois covers establishing a product for H^* on the inertia variety level, i.e. a G-Frobenius algebra as defined in [28, 29], which is commonly referred to as the Fantechi–Göttsche ring. In [24], we put this global structure back into a moduli space setting and proved the trace axiom. The multiplication on the Chow ring A^* for the inertia stack was defined in [3, 4]. The representation of the obstruction bundle in terms of the S_m and hence the passing to the differentiable setting as well as the two flavors of K-theory stem from [25].

The following is the key diagram:

$$\begin{array}{ccc} X^{m_1} & X^{m_2} & X^{m_3^{-1}} \\ e_1 \nwarrow & \uparrow e_2 & \nearrow \check{e}_3 \\ & X^{\mathbf{m}} & \end{array} \tag{4.9}$$

Here, we used the notation of [25], where $e_3 : X^{\mathbf{m}} \to X^{m_3}$ and $i_3 : X^{m_3} \to X$ are the inclusion, $\vee : I(X) \to I(X)$ is the involution which sends the component X^m to $X^{m^{-1}}$ using the identity map and $\check{i}_3 = i_3 \circ \vee$, $\check{e}_3 = \vee \circ e_3$. This is short-hand notation for the general notation of the inclusion maps $i_m : X^m \to X$, $\check{i}_m := i_m \circ \vee = i_{m^{-1}}$.

Theorem 4.3. *The cases in which $\mathcal{F}$ equals H^* and K^{top} yield G-Frobenius algebras. In the cases of A^* and K_0 the stringy functors are still G-Frobenius algebra* *objects. The co-multiplication is given by*

$$\Delta(\mathcal{F}_{m_3}) = \sum_{m_1,m_2\,:\,m_1^{-1}m_2^{-1}=m_3} (\check{e}_{1*} \otimes \check{e}_{2*})\Delta_{X^{\mathbf{m}}*}(e_3^*(\mathcal{F}_3)\mathrm{Eu}(\mathcal{R}(\mathbf{m})), \tag{4.10}$$

where $\Delta_{X^m} : X^m \to X^m \times X^m$ *is that diagonal map.*

The F-traces $\tau(\lambda_c\phi_g, h) := \tau(\lambda_c \circ \rho(\,ghg^{-1}\llcorner_g))$ *give the trace elements which were part of the definition of pre-Frobenius algebra structures defined in* [25].

Proof. The first part about H^* and K^{top} is contained in [25]. For A^* or K_0 the verification of the Frobenius condition (3.2) is somewhat tedious but straightforward using analogous arguments as in Proposition 3.7. We will calculate the trace elements. We fix $a, b \in G$ and $v_{[a,b]} \in A_{[a,b]}$ and calculate the F-trace $\tau(\lambda_{v_{[a,b]}}\phi_b, a)$. For this, we will need to set up some notation and recall some results from [25]. We will use the following notation analogous to [25], $\mathbf{m}' = ([a, b], bab^{-1}, a)$, $H' := \langle [a, b], bab^{-1}\rangle \subset H := \langle a, b\rangle$. We will also need the commutative diagram

$$\begin{array}{ccc} X^H & \xrightarrow{j_2'} & X^{H'} \\ {\scriptstyle j_1'}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_2'} \\ X^a & \xrightarrow{\Delta_1'} & X^{bab^{-1}} \times X^a \end{array}$$

where j_1' and j_2' are the inclusion morphisms, Δ_2' is the diagonal map, and Δ_1' is the composition

$$X^a \xrightarrow{\Delta_{X^a}} X^a \times X^a \xrightarrow{\phi(b)\times\vee} X^{bab^{-1}} \times X^{a^{-1}}.$$

We denote the excess intersection bundle by $\mathscr{E}'$. Also, we recall that for a triple product $v_{m_1} * v_{m_2} * v_{m_3}$, we have a special formula which actually is the reason for associativity.

Let $\mathbf{m} = \langle m_1, m_2, m_3, m_4 = (m_1m_2m_3)^{-1}\rangle$, and $\mathbf{m}' = \langle m_1, m_2, (m_1m_2)^{-1}\rangle$ $X^{H'} := X^{\mathbf{m}}$ and as usual let $e_i : X^{H'} \to X^{m_i}$ be the inclusions and $\check{e}_i = \vee \circ e_i$, then we have

$$v_{m_1} * v_{m_2} * v_{m_3} = \check{e}_{m_4*}\left[\left(\prod e_i^*(v_{m_i})\mathrm{Eu}(\mathcal{R}(\mathbf{m})\right)\right],$$

where $\mathcal{R}(\mathbf{m}) = \bigoplus S_{m_i} \ominus N(X^H)$.

Let $p_V : V \to \mathrm{pt}$ be the projection to a point. In our case, $\mathbf{m} = ([a, b], bab^{-1}, a^{-1}, e)$ and $\mathbf{m}' = ([a, b], bab^{-1}, a^{-1})$ and $H = \langle a, b\rangle$. Let 1_V be the unit in $\mathcal{F}(V)$. Then

$$\Delta(1_X) = \sum_h e_{h*} \otimes e_{h^{-1}*}(e_h^*(1_X)\mathrm{Eu}(\mathcal{R}((h, h^{-1}, e))) = \sum_h (\mathrm{id} \otimes \vee)\Delta_{X^h}(1_{X^h}).$$

The bi-degree (h, h^{-1})-part is just given by $(\mathrm{id} \otimes \vee)\Delta_{X^h}(1_{X^h})$

$$\begin{aligned} \tau(\phi(b), a) &= p_{X*}[\check{e}_{m_4*}[e_1^*(v_{[a,b]}\Delta_2'(\Delta_1'(1))\mathrm{Eu}(\mathcal{R}(\mathbf{m}')))]] \\ &= p_{X^H*}[e_1^*(v_{[a,b]}j_{2*}'(j_1'^*(1_{X^h})\mathrm{Eu}(\mathscr{E}')\mathrm{Eu}(\mathcal{R}(\mathbf{m}'))))] \\ &= p_{X^{H'}*}[v_{[a,b]}|_{X^{H'}}\mathrm{Eu}(\mathscr{E}')j_2'^*(\mathrm{Eu}(\mathcal{R}(\mathbf{m}')))] \end{aligned}$$

$$= p_{X^{H'}*}[v_{[a,b]}|_{X^{H'}}\mathrm{Eu}(\mathscr{E}' \oplus j_2'^*(\mathcal{R}(\mathbf{m}')))]$$
$$= p_{X^{H'}*}[v_{[a,b]}|_{X^{H'}}\mathrm{Eu}(TX^H \oplus S_{[a,b]}|_{X^H})], \tag{4.11}$$

which is the expression of [25]. Here, the last equality follows from the equality of the bundles $\mathscr{E}' \oplus j_2'^*(\mathcal{R}(\mathbf{m}')) = TX^H \oplus S_{[a,b]}|_{X^H}$ which fittingly was proved in [25, Theorem 5.5].

The traces $\tau(\lambda_v\phi_b, a)$ will of course be zero if v is of pure G-degree different from $[a,b]$. □

Proposition 4.4. *Given (X,G) and (Y,G), $X\times Y$ has a diagonal G action and $\mathcal{F}_{\mathrm{stringy}}((X\times Y,G)) = \mathcal{F}_{\mathrm{stringy}}((X,G))\widehat{\otimes}\mathcal{F}_{\mathrm{stringy}}((Y,G))$ where $\mathcal{F}_{\mathrm{stringy}}$ is the global stringy version of any of the functors $\mathcal{F}$ as defined in* [25].

Proof. Straightforward by the Künneth formula or relevant versions thereof. □

4.3. *The stack case*

In [25], a version of stringy K-theory or Chow for general stacks was developed as well. The important thing about the stringy K-theory in this case, which was also called full orbifold K-theory is that is it usually bigger than the global K-theory. In particular for a stack $\mathfrak{X}$ it was defined that $K_{\mathrm{full}}(\mathfrak{X}) := K(\mathfrak{I}\mathfrak{X})$ where $\mathfrak{I}\mathfrak{X}$ is the inertia stack. For a global quotient stack, we also defined $K_{\mathrm{small}}([X/G]) := K_{\mathrm{global}}(X,G)^G$. Notice that this is actually presentation independent [25].

In particular for a global quotient, three theories where introduced which are *additively* over given $\mathbb{C}$ as follows:

$$K_{\mathrm{global}}((X,G)) := K(I(X,G))) \cong \bigoplus_{g\in G} K(X^g), \tag{4.12}$$

$$K_{\mathrm{full}}([X/G]) := K(\mathfrak{I}[X/G]) \cong \bigoplus_{[g]} K([X^g/Z(g)]), \tag{4.13}$$

$$K_{\mathrm{small}}([X/G]) := K_{\mathrm{global}}((X,G))^G \cong \bigoplus_{[g]} K(X^g)^{Z(g)}. \tag{4.14}$$

Here, these are only *linear isomorphism* and the product is the one given by the push-pull formula (4.7). Notice they are all different. It is however the case that K_{small} is a subring of K_{full} (see [25]).

4.4. *Comparing the different constructions in the case of a global quotient*

As mentioned above for global quotient stacks, we have $K_{\mathrm{small}}([X/G]) \cong K(X,G)^G$ which is isomorphic to A^* or H^*, but also we have $K_{\mathrm{full}}([X/G])$, which is usually much bigger. Notice that $K_{\mathrm{small}}(I(X,G),G)$ and $K_{\mathrm{full}}[(X/G)]$ are of the same size but have different multiplications that is they *are additively isomorphic, but not multiplicatively.*

Proposition 4.5. *Additively,*

$$K_{\mathrm{global}}(I(X,G),G) = K(\amalg_{x\in G}(\amalg_{g\in G} X^g)^x \tag{4.15}$$

$$= \bigoplus_{g\in G, x\in Z(g)} K(X^{\langle g,x\rangle}) \tag{4.16}$$

$$\tag{4.17}$$

and for $\prod x_i = 1$, *and* $g : x \in Z(g), h : y \in Z(h)$, *the multiplication is given by*

$$\begin{aligned}\mathcal{F}_{g,x_1} * \mathcal{F}_{h,x_2} &= \check{e}_{x_3*}(e^*_{x_1}(\mathcal{F}_{g,x_1})e^*_{x_2}(\mathcal{F}h, x_2)\mathcal{R}((x_1, x_2, (x_1x_2)^{-1})) \\ &= \delta_{g,h}\, \mathcal{F}_{g,x_1} *_g \mathcal{F}_{g,x_2},\end{aligned} \tag{4.18}$$

where $*_g$ *is the multiplication on* $K_{\mathrm{global}}(X^g, Z(g))$, *that is, as rings*

$$K_{\mathrm{global}}(I(X,G),G) = \bigoplus_{g\in G} K_{\mathrm{global}}(X^g, Z(g)). \tag{4.19}$$

Proof. Notice that if $g \neq h$, then the pull-backs land in different components, so that the product is zero. In case one pulls back to the same component $(X^g)^{\langle x,y\rangle}$, the obstruction bundle is equal to that of $K_{\mathrm{global}}(X^g, Z(g))$, since the respective maps are given by $e_i : (X^g)^{\langle x_1,x_2\rangle} \to (X^g)^{x_i}$. □

Corollary 4.6. *Given* (X,G) *and* (Y,G), $X \times Y$ *has a diagonal* G *action and* $K_{\mathrm{global}}(I(X\times Y,G),G) = K_{\mathrm{global}}(I(X,G),G)\widehat{\widehat{\otimes}}K_{\mathrm{global}}(I(Y,G),G)$ *with the diagonal product structure.*

Proof. Using the Proposition 4.5 above, Proposition 4.4 and the definition of $\widehat{\widehat{\otimes}}$

$$\begin{aligned}K_{\mathrm{global}}(I(X\times Y,G),G) &= \bigoplus_{g\in G} K_{\mathrm{global}}((X\times Y)^g, Z(g)) \\ &= \bigoplus_{g\in G} K_{\mathrm{global}}(X^g, Z(g))\widehat{\otimes}K_{\mathrm{global}}(Y^g, Z(g)) \\ &= K_{\mathrm{global}}(I(X,G),G)\widehat{\widehat{\otimes}}K_{\mathrm{global}}(I(Y,G),G).\end{aligned} \tag{4.20}$$

□

Corollary 4.7. *Denote the set of double conjugacy classes of* $G\times G$ *by* $C^2(G)$. *Additively,*

$$\begin{aligned}K_{\mathrm{small}}(I(X,G),G) &= K_{\mathrm{global}}(I(X,G),G)^G \\ &= \left[\bigoplus_{(g,x)\in G\times G, x\in Z(g)} K((X^g)^x)\right]^G \\ &= \bigoplus_{[g,x]\in C^2(G), x\in Z(g)} K(X^{\langle g,x\rangle})^{Z(g,x)},\end{aligned} \tag{4.21}$$

and as rings

$$K_{\text{small}}(I(X,G),G) = \bigoplus_{[g]\in C(G)} K_{\text{small}}(X^g, Z(g)). \tag{4.22}$$

□

Remark 4.8. On the other hand, we have *additively*

$$\begin{aligned} K_{\text{full}}([X/G]) &= \bigoplus_{[g]\in C(G)} K([X^g/Z(g)]) \\ &= \bigoplus_{[g]\in C(G)} K_{Z(g)}(X^g) \\ &= \bigoplus_{[g]\in C(G),[x]\in C(Z(g))} K((X^g)^x)^{Z(g,x)} \\ &= \bigoplus_{[g,x]\in C^2(G)} K(X^{\langle g,x\rangle})^{Z(g,x)}. \end{aligned} \tag{4.23}$$

Remark 4.9. Both versions above are hence additively isomorphic to the sum over double twisted sectors. In particular, if G is *Abelian*, then as vector spaces both versions above are additively given by the direct sum $\bigoplus_{G\times G}(K(X^{\langle g,x\rangle}))^G$.

4.5. *The second and third appearance of the Drinfel'd double*

Before going on to the twisting, it will be instructive to work out the two theories on the simplest example $[\mathrm{pt}/G]$. For both $K_{\text{full}}([\mathrm{pt}/G])$ and $K_{\text{global}}(I(X,G),G)$, we find the Drinfel'd double, be it in different guises.

Proposition 4.10. $K_{\text{global}}(I(\mathrm{pt},G),G) = D(k[G])^{\text{comm}}$.

Proof. By Proposition 4.5

$$K_{\text{global}}(I(\mathrm{pt},G),G) = \bigoplus_{g\in G} k[Z(g)] = \bigoplus_{g\in G, z\in Z(g)} k1_{g,x}, \tag{4.24}$$

where we have chosen $1_{x,g}$ for the bi-degree (g,x) part. Note, all the obstruction bundles vanish, since all the normal bundles vanish and the multiplication is given by

$$1_{g,x}1_{h,y} = e_{xy*}(e_x^*(1_{g,x})e_y^*(1_{h,y})) = \delta_{g,h}1_{xy,g} \tag{4.25}$$

so the multiplication is just that of $k[Z(g)]$. □

Corollary 4.11. *Since* $K_{\text{global}}(I(\mathrm{pt},G),G)$ *is a sum of free* $Z(g)$ *Frobenius algebras as needed in Definition* 3.23 *so we can DPR induce to obtain* $D(k[G])$.

Corollary 4.12. *As groupoid algebras, the* G*-module* $K_{\text{global}}(I(\mathrm{pt},G),G)$ *is Morita equivalent to* $D(k[G])$.

Proof. If we consider the G action, we see that it permutes the sectors in a given conjugacy class. The G-action on a module is completely determined via DPR induction. In the groupoid language, $(I(\mathrm{pt}, G), G)$ is the disjoint union of groupoids $[\mathrm{pt}/Z(g)]$ and the G-action adds the morphisms $*_g \xrightarrow{h} *_{hgh^{-1}}$ where $*_g$ denotes the different objects of the groupoid. This is now Morita equivalent to the loop groupoid of $[\mathrm{pt}/G]$ and hence the result follows. $\square$

See [35] for similar considerations.

Theorem 4.13. $K_{\mathrm{full}}([\mathrm{pt}/G]) \cong \mathrm{Rep}(D(k[G]))$.

Remark 4.14. We were informed by C. Teleman, that a similar formula at least additively for the case of $[\mathrm{pt}/G]$ can be deduced from the work of Freed–Hopkins–Teleman [20, 21].

Notation 4.15. In order to do the calculations, we will use the standard notation [13, 14, 17]. Let ${}^{A}g$ be a system of representatives of conjugacy classes in $C(G)$, which we will consider to be indexed by A. Furthermore, let α be an irreducible representation of $Z({}^{A}g)$, then we get an irreducible representation π_{α}^{A} of $D(k[G])$ by using DPR induction.

Proof of Theorem 4.13. For this, we notice that the inertia stack $\mathfrak{I}[\mathrm{pt}/G] = \coprod_{[g]\in C(G)}[\mathrm{pt}/Z(g)]$ and hence

$$K_{\mathrm{full}}([\mathrm{pt}/G]) = \bigoplus_{[g]\in C(G)} K([\mathrm{pt}/Z(g)]) = \bigoplus_{[g]\in C(G)} K_{Z(g)}(\mathrm{pt}) = \bigoplus_{[g]\in C(G)} \mathrm{Rep}(Z(g)).$$

The product is given by

$$\alpha_{[{}^{A}g]} * \beta_{[{}^{B}g]} = \sum_{m_1\in[{}^{A}g], m_2\in[{}^{B}g]} \frac{|Z(m_1m_2)|}{|G|} \mathrm{Ind}_{\mathrm{Z}(\langle \mathrm{m}_3^{-1}\rangle)}^{\mathrm{Z}(\langle \mathbf{m}\rangle)} (\mathrm{Res}_{\mathrm{Z}(\langle \mathrm{m}_1\rangle)}^{\mathrm{Z}(\langle \mathbf{m}\rangle)}(\alpha_{\mathrm{m}_1}) \otimes \mathrm{Res}_{\mathrm{Z}(\langle \mathrm{m}_2\rangle)}^{\mathrm{Z}(\langle \mathbf{m}\rangle)}(\beta_{\mathrm{m}_2})). \tag{4.26}$$

Notice that each $\mathrm{Rep}(H)$ has a non-degenerate pairing which is essentially given by the trace:

$$\eta(\rho_1, \rho_2) := \frac{1}{|H|}\sum_{h\in H} \mathrm{tr}(\rho_1(h))\mathrm{tr}(\rho_2^*(h)),$$

and with this pairing, there is an sesquilinar isomorphism of the Frobenius algebras $(K_G(\mathrm{pt}), \chi)$ and $(\mathrm{Rep}(G), \eta)$. What we mean by this is that we can compute the structure constants of the multiplication for a fixed basis of irreducible representations using either metric.

Furthermore, Frobenius reciprocity holds for a subgroup $H \subset K$

$$\eta_K(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{K}}(\rho_1), \rho_2) = \eta_{\mathrm{H}}(\rho_1, \mathrm{Res}_{\mathrm{H}}^{\mathrm{K}}(\rho_2)).$$

Hence, we obtain

$$\begin{aligned}
&\eta(\alpha_{[{}^Ag]} * \beta_{[{}^Bg]}, \nu_{[{}^Cg]}) \\
&= \sum_{\substack{[m_1,m_2,m_3] \\ m_1\in[{}^Ag], m_2\in[{}^Bg], \\ m_3\in[{}^Cg], \prod m_i=1}} \eta_{Z(\langle m_3\rangle)}\Big(\mathrm{Ind}^{\mathrm{Z}(\langle \mathbf{m}\rangle)}_{\mathrm{Z}(\langle \mathrm{m}_3^{-1}\rangle)}\Big(\mathrm{Res}^{\mathrm{Z}(\langle \mathbf{m}\rangle)}_{\mathrm{Z}(\langle \mathrm{m}_1\rangle)}(\alpha_{\mathrm{m}_1}) \\
&\qquad \otimes \mathrm{Res}^{\mathrm{Z}(\langle \mathbf{m}\rangle)}_{\mathrm{Z}(\langle \mathrm{m}_2\rangle)}(\beta_{\mathrm{m}_2})\Big), \nu_{\mathrm{m}_3}\Big) \\
&= \sum_{\substack{[m_1,m_2,m_3] \\ m_1\in[{}^Ag], m_2\in[{}^Bg], \\ m_3\in[{}^Cg], \prod m_i=1}} \eta_{Z(\langle \mathbf{m}\rangle)}\Big(\mathrm{Res}^{\mathrm{Z}(\langle \mathbf{m}\rangle)}_{\mathrm{Z}(\langle \mathrm{m}_1\rangle)}(\alpha_{\mathrm{m}_1}) \\
&\qquad \otimes \mathrm{Res}^{\mathrm{Z}(\langle \mathbf{m}\rangle)}_{\mathrm{Z}(\langle \mathrm{m}_2\rangle)}(\beta_{\mathrm{m}_2}), \mathrm{Res}^{\mathrm{Z}(\langle \mathbf{m}\rangle)}_{\mathrm{Z}(\langle \mathrm{m}_3\rangle)}(\nu_{\mathrm{m}_3})\Big) \\
&= \sum_{\substack{[m_1,m_2,m_3] \\ m_1\in[{}^Ag], m_2\in[{}^Bg], \\ m_3\in[{}^Cg], \prod m_i=1 \\ h\in Z(g_1,g_2)}} \frac{1}{|Z(m_2,m_2)|}\mathrm{tr}(\alpha_{m_1}(h))\mathrm{tr}(\beta_{m_2}(h))\mathrm{tr}(\nu^*_{m_3}(h)) \\
&= \frac{1}{|G|}\sum_{\substack{m_1\in[{}^Ag], m_2\in[{}^Bg], \\ m_3\in[{}^Cg], \prod m_i=1 \\ h\in Z(g_1,g_2)}} \mathrm{tr}(\alpha_{m_1}(h))\mathrm{tr}(\beta_{m_2}(h))\mathrm{tr}(\nu^*_{m_3}(h)),
\end{aligned} \tag{4.27}$$

for the three-point functions, which agrees with the three-point functions in the case of the Drinfel'd double calculated in [17]. The two-point functions then also coincide, since we can take one representation to be identity, *viz.* the trivial representation on the identity sector. □

5. Twisting

In this section, we will be concerned with twisting of the above structures. This can actually be done on three levels in two different but equivalent fashions. For the twisting, we can concern ourselves as above with (X, G) and $(I(X, G), G)$, where we will consider twisting $K_{\text{global}}(X, G), K_{\text{global}}(I(X, G), G)$ and $K_{\text{full}}([X/G])$. The first two are of course isomorphic to the global orbifold Chow ring or Cohomology ring.

5.1. *Geometric twisting: Gerbe twisting*

In this subsection, we give a geometric interpretation of the twistings in terms of gerbes.

Assumption. We will only consider global quotients (X, G) and gerbes equivariantly pulled back from a point. This means in particular that we can think of 0-, 1-, and 2-gerbes as elements in $Z^{1,2,3}(G, k^*)$. These gerbes are necessarily flat.

Remark 5.1. It is well-known that there is a transgression of an n-gerbe on a stack $\mathfrak{X}$ to an $(n-1)$ gerbe on its inertia $\mathfrak{I}\mathfrak{X}$.

5.2. *Line bundle twisting*

Given a line bundle $\mathscr{L}_Y$ on Y, there are basically two "twists" we can do. One in K-theory and one in cohomology, which are as follows. For cohomology, we can consider cohomology with coefficients in the line bundle $H^*(Y, \mathscr{L}_Y)$ and in K-theory, we have an endomorphism

$$K(Y) \xrightarrow{\sim} K(Y), \mathcal{F} \mapsto \mathcal{F} \otimes \mathscr{L}_Y. \tag{5.1}$$

We will use the notation $K(Y)_{\mathscr{L}}$ to denote the twisted side.

Remark 5.2. One way to view this is that the line bundles $\mathscr{L}$ are gauge degrees of freedom.

If we can choose a global section s of $\mathscr{L}$, then we get an isomorphism

$$H^*(Y,k) \to H^*(Y,\mathscr{L}); \quad v \mapsto v \cdot s. \tag{5.2}$$

Given line bundles $\mathscr{L}, \mathscr{L}'$ and $\mathscr{L}''$ on Y and an isomorphism $\mu : \mathscr{L} \otimes \mathscr{L}' \to \mathscr{L}''$, we get the following multiplicative maps:

$$\begin{gathered} H^*(Y,\mathscr{L}) \otimes H^*(Y,\mathscr{L}') \xrightarrow{\cup} H^*(Y, \mathscr{L} \otimes \mathscr{L}') \xrightarrow{\mu_*} H^*(Y,\mathscr{L}''), \\ K(Y)_{\mathscr{L}} \otimes K(Y)_{\mathscr{L}'} \to K(Y)_{\mathscr{L} \otimes \mathscr{L}'} \to K_{\mathscr{L}''}(Y), \\ (\mathcal{F} \otimes \mathscr{L}) \otimes (\mathcal{F}' \otimes \mathscr{L}') \mapsto \mathcal{F} \otimes \mathcal{F}' \otimes (\mathscr{L} \otimes \mathscr{L}') \to \mathcal{F} \otimes \mathcal{F}' \otimes \mathscr{L}''. \end{gathered} \tag{5.3}$$

Remark 5.3. If Y has a G action and the line bundles are equivariant line bundles, then the maps above carry over to the G-equivariant case.

Caveat. Equation (5.2) in the G-equivariant setting is only an isomorphism on the level of vector spaces. If the bundle $\mathscr{L}$ is trivial but the G-module structure is given by a character χ, then the G-module structure will be twisted by χ upon tensoring with $\mathscr{L}$.

5.3. *0-Gerbe twisting: Ramond twist*

By definition, a 0-gerbe is nothing but a line bundle on the stack and if we are dealing with a global quotient (X, G), using the assumption above, we get a trivial line bundle $\mathscr{L}$ on X, which is equivariant, but not necessarily equivariantly trivial.

If we fix a trivialization of the line bundle, *viz.* choose a global section v. This induces an isomorphism

$$\mu : \mathscr{L} \otimes \mathscr{L} \to \mathscr{L}; \quad v \otimes v \mapsto v. \tag{5.4}$$

The equivariance of this line bundle is expressed by isomorphisms

$$g^*(\mathscr{L}) \cong \mathscr{L}; \quad v \mapsto \chi(g)v; \quad \chi \in Z^1(G, k^*) = \mathrm{Hom}(G, k^*). \tag{5.5}$$

In terms of the twisting using μ, we can twist as described in the paragraph above. In this case, the G-action will be twisted by the character χ as will be the metric. This will "destroy" the properties of a pure G-Frobenius algebra (for instance, axiom T will cease to hold), but we will almost end up with a G-Frobenius algebra which is twisted by the character χ. This will indeed be the case, if we had started out with a χ^{-1} twisted Ramond model [28, 29]. In the current A-model setting, we will always have invariant metrics and strict self-invariance (axiom (T)). This type of twist is, however, very important in the B-model setting as it is not guaranteed that the objects have invariant pairings and self-invariance [28, 29, 31]. Hence we can view the 0-gerbe twisting as a twisting to the Ramond model and hence as a spectral flow [12, 28, 29, 31].

5.4. *1-Gerbe twisting: Discrete torsion*

This twisting has been investigated the most and goes under the name of discrete torsion. We shall disentangle the definitions so as to show that the resulting algebraic structure is that of [30]. This exposition owes a lot to [34, 23].

A 1-Gerbe $\mathscr{G}$ on (X, G) which is equivariantly pulled back from a point is given by fixing the (a) isomorphism $\mathscr{L}_g : g^*(\mathscr{G}) \xrightarrow{\sim} \mathscr{G}$ which are in turn given by line bundles $\mathscr{L}_g$ and (b) isomorphisms $\psi(g,h) : \mathscr{L}_g \otimes \mathscr{L}_h \to \mathscr{L}_{gh}$ which are associative.

Notice since the gerbe is trivial on X, so are the line bundles. In order to go on, we also choose sections s_g of $\mathscr{L}_g$. Then, in this basis, the morphisms $\psi(g,h)$ are given by their matrix entry $\alpha(g,h) \in Z^2(G, k^*)$. Notice that a different choice of sections changes α by a co-boundary.

Remark 5.4. Notice that the line bundles $\mathscr{L}_g|_{X^g}$ are actually $Z(g)$ equivariant line bundles. Furthermore, fixing the sections s_g we see that the isomorphisms are given by the characters $\epsilon_g(h) = \alpha(g,h)/\alpha(h,g)$ which are the famous discrete torsion co-cycles (see e.g. [30] for a full list of references). Furthermore, $\epsilon(g,h) := \epsilon_g(h)$ is even a bi-character when restricted to commuting elements (see e.g. [30]). This means that as $Z(m_1) \cap Z(m_2)$ modules $\mathscr{L}_{m_1} \otimes \mathscr{L}_{m_2}|_{X^{\mathbf{m}}} \cong \mathscr{L}_{m_1 m_2}|_{X^{\mathbf{m}}}$.

5.4.1. *Cohomology*

We can now set $\mathscr{H}^{\mathscr{G}}(X, G) := \bigoplus H^*(X^g, \mathscr{L}_g|_{X^g})$. For the multiplication, we can use the standard push-pull mechanism in a slightly modified version: for $v_{m_i} \in H^*(X^{m_i}, \mathscr{L}_{m_i}|X^{m_i})$

$$v_{m_1} *_{\mathscr{G}} v_{m_2} := \check{e}_{m_3 *}(\psi_*(m_1, m_2)_{X^{\mathbf{m}}}[e_1^*(v_{m_1}) e_2^*(v_{m_2})] \mathrm{Eu}(\mathcal{R}(\mathbf{m}))). \tag{5.6}$$

Notice that the result indeed lies in $H^*(X^{m_1 m_2}, \mathscr{L}_{m_1 m_2}|_{X^{m_1 m_2}})$ due to the projection formula.

Given the section s_g, we get isomorphism of the $\lambda_g : H^*(X^g, \mathscr{L}_g|_{X^g}) \xrightarrow{\sim} H^*(X^g)$ additively and this induces a new twisted multiplication on $A := \bigoplus H^*(X^g)$ via

$$\begin{aligned} v_{m_1} *_\alpha v_{m_2} \\ &:= \lambda^{-1}_{m_1 m_2} \circ \check{e}_{m_3 *}(\psi_*(m_1, m_2)|_{X^{\mathbf{m}}}[e_1^*(\lambda_{m_1}(v_{m_1}))e_2^*(\lambda_{m_1}(v_{m_2}))]\mathrm{Eu}(\mathcal{R}(\mathbf{m}))) \\ &= \alpha(m_1, m_2)\, v_{m_1} * v_{m_2}. \end{aligned} \tag{5.7}$$

That is the algebraic twist of [30] and Sec. 5.7 above.

Of course, we could have alternatively discussed the Chow ring A^* in the same way.

5.4.2. *K-theory I: Twisted multiplication*

In the case of K-theory using the standard formalism, we will obtain morphisms

$$K(X^{m_1})_{\mathscr{L}_{m_1}|_{X^{m_1}}} \otimes K(X^{m_2})_{\mathscr{L}_{m_2}|_{X^{m_2}}} \to K(X^{m_1 m_2})_{\mathscr{L}_{m_1 m_2}|_{X^{m_1 m_2}}}. \tag{5.8}$$

Considering the direct sum of twisted K-theories

$$K^{\mathscr{G}}_{\mathrm{global}}(X, G) := \bigoplus_{m \in G} K(X^m) \otimes \mathscr{L}_m, \tag{5.9}$$

we hence obtain a multiplication using the push-pull formalism of Eq. (4.7) analogously to the above.

By choosing sections, we again get a twisted version of the multiplication

$$\mathcal{F}_{m_1} *_\alpha \mathcal{F}_{m_2} = \alpha(m_1, m_2)\, \mathcal{F}_{m_1} * \mathcal{F}_{m_2}, \tag{5.10}$$

where a different choice of sections results in a change of α by a co-boundary.

Remark 5.5. There are several aspects, though not all, of the considerations above which have been previously discussed and also there have been related discussions which we would like to address briefly:

- It was shown in [5] that the *additive* α twisted K-theory of (X, G) as defined via projective representations is given by $K^\alpha \cong \bigoplus_{[g]}(K(X^g) \otimes \mathscr{L}_g)^{Z(g)}$ where $\mathscr{L}_g$ was considered as a G module via the discrete torsion co-cycle $\epsilon(g, h) = \alpha(g, h)/\alpha(h, g)$ for $[g, h] = e$. There is no obvious multiplicative structure on this space as remarked in [5], but the formalism above does give it a multiplicative structure.
- We would also like to note that in [32], an additive theory for a gerbe twist was constructed and it was shown that in the case of a global quotient with a gerbe pulled back from a point the gerbe twisted K-theory and the Adem–Ruan twisted theory as cited above coincide.
- Our geometric twisting above coincides with the algebraic twisting of GFAs considered in [30, 25] — see Sec. 5.7 below. Hence, the formula above and the Chen character of [25] answer the question of Thaddeus [34] about the relation of the two types of possible twists by line bundles in Cohomology *versus* K-theory.

5.4.3. *K-theory II: Twisted K-theory*

Another standard thing to do with a flat gerbe, that is a 2-cocycle $\theta \in H^2(Y, k^*)$ is to regard the twisted K-theory $K^\theta(Y)$. In our case of a global orbifold, given α as above we will study the twisted equivariant K-theory $K^\alpha_G(X)$ which by definition is the twisted K-theory of the stack $K^\alpha([X/G])$.

In this interpretation, one cannot see any type of multiplication. It is basically the same problem as in the case of a 0-gerbe. The natural product goes from $K^\alpha(Y) \otimes K^\beta(Y) \to K^{\alpha\beta}(Y)$. We will get back to this in the 2-gerbe twisting.

5.4.4. *Twisted group ring*

It is again useful to look at the details in the case of (pt, G). Here $K^\alpha([\mathrm{pt}/G]) = \mathrm{Rep}^\alpha(G)$ that is the ring of projective representation with cocycle α.

On the other hand, the global orbifold K-theory with an α twist $K^\alpha_{\mathrm{small}}(\mathrm{pt}, G) = k^\alpha[G]$ and the G invariants by the conjugation action are isomorphic to $\mathrm{Rep}^\alpha(G)$ [26].

Here, the multiplication is the one in $k^\alpha[G]$ which is just the one of $k[G]$ twisted by α.

5.5. *2-Gerbe twisting*

Finally, we wish to discuss twisting by a gerbe of the type $\beta \in Z^3(G, k^*)$. This type of gerbe is also the one we used to twist the Drinfel'd double and indeed there is a connection.

We can transgress the equivariant 2-gerbe to an equivariant 1-gerbe $\mathscr{G}$ on $\mathfrak{I}X$ and actually even to a 1-gerbe over $(I(X, G), G)$. Here the gerbe is characterized by a set of line bundles, which provide the isomorphisms $\mathscr{L}_{g,x} : x^*(\mathscr{G}|_{X^g}) \xrightarrow{\sim} \mathscr{G}|_{X^{x^{-1}gx}}$ together with associativity isomorphisms $\theta_g(x, y) : \mathscr{L}_{g,x} \otimes \mathscr{L}_{h,y} \to \mathscr{L}_{g,xy}$ if $g = x^{-1}gx$.

The condition of coming from a 2-gerbe expresses itself in a constraint on the θ_g. In particular, it means (see e.g. [35]) the θ_g are given by Eq. (2.3).

5.5.1. *2-Gerbe twisted K-theory I: Twisting on $K_{\mathrm{global}}((I(X, G), G))$*

Now, we are in a situation in which we can twist.

First of all, there is a naïve twisting on $K_{\mathrm{global}}((I(X, G), G)$ by the various θ_g transgressed from β; see Sec. 5.7.2 below where we give a more detailed description of this type of twist. In the case of (pt, G) with G Abelian, this yields a geometric incarnation of $D^\beta(k[G])$. In the general group case, we get a Morita equivalent subalgebra just as in the untwisted case.

5.5.2. *2-Gerbe twisted K-theory II: Twisting on $K_{\mathrm{full}}([X/G])$*

More importantly, however, there is a twisting for the full K-theory.

Definition 5.6. Given $\beta \in Z^3(G,(k^*)$, we define the twisted full K-theory $K^\beta_{\text{full}}([X/G])$ using the co-product and the obstruction, that is, the multiplication which is induced by

$$\mathscr{F}_g \cdot \mathscr{F}_h := e_{3*}(e_1^*(\mathscr{F}_g) \otimes^\gamma e_2^*(\mathscr{F}_h) \otimes \mathrm{Obs}_K(g,h)). \tag{5.11}$$

See [25] for details on how this global formula relates to the inertia stack setting.

Here, we use the co-product in $D^\beta(k[G])$ which is given by γ defined above by Eq. (2.5) to define the action of $Z(g,h)$ on the tensored bundle. This means that if for $x \in Z(g,h)$ $\phi_x : x^*(\mathscr{F}_g) \to \mathscr{F}_g$ and $\psi_x : x^*(\mathscr{F}_h) \to \mathscr{F}_h$ are the isomorphisms given by the equivariant data, then the isomorphism of $x^*(e_1^*(\mathscr{F}_g) \otimes^\gamma e_2^*(\mathscr{F}_h)) \cong e_1^*(\mathscr{F}_g) \otimes^\gamma e_2^*(\mathscr{F}_h)$ is chosen to be $\gamma_x(g,h)\phi_x|_{X^{g,h}} \otimes \psi_x|_{X^{g,h}}$ where γ is defined by Eq. (2.5).

Remark 5.7. For an interesting, different and independent approach, we refer the reader to [6]. Here, the authors consider a twist which is on the full K-theory of the inertia stack $K_{\text{full}}(\mathfrak{I}\mathfrak{X})$ and does not seem to use a co-product structure. The latter is key to the braided associativity.

5.6. *Case of a point*

Restricting to a point, we obtain the analog of Theorem 4.13:

Theorem 5.8. *$K^\beta_{\text{full}}([X/G])^\beta \cong D^\beta(k[G])$.*

Proof. Analogous to Theorem 4.13, using the calculations of [17, 14, 13]. □

Remark 5.9. Here, we see that we essentially get the [14] realization of the 2D calculation of [13], which is astonishing and inspiring. Using this insight, we can also understand why the twist already works on the level of the global quotient stack itself. The point is that applying the full stringy K-theory functor already entails moving to the inertial stack. This can be interpreted as moving to the loop space and hence evaluating the correlation functions on $\sigma \times S^1$, *viz.* the procedure described in [14]. This explains why the $(1+1)$-dimensional theory has the flavor of a $(2+1)$-dimensional theory.

Remark 5.10. This theorem is mathematically astonishing in the sense that the resulting structure is neither commutative nor associative in general. *We will get an essentially non-associative algebra unless $\beta \equiv 1$.* But it is of course associative and commutative in the sense of braided monoidal categories. We hope that we have motivated the appearance of braided monoidal categories already through the definition of Frobenius traces and objects. Moreover, if one reads for instance Moore and Seiberg's work on classical and quantum field theory, one sees that the fusion ring is actually not expected to be associative and commutative. However, the fusion and braiding operators satisfy pentagon and hexagon relations. Only the dimensions of the intertwiners lead to such an algebra on the nose. In case of the objects themselves, one should actually expect that one has to go to the braided picture.

5.6.1. *Verlinde algebra*

We can get an associative algebra by introducing a basis of irreducible representations $V_i, i \in I$ and using the dimensions of the intertwiners as the structure coefficients. That is, if $V_i \otimes V_j = \bigoplus_k V_{ij}^k \otimes V_k$, where V_{ij}^k is the space of intertwiners or multiplicity, set $c_{ij}^k = \dim(V_{ij}^k)$. Then, the Verlinde ring is just $k[v_i, i \in I]$ where the v_i are now formal variables with the multiplication $v_i v_j = \sum_k c_{ij}^k v_k$.

5.7. *Algebraic twisting*

In this section, we give a purely algebraic version of the twistings. This allows us among other things to connect the 1-gerbe twistings to the discrete torsion twistings used in [30, 25].

5.7.1. *Algebraic twisting I: Discrete torsion*

We briefly recall the twisting by discrete torsion in the G-Frobenius algebra case.

In [30], we defined the twisting of G-Frobenius algebras via

$$A \rightsquigarrow A^\alpha := A \widehat{\otimes} k^\alpha[G]. \tag{5.12}$$

This provides an action of the group $Z^2(G, k^*)$ on the set of GFAs. Notice that two twists A^α and A^β are isomorphic if and only if $[\alpha] = [\beta] \in H^2(G, k^*)$. It is clear that this extends to G-Frobenius algebra objects.

Proposition 5.11. *The algebraic twist and the geometric twist coincide, that is for $\alpha \in Z^2(G, k^*)$*

$$(\mathcal{F}_{\text{stringy}}(X, G))^\alpha = \mathcal{F}^\alpha_{\text{stringy}}(X, G). \tag{5.13}$$

Proof. Straightforward from the definition and Sec. 5.4. $\square$

5.7.2. *Algebraic twisting II: Twisting on $I(X, G)$ and the second appearance of the twisted Drinfel'd double*

Notice that by Proposition 4.5 $K_{\text{global}}(I(G, X), G)$ splits as a direct sum of rings indexed by $g \in G$, each of which is a $Z(g)$-Frobenius algebra. If G is Abelian, then all the $Z(g) = G$. It is hence possible to twist each G-Frobenius algebra separately by a discrete torsion $\theta_g \in Z^2(G, k^*)$. In the non-Abelian case, the twists cannot be chosen arbitrarily, since they have to be compatible with the G action that acts by double conjugation. This means that one has the free choice of a twist for each conjugacy class $[g]$, that is, co-cycles $\theta_g \in Z^2(Z(g), k^*)$, such that θ_g and $x^*(\theta_{xgx^{-1}})$ are cohomologous for all $x \in X$.

In this situation, we can also ask that the θ_g be even more coherent, that is, that they stem from a $\beta \in Z^3(G, k^*)$). In this case, we basically obtain an identification of $K_{\text{global}}(I(\text{pt}, G), G)$ with the Drinfel'd double.

Definition-Proposition 5.12. For $\beta \in Z^3(G, k^*)$

$$K^{\beta}_{\text{global}}(I(X,G),G) := \bigoplus_{g\in G} K^{\theta_g}_{\text{global}}(X^g, Z(G)) \tag{5.14}$$

$$= \bigoplus_{g\in G} K_{\text{global}}(X^g, Z(G))\widehat{\otimes} k^{\theta_g}[Z(g)] \tag{5.15}$$

$$= K_{\text{global}}(I(X,G),G)\widehat{\widehat{\otimes}} D^{\beta}(k[G]). \tag{5.16}$$

Proof. The proposition part is Eq. (5.16). In view of Proposition 5.11, this follows from the fact that the components (g, x) of K_{global} are only non-empty if $x \in Z(g)$. The restriction to the corresponding subspace of $D^{\beta}(k[G])$ is given by $\bigoplus_g k^{\theta_g}[Z(G)]$. □

Corollary 5.13. *There is an action of* $Z^3(G, k^*)$ *on* $K_{\text{global}}(I(X,G),G)$ *obtained by tensoring with* $\widehat{\widehat{\otimes}} D^{\beta}(k[G])$.

Proof. Directly from the above and Lemma 2.13. □

We can thus twist $K_{\text{global}}(I(X,G),G)$ via the procedure above and hence have a completely analogous story to the twists of $K_{\text{global}}(X,G)$ by discrete torsion analyzed in [30], but now one gerbe level higher.

If $K_{\text{global}}(I(X,G),G)$ is free in the sense that all the $Z(g)$-Frobenius algebras are free, we can further DPR algebra induce as discussed in Sec. 3.5.

Theorem 5.14. *We have the following identifications*:

$$\left(K^{\beta}_{\text{global}}(I(\text{pt},G),G)\right) = D^{\beta}(k[G])^{\text{comm}} \tag{5.17}$$

and

$$\text{Ind}^{\text{DPR}}\left(\text{K}^{\beta}_{\text{global}}(\text{I}(\text{pt},\text{G}),\text{G})\right) = \text{D}^{\beta}(\text{k}[\text{G}]). \tag{5.18}$$

Proof. Straightforward computation. □

5.7.3. *Algebraic twisting III*

In contrast to the previous twistings, the full orbifold K-theory twisting cannot just be reduced to an algebraic twisting. This can only be done additively in general. In the trival G-action case however, the twists by $\beta \in Z^3(G,k^*)$ again have a purely algebraic description.

Proposition 5.15. *Given a global quotient stack $\mathfrak{X} = [X/G]$ and a class $\beta \in Z^3(G,k^*)$, we have* additively

$$K^{\beta}_{\text{full}}(\mathfrak{X}) := \bigoplus_{[g]} K^{\theta_g}[X^g/Z(g)] \tag{5.19}$$

$$= \bigoplus_{g} (K_{\text{global}}((X^g, Z(g)))\widehat{\otimes} k[Z(g)])^{Z^g} \tag{5.20}$$

but the multiplication is the one defined by Eq. (5.11).

Proof. This follows from the fact that additively $K_H(Y) \cong K_{\text{global}}((Y,H))^H$. □

5.7.4. *Trivial action case*

In the case of a trivial G-action, the multiplication becomes particularly transparent.

Theorem 5.16. *Let $\mathfrak{X} = [X/G]$, where X has a trivial G action, then*

$$K^{\beta}_{\text{full}}(\mathfrak{X}) \cong \bigoplus_{[g]} K(X) \otimes \text{Rep}^{\theta_g}(Z(g)) \tag{5.21}$$

$$\cong K(X) \otimes \text{Rep}((D^{\beta}(k[G]))), \tag{5.22}$$

where in the last line the algebra structure is the tensor product and in the second line we have the following multiplication:

$$\mathscr{F}_g \otimes \rho * \mathscr{F}_h \otimes \rho' := \mathscr{F}_g * \mathscr{F}_h \otimes \rho * \rho', \tag{5.23}$$

where $\mathscr{F}_g * \mathscr{F}_h = \mathscr{F}_g \otimes \mathscr{F}_h \in K(X)$ and $\rho * \rho'$ is induced by $\text{Res}^{\mathrm{D}^{\beta}(\mathrm{k[G]})}_{\mathrm{Z(gh)}}(\text{Ind}^{\text{DPR}}(\rho) \otimes \text{Ind}^{\text{DPR}}(\rho'))$ *using the braided structure of $D^{\beta}(k[G])$ and Theorem* 2.7.

Proof. First we calculate using that for a trivial action all $X^g = X$:

$$K^{\beta}_{\text{full}}(\mathfrak{X}) = \bigoplus_{[g]} K^{\theta_g}[X^g/Z(g)] \tag{5.24}$$

$$\cong \bigoplus_{[g]} K^{\theta_g}_{Z(g)}(X) \tag{5.25}$$

$$\cong \bigoplus_{[g]} K(X) \otimes \text{Rep}^{\theta_g}(Z(g)) \tag{5.26}$$

$$\cong K(X) \otimes \bigoplus_{[g]} \text{Rep}^{\theta_g}(Z(g)) \tag{5.27}$$

$$\cong K(X) \otimes \text{Rep}((D^{\beta}(k[G]))), \tag{5.28}$$

where the second line is by Grothendieck, the third line follows from e.g. [5, Lemma 7.3] and the fourth line uses Theorem 2.7.

Now, for the multiplicative structure, we notice that since the inclusions e_i are all the identity, on the factors of $K(X)$ the multiplication boils down to the tensor product, whereas the product on the representations rings goes through the induction process and uses the co-cycle γ. This of course is nothing but the description in terms of $D^\beta(k[G])$. □

5.8. *Alternative description using modules*

Theorem 5.16 above can nicely be seen in the module language.

5.8.1. *Equivariant K-theory in the module language*

We first recall the setup of G-equivariant K-theory in terms of modules. See [1, 7]: $K_G(X) \cong B_{\text{proj.,fin. gen.}}$-mod where B is $C^\infty(X) \rtimes G$ with the multiplication $(a, g) \cdot (a', g') = (ag(a'), gg')$ and the modules are projective finitely generated.

In order to twist with a 1-gerbe $\alpha \in Z^2(G, k^*)$ following Atiyah–Segal, we give a new multiplication on B via

$$(a, g) \cdot (a', g') = (ag(a'), \alpha(g, g')gg'). \tag{5.29}$$

We call the resulting ring B^α. Now the twisted K-theory is given by the projective finitely generated B^α-modules.

The naive tensor structure which sends the α twisted K-theory times the β twisted K-theory to the $\alpha\beta$ twisted K-theory uses the A module structure induced by the multiplication map $A \otimes A \to A$ and the co-product $\Delta : k[G] \to k[G] \otimes k[G]$ given by $\Delta(g) = g \otimes g$.

Remark 5.17. In the algebraic category, we can use $\mathcal{O}_X$ instead of $C^\infty(X)$.

5.8.2. *Remarks on the 2-gerbe twisted case*

For the 2-gerbe β twisted K-theory, we can describe the K-theory additively as follows. Let $A_g = C^\infty(X^g)$, let the θ_g be defined via Eq. (2.3) and we define $B_g^{\theta_g} = A_g \rtimes Z(g)$ with the multiplication as in Eq. (5.29). Set $B = \bigoplus_{[g]} B_g^{\theta_g}$, then additively $K^\beta_{\text{full}}([X/G]) \cong B_{\text{proj.,fin. gen.}}$-mod.

To describe the multiplicative structure, we would have to define a co-product on B which incorporates the G-grading, the obstruction and the twisting. This should also be possible for a general stack or groupoid and a 2-gerbe. The full analysis is beyond the scope of the present considerations, but we plan to return to this in the future.

In the special case of a trivial G action, the construction has can be made fully explicit.

5.8.3. *Trivial G-action*

In the trivial G action case, like the case $[\mathrm{pt}/G]$, there are no obstructions and we can give a full description of the theory in module terms: Let $A = C^\infty(X)$ *and* assume X has a trivial G action then $C^\infty(I(X,G)) = \bigoplus_{g\in G} A$. Now although the G action on X is trivial, it is not trivial on $I(X,G)$, since it permutes the components.

It is easy to check that in the trivial G action case, the algebra is

$$B = \bigoplus_{[g]\in G} C^\infty(X) \rtimes k[Z(g)] \sim_{\mathrm{Morita}} A \otimes D(k[G]), \tag{5.30}$$

where the product structure on the K-theory is given via the co-multiplication of $D(k[G])$.

Similarly, twisting with β we obtain the following.

Proposition 5.18.

$$B^\beta \sim_{\mathrm{Morita}} A \otimes D^\beta(k[G]),$$

and the product structure on the K-theory of projective finitely generated B^β-modules is given via the co-multiplication of $D^\beta(k[G])$.

By considering the braided category projective finitely generated B^β modules, we hence obtain a generalization of the Theorem of $[\mathrm{pt}/G]$ to the case of a trivial G-action which is analogous to Theorem 5.16.

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References

[1] M. F. Atiyah, *K-Theory,* (W. A. Benjamin Inc., New York-Amsterdam 1967).
[2] D. Altschuler, A. Coste and J.-M. Maillard, Representation theory of twisted group double, *Ann. Fond. Louis de Broglie* **29**(4) (2004) 681–694.
[3] D. Abramovich, T. Graber and A. Vistoli, Algebraic orbifold quantum products, in *Orbifolds in Mathematics and Physics* (Madison, WI, 2001), Contemporary Mathematics, Vol. 310 (American Mathematical Society, Providence, RI, 2002), pp. 1–24.
[4] D. Abramovich, T. Graber and A. Vistoli, Gromov–Witten theory of Deligne–Mumford stacks, preprint, math.AG/0603151.
[5] A. Adem and Y. Ruan, Twisted orbifold K-theory, *Commun. Math. Phys.* **273**(3) (2003) 533–556.
[6] A. Adem, Y. Ruan and B. Zhang, A stringy product on twisted orbifold K-theory, preprint, math.AT/0605534.
[7] M. Atiyah and G. Segal, Twisted K-theory and cohomology, in *Inspired by S. S. Chern*, Nankai Tracts in Mathematics, Vol. 11 (World Scientific Publishing, Hackensack, NJ, 2006), pp. 5–43.
[8] P. Bántay, Orbifolds and Hopf algebras, *Phys. Lett. B* **245** (1990) 477–479.

[9] P. Bántay, Orbifolds, Hopf algebras, and the Moonshine, *Lett. Math. Phys.* **22**(3) (1991) 187–194.
[10] P. Bántay, Symmetric products, permutation orbifolds and discrete torsion, *Lett. Math. Phys.* **63**(3) (2003) 209–218.
[11] W. Chen and Y. Ruan, A new cohomology theory for orbifold, *Commun. Math. Phys.* **248**(1) (2004) 1–31.
[12] R. Dijkgraaf, E. Verlinde and H. Verlinde, Topological strings in $d < 1$, *Nuclear Phys. B* **352**(1) (1991) 59–86.
[13] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, The operator algebra of orbifold models, *Commun. Math. Phys.* **123** (1989) 485–526.
[14] R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, *Commun. Math. Phys.* **129** (1990) 393–429.
[15] L. Dixon, J. Harvey, C. Vafa and E. Witten, Strings on orbifolds, *Nuclear Phys. B* **261** (1985) 678–686.
[16] L. Dixon, J. Harvey, C. Vafa and E. Witten, Strings on orbifolds II, *Nuclear Phys. B* **274** (1986) 285–314.
[17] R. Dijkgraaf, V. Pasquier and P. Roche, Quasi hopf algebras, group cohomology and orbifold models, *Nucl. Phys. Proc. Suppl.* **18B** (1990) 60–72.
[18] D. S. Freed, Higher algebraic structures and quantization, *Commun. Math. Phys.* **159**(2) (1994) 343–398.
[19] B. Fantechi and L. Göttsche, Orbifold cohomology for global quotients, *Duke Math. J.* **117** (2003) 197–227.
[20] D. S. Freed, M. J. Hopkins and C. Teleman, Twisted K-theory and loop group representations, preprint, math.AT/0312155.
[21] D. S. Freed, M. J. Hopkins and C. Teleman, Loop Groups and Twisted K-Theory II, preprint, math.AT/0511232
[22] D. S. Freed and F. Quinn, Chern–Simons theory with finite gauge group, *Commun. Math. Phys.* **156**(3) (1993) 435–472.
[23] N. Hitchin, Lectures on special Lagrangian submanifolds, in *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds* (Cambridge, MA, 1999), AMS/IP Studies in Advanced Mathematics, Vol. 23 (American Mathematical Society, Providence, RI, 2001), pp. 151–182.
[24] T. Jarvis, R. Kaufmann and T. Kimura, Pointed admissible G-covers and G-equivariant cohomological field theories, *Compositio Math.* **141** (2005) 926–978.
[25] T. Jarvis, R. Kaufmann and T. Kimura, Stringy K-theory and the Chern character, *Inv. Math.* **168** (2007) 23–81.
[26] G. Karpilovsky, *Projective Representations of Finite Groups* (Dekker, 1985).
[27] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics, Vol. 155 (Springer-Verlag, New York, 1995).
[28] R. M. Kaufmann, Orbifold Frobenius algebras, cobordisms, and monodromies, in, *Orbifolds in Mathematics and Physics*, eds. A. Adem, J. Morava, and Y. Ruan, Contemporary Mathematics, Vol. 310 (American Mathematical Society, Providence, RI, 2002), pp. 135–162.
[29] R. M. Kaufmann, Orbifolding Frobenius algebras, *Int. J. Math.* **14** (2003) 573–619.
[30] R. M. Kaufmann, The algebra of discrete torsion, *J. Algebra* **282** (2004) 232–259.
[31] R. M. Kaufmann, Singularities with symmetries, orbifold Frobenius algebras and mirror symmetry, in *Gromov-Witten Theory of Spin Curves and Orbifolds*, Contemporary Mathematics, Vol. 403 (American Mathematical Society, Providence, RI, 2006), pp. 67–116.

[32] E. Lupercio and B. Uribe, Gerbes over orbifolds and twisted K-theory, *Commun. Math. Phys.* **245**(3) (2004) 449–489.
[33] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Commun. Math. Phys.* **123**(2) (1989) 177–254.
[34] M. Thaddeus, to be published in Lectures Notes in Mathematics (Springer, forthcoming).
[35] S. Willerton, The twisted Drinfeld double of a finite group via gerbes and finite groupoids, preprint, arXiv:math/0503266v1.