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# Math by Pure Thinking: R First and the Divergence of Measures in Hegel's Philosophy of Mathematics

Ralph M. Kaufmann and Christopher Yeomans

*Abstract:* We attribute three major insights to Hegel: first, an understanding of the real numbers as the paradigmatic kind of number (which also accords with their role in physical measurement); second, a recognition that a quantitative relation has three elements (the two things being related and the relation itself), which is embedded in his conception of measure; and third, a recognition of the phenomenon of divergence of measures such as in second-order or continuous phase transitions in which correlation length diverges (e.g., the critical point of water at which the reciprocal size of the droplets diverges). For ease of exposition, we will refer to these three insights as the R First Theory, Tripartite Relations, and Divergence of Measures. Given the constraints of space, we emphasize the first and the third in this paper.

## 1. Introduction

In this essay, we will discuss the mathematical concepts analyzed by G. W. F. Hegel in his *Logic*, particularly as presented in the *Encyclopedia*.[1] In crucial respects, he is foreshadowing many mathematical concepts around real numbers, relations, and topology (which are in modern times described by set theory), predating by several decades the usual foundations laid by Cantor and Dedekind and used today. In this respect, we agree with Reinhold Baer:

> For mathematicians of subsequent generations much in Hegel appears fuzzy that was simply the highest wisdom of his time, but a closer look reveals here and there that Hegel, had he not so staunchly believed his contemporaries but rather thought his thoughts straight through to their conclusions, would have anticipated many of the highest achievements of the last 80 years (Baer 1932: 5).

What is even more striking is that Hegel does this not using symbolic notation but rather conceptual thought, which proceeds by using 'definitions whose content is not accepted merely as something that we come across, but is recognized as grounded in free thinking, and hence at the same time as grounded within itself' (§99Z). Our goal is to match these lines of thought to mathematical concepts which are current today. In this fashion, we hope to elucidate the complex thoughts that Hegel presents to us in a more modern language. By interpreting this text in a more formalized mathematical setting, we also wish, vice-versa, to underscore his

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fundamental insights and intricate expression of these concepts in natural language without symbolic notation. For the reader with a background in mathematics, this presents a nice opportunity to gain insight into the thought process which goes into these constructions, but which has been axiomatized away by more modern mathematicians, leaving only the names (such as class or relation) as a reminder. The astonishing outcome is that Hegel's ideas represent formal mathematical and physical notions concerning real numbers, arithmetic, phase transitions, and their properties very well. He furthermore penetrates several technical subtleties by pure reasoning.

With respect to this comparative background, we attribute three major insights to Hegel: first, an understanding of the real numbers as the paradigmatic kind of number (which also accords with their role in physical measurement); second, a recognition that a quantitative relation has three elements (the two things being related and the relation itself), which is embedded in his conception of measure; and third, a recognition of the phenomenon of divergence of measures such as in second-order or continuous phase transitions in which correlation length diverges[2] (e.g., the critical point of water at which the reciprocal size of the droplets diverges). For ease of exposition, we will refer to these three insights as the **R** First Theory, Tripartite Relations, and Divergence of Measures. Given constraints of space, we emphasize the first and the third in this paper.

Concerning the **R** First Theory, Hegel foreshadows the development of real numbers. Real numbers were only first rigorously defined in the early 1860s, so it is not surprising that Hegel uses neither that terminology nor the specific mathematical devices and notation that were developed later. But Hegel's 'quantity' and its associated logical categories represent another terminology for describing many of the mathematical features that since the 1860s have been taken to be essential to analysis and number theory and to the theory of the reals in particular. So, on our view, it is not just that Hegel's conceptual analysis of quantity provides insight into the significance of real numbers that is lost with axiomatization but also that we can gain insight into the significance and importance of Hegel's discussion of quantity when we see him as attempting to conceptualize a logical object that more modern mathematicians take to be the real numbers.[3]

Traditionally, real numbers are associated with a line **L**, and Hegel takes this as a starting point. Actually, Hegel is more careful, since we do not encounter lines directly but starts with space. On the contemporary mathematical understanding of space, its key properties are that it is homogeneous and that we can measure inside it. The measurement of the distance between two points then naturally yields the restriction to a line following classical geometry. But to simplify the exposition and highlight the intuitive stakes of the argument, we will treat Hegel as beginning with **L**. From this starting point, we can map the train of thought in the *Encyclopedia Logic* onto subsequent developments in mathematics after Hegel. This double development can be summarized as a kind of progression in a U shape moving from the geometric concept of a line **L** to the axiomatic reconstruction of it as the real numbers or, with more technical precision, one-dimensional affine space:[4]

(1) Start out with the line **L,** which has certain properties like homogeneity and completeness.
(2) Identify it with numbers (let's call them measurement numbers or *Maßzahlen*).
(3) Realize that by picking a unit, the natural numbers appear. These unit natural numbers carry the natural arithmetic operations of addition, multiplication, and their inverses, which makes them a measurement copy of the rationals **Q** .[5]
(3′) Make the identification of the unit natural numbers with counting numbers (*Zählzahlen*) and construct the rationals **Q** (*Rechenzahlen*) by the same arithmetic operations thereby identifying the two versions of rationals (i.e., the measurement and counting rationals).[6]
(2′) Axiomatically introduce new numbers to **Q** in order to complete it to **R**.[7]
(1′) Show that these numbers are complete and homogeneous and hence give a model for the line. Technically, one introduces the affine line.

| | | | |
|---|---|---|---|
| 1 | | | 1' |
| 2 | | | 2' |
| | 3 | 3' | |

If one just regard the ascending part of the 'U'—i.e., 3' ➔ 1′—the axiomatic construction is independent of geometry and intuition, which is stressed by Dedekind (1963: 37). Nevertheless the conceptual origin of the reals is geometric (Dedekind 1963: 9, Dominguez 1999: 140), which is represented in Hegel in the descending part of the 'U', i.e., 1 ➔3. Furthermore, in this transition from the descending to the ascending side, we have 'lost' the actual nature of the line by replacing it with an axiomatic mathematical concept. As Gauss puts it:

> According to my innermost conviction, the theory of space (*Raumlehre*) has a priori a completely different position in our knowledge as the pure theory of magnitudes (*Grössenlehre*); our cognizance of the former absolutely lacks that complete conviction of its necessity (as well as its absolute truth) that is proper to the latter. We must humbly admit that if number is a mere product of our minds, space also has a reality outside our minds to which we cannot completely prescribe its laws a priori (Gauss 1863/1929: Vol. 8, 201).[8]

Hegel sees the potential for this loss (though his understanding of the difference between space and magnitude is different from Gauss). His strategy, which we call the **R** First Theory, is to avoid it by elevating **L** to be the real representative of numbers (*Maßzahlen*) and then following the descending part of the 'U' to construct all the other numbers (*Zählzahlen* and *Rechenzahlen*).

The text is organized as follows. The main part of the text is the second section in which we follow Hegel's discussion in *Encyclopedia Logic* (as supplemented by the *Science of Logic*) and represent his exposition in modern formal terms. Then, a brief section follows which summarizes the philosophical significance of Hegel's understanding as reflected through the lens of contemporary developments. Finally, we add a section about the mathematical concepts for the reader's reference. In the main text, we relegate more specific technical details to the footnotes.

## 2. Interpretation of §§99—111 of the *Encyclopedia Logic* (Quantity and Measure)

### *2.1. § 99*

In order to measure something, we need to abstract from all other aspects except for the quantity that we are measuring, and this abstraction is central to Hegel's definition of quantity in this section: '*Quantity* is pure being in which determinacy is posited as no longer one with being itself, but as *superseded* or *indifferent*.' In doing so, we isolate and hence separate this quantity from the object. The object needs to retain its identity regardless of the outcome of the measurement, viz. the magnitude (see Hegel's first remark to this section). As a concept, quantity pertains to existing objects but disregard their qualitative identity. Moreover, in obtaining a number or quantity, one has to disregard the differences of the constituents one measures, sums, or integrates (the constituents are thus considered as indifferent ('*gleichgültig*')).[9]

Thus, though Hegel clearly starts his discussion with neither measure nor measurement, he nonetheless starts with a feature of quantity that is essential for measurability. One basic premise that we encounter in today's view of physics is that observable quantities are measured by real numbers. Fixing such a quantity (e.g., force, magnetic field, spin, and so on) by performing a measurement of it on a given object yields its magnitude (*Größe*), which is a specific real number (i.e., a '*determinate* quantity').[10] As Hegel puts it in his lectures, 'when we employ quantitative determinations in our observation of the objective world, it is in fact always already measure that we keep in mind as the aim of such employment' (EL§106Z). In the wider context of Hegel's thinking about quantity, he makes the **R** First Theory even more explicit in his discussion of mechanics in the philosophy of nature: 'The true philosophical science of mathematics as the theory of magnitude (*Größenlehre*) would be the science of measure; but this presupposes already the real particularity of things, which is first available in concrete nature. But owing to the *external* nature of magnitude it would be perhaps the most difficult science' (EN§259R). In this respect, the broader argument from quantity to measurement is to show that the determination of magnitudes is ultimately parasitic on the determining of magnitude. One measures certain quantities and obtains a real number that is a real quantity. Now, the result of the measurement is the magnitude, which is a specific real number. The quantity that is measured does not depend on the outcome of this particular measurement, thereby making its existence and definition independent of the process and the outcome.[11] Moreover, intrinsically, such a quantity may take on different values, as determined by a measurement, without changing its identity. So, paradoxically, the independence of quantity from any particular measurement is essential to the role that quantity plays within measurement.

These two forms of indifference—of the measured object to its quantity and of the quantity itself to its specific value at a given time—are emphasized by Hegel (in reverse order) in an introductory paragraph added in the second edition of the *Science of Logic*: 'such a limit, the indifference of the limit as limit and the indifference of the something to the limit, constitutes the *quantitative* determinateness of the something' (WL 21:173). The indifference of the object to its quantitative determination is primarily spelled out in this and the next section of the *Encyclopedia*, on pure quantity; the second indifference of the quantity to its own value is spelled out in the following discussion on quantum. In modern mathematical language, the first indifference is homogeneity; the second is the continuity of variation that is essential to completeness. Since these two features are the crucial features of affine space, this shows Hegel has begun his discussion of quantity with a logical structure that contemporary mathematicians would consider a formal geometrical concept of **L**.

Hegel's remark and much of the addition to this section can be thought of as relating to the real numbers, since they deal with particular features that are distinctive of the real numbers. So, for example, the fact that real numbers are ordered and one can freely move to bigger or smaller values is deemed to be crucial ('was *vermehrt* oder *vermindert* werden kann').[12] And indeed, one of the trademarks of real numbers is that they are an ordered field. This entails the operations mentioned above, but additionally an order, that is the relation <, as well as compatibility between the two, which means that the order preserves the homogeneity.

Furthermore, in the remark, Hegel gives another example of the indifference or *Gleichgültigkeit* of quantity in claiming that this is what we have in mind when we understand the absolute as matter. In modern terms, this can be summarized in the fact that the set of real numbers **R** is homogeneous with respect to addition—in fact it is a principal homogeneous space for the operation of addition (see §4). This pertains to the role of the real numbers as measuring inside time **R** and space, which is $\mathbf{R}^3$.[13] Non-technically, 'homogeneous' means that 'it is the same everywhere'. Mathematically, this is given by invariance of structures under shifts, called translations, and stretching, called dilations. Space is homogeneous under the continuous action of addition or rather translation and multiplication or rather dilation. One could call this feature of the reals the indifferent absolute, as Hegel does ('*das Absolut-Indifferente*').[14] Hegel thus rightly realizes that there is a way to 'find' the reals starting from space (i.e., 1 ➔ 2 in the 'U'). Interestingly enough, the concept of magnitude in space involves assigning a real number to distances, which is invariant under translation and rotation. This means that these magnitudes disregard all differences between individual different points. Nevertheless, there is a quantity given by two points, which is their difference.[15]

In Hegel's words, this expresses that the reals themselves can be considered as examples of quantity but only as *pure quantity*. It may be interesting to remark that here, Hegel foreshadows the fact that if one generalizes (as Cantor did) the quantity of natural numbers as the cardinality of finite sets to the cardinality of the reals, one obtains a new concept not useful for measurement. The questions occurring in measurement are not 'How many points in an interval?' or 'How long are the reals?' but 'How long is an interval?'[16] Indeed, Hegel operates with a provisional definition according to which the absolute is pure quantity. Translated to this situation, one could say that the reals are not to be measured in a physical way, but they are underlying measurement: '*Otherwise* pure space, time, etc. could be taken to be examples of quantity, insofar as the real is supposed to be grasped as an *indifferent* filling of space or time [***Sonst*** *können der reine Raum, die Zeit usf. als Beispiele der Quantität genommen warden …*] (emphasis ours).' Again, this makes the point that Hegel's argument has contemporary significance with respect to the internal connection between measurement and real numbers, even though the category of pure quantity itself is identical neither with measurement nor with real numbers, each of which is a more concrete concept. In this opening section of Hegel's discussion, pure quantity is identifiable with homogeneity in the contemporary mathematician's sense.

## *2.2. § 100*

Here, Hegel seems to struggle to combine the two faces of the reals that come out in the two definitions reviewed in more technical detail in §4, below. First, the reals are a continuum or the continuum, which is represented by drawing the real number line. In mathematical terms, this feature is captured by the axiomatics of Dedekind cuts as a method of constructing **R**. This, however, says nothing a priori about their algebraic structure, such as addition and multiplication, which are introduced in a rather complicated fashion. On the other hand, they can be constructed as a completion of the rational numbers **Q**, which makes their algebraic properties easier to understand.[17] A basic feature of the algebraic construction is that it starts with the natural numbers, which are discrete (so are the rationals when viewed from an arithmetic angle).[18] Hence in modern axiomatics, we construct something continuous (**R**) from something discrete (**Q**). The remnant of discreteness is apparent in the embedding of the natural numbers into **R** and to its fullest extent given by viewing **R** simply as a set of points underlining the line with arithmetic operations. Here is the apparent paradox. It is appropriate that the paradox would come up at this point, because here Hegel is attempting to understand the second form of indifference noted in the previous section—i.e., the indifference of a quantity to its own determinate magnitude. This mode of indifference tracks the modern mathematical characteristic of continuity of variation (essential to the completeness of **R)** which is, in turn, precisely what both constructions, methods of **R** discussed in this paragraph, attempt to demonstrate.

These two aspects of **R** are similarly emphasized in the greater *Logic*: quantity is at first 'a limit which is just as much no limit … the repulsion of the many ones which is immediate non-repulsion, their continuity' (WL 21:173) and in more detail in the section on pure quantity: '*Continuity* is therefore simple, self-same reference to itself unbroken by any limit or exclusion – *not*, however, *immediate* unity but the unity of ones which have existence for themselves. Still contained in it is the *outside-one-another* of *plurality*, though at the same time as something without distinctions, *unbroken*' (WL 21: 176).

Hegel expresses the first aspect of the paradox in the greater *Logic* by saying that we understand the very relation itself between continuity and discreteness first as a kind of continuity and then as a kind of discreteness. It is precisely by these explicit modes of understanding that we generate *magnitude* rather than simple *quantity*, which is a way of putting into pure thought the way that the two construction methods outlined above (Dedekind cuts and algebraic construction from **Q**) each generate a distinct conception of the concept of **R** that emphasizes different aspects of **R**. The former emphasizes the continuous nature of **R** presupposed by its role in physical measurement; the latter emphasizes the conceptual relations between distinct numbers and therefore articulates the non-immediate aspect of number and arithmetic. But Hegel is at pains to say that both sides are present in both continuous and discrete magnitude, with the difference between the two lying in which is emphasized and which remains implicit (WL 21:189–90). Of course both constructions are isomorphic and in the end contain both arithmetic operations of a field and a metric with respect to which they are complete.[19]

Hegel resolves this nicely by calling these aspects rather than species of magnitude.[20] In modern terms, we could say that if we look at a natural number, it is a multiple of 1 and hence made up out of units.[21] On the other hand, when embedding **N** into **R,** a natural number is just a real number and hence part of the continuum and thus 'infinitely divisible'. [22] Hegel's example, in the Addition of people in the room or the division of space into '*Raumpunkte*' is right along this line. Mathematically, this is captured by the embedding of $\mathbf{Z}^3$ (i.e., the points of $\mathbf{R}^3$ with integer coordinates) into the ambient $\mathbf{R}^3$, (i.e., space).[23] In elaborating in the greater *Logic*, Hegel says that in contrast to a false naturalism that would attempt to build up reality by a process of '*composition* [Zusammensetzung]' of many exclusive points, 'Mathematics, on the contrary, rejects a metaphysics that would make time *consist* of points of time; space in general, or the line in the first instance, of points of space … It gives no credit to such discontinuous ones' (WL 21: 178).

To connect this distinction between continuous and discrete magnitudes to the 'U', we can say that it gets at one of the fundamental differences between (1) and (1′). That is, another reason for the disparity between the real numbers just as a set, say of Dedekind cuts (i.e., (1′)), and the number line **L** (i.e., (1)), is that the latter is continuous and the former is without any extra data discrete. In order to reintroduce this feature, one has to put a topology on **R**, and this is a choice to be made by the mathematician. There are indeed two natural choices, the discrete topology in which every number is separate from every other and the metric topology, which captures the continuity. This construction is again courtesy of a metric or a distance function, which are mathematical versions of measurement. Hegel realizes that the possibility of what contemporary mathematicians would describe as the choice of a topology (metric or discrete) is contained in the concept of numbers and by making them aspects of numbers hence foreshadows this development in topology as well. In Hegel's terms, these are the two sides of the same coin given by repulsion and attraction: 'As the proximate result of being-for-itself, quantity contains within itself as ideal elements both sides of its process (repulsion and attraction); hence it is both continuous and discrete' (EL§100Z).

*2.3. §101*

A nice mathematical version of quantum as 'quantity essentially posited with the excluding determinacy that it contains …; limited (*begrenzte*) quantity' is as follows. First, if we have a set X, then any posited subset A of X gives rise to the complement X\A of A in X and A=X\(X\A), i.e., the 'excluding determinacy'. Now, in order to be '*begrenzt'*—that is bounded—one needs a notion of distance or a metric on the set. Per Hegel, this should be an ordered one ('was vermehrt oder vermindert werden kann'). Typical bounded sets on the reals, as given by their bound, are open and closed intervals [*a*,*b*] and (*a*,*b*). There are more bounded sets, in the technical sense, but if one would use the bound as a defining property, these are the natural ones.[24] This dichotomy between open and closed intervals and their boundaries foreshadows the difference between intensive and extensive magnitudes and the need for a limit.

Following up on the need for a metric, we can understand Hegel's quantum as simply a unit of measurement, in the modern mathematical sense of that term. In the greater *Logic*, he identifies it from the first sentence of the section with number (*Zahl*) (WL 21: 193). In modern terms, this can be incorporated into the fact that the reals viewed as the real line only are an affine space and hence admit a scaling and translation action. In order to break the translation action, one uses a fixed point, which is usually taken to be 0. In order to break this scaling action, one needs a unit. Incidentally, the unit is usually taken to be 1, which has a certain arithmetic property as mentioned above.

Though the discussion in the lesser *Logic* is very short, this basic idea is filled out in Hegel's expression in the greater *Logic* that quantum or number under the aspect of unit ('das Eins') is '(α) *self-referring*, (β) *enclosing*, and (γ) other-excluding limit' (WL 21:194). That is, the unit (α) characterizes its own extent qua continuous; (β) ties all other natural numbers together in a determinate pattern in virtue of attributing to them a determinate multiple of its own continuous extent; and (γ) breaks the scaling action precisely by being different from other possible units among those that it characterizes in its second aspect. [25]

A good mathematical model of Hegel's 'unit' is the subset of affine space which is between two points, say O and P. Then, we can identify the line with **R** by sending O to 0 and P to 1, fixing both the origin and the unit. Now the quantum for Hegel is the unit interval [0,1]. As a subset of **R**, it is unit that is *das Eins* but also a set, which is possibly the union of some other sets into which it can decompose ('*zerfällt*'). As we shall argue below, it is even more the unit vector, which can be defined as the equivalence class of this interval under the action of translation and the additional choice of an orientation, that is a magnitude and a direction. Fixing this identification with **R** once and for all the other quanta can be thought of as vectors, which can be represented by some interval [0,a].[26] This then fits with the notion of negative quanta, which are briefly—almost parenthetically—introduced by Hegel; these are the same quanta, only with the opposite direction.

Philosophically speaking, this shows Hegel at work on a flat ontology in which the category of significance is a unit with the same ontological status as those it is used to measure. Materially speaking, this unit could be something like the standard meter bar in Paris or the subsequent re-definitions of the meter in terms of the wavelength and then speed of light. In these cases, something serves as a standard for other things of the same sort (i.e., physical phenomenon). But what is important is the way it shows how many complex and reflexive conceptual relations have to be built into such a flat ontology, even prior to the introduction of distinct levels such as essence/appearance, substance/accident or universal/individual/particular. Paradoxically, the benefit of exploring such a flat ontology is that it is precisely there that issues of units of significance must be dealt with directly, as it were, rather than demoting them to considerations of appearances or subjective takes on the objective. This brings out the way in which Hegel's philosophy of mathematics is also an internal criticism of materialism for underestimating the magnitude of the challenge generated by its ontological commitments.

Mathematically speaking, this connection between limit and unit is an important one and helps to show why it is wrong to think that Hegel either did[27] or should have[28] started from **N** on his way to **R**. The connection is represented here by the fact that limit is essential to *Dasein*, and Hegel writes that 'Quantum is the *existence* (Dasein) of quantity' (§101Z) and that quantum is 'limited quantity'. The term '*begrenzt*' foreshadows the limiting thing, the limit or '*Grenze*' which will be used in §103. Limits *in general* exist in **R** but not in **N** or **Q**, and the existence of limits is the key characteristic of **R**, so here we find evidence that Hegel is looking to find that conceptual structure that contemporary mathematicians understand to be **R**.[29] The limiting process is also, in turn, presupposed by measurement. Following the explicit dialectic of Hegel's text, we come to quantity out of concern to determine the status of the limit that is constitutive of quality, which is the ontological correlate of the physicist's concern to have existent limits for measurement.

*2.4. §102*

Here, Hegel draws out the aspect of number that was implicit in the description of number as '(β) *enclosing*', above, i.e., that there must be a multiplicity to be enclosed. So whereas the unity ('*Einheit*') represents number under the aspect of continuity and thus emphasizes those aspects of number also highlighted by the construction of **R** by Dedekind cuts, amount (('*Anzahl*') or, as di Giovanni translates it 'the how many times') represents number under the aspect of discreteness and thus emphasizes those aspects of number highlighted by the constructions of **R** out of the completion of **Q**. These come out in his construction of the different forms of calculation.

Breaking the symmetry of the translation action by picking an origin and a unit one can identify the continuum as a line with the real numbers.[30] Now as numbers, the reals have certain properties. A quick mathematical construction starts from the natural numbers. Hegel conceptualizes the arithmetic operations on the natural numbers that are passed on to the reals in 3′ ➔ 2′ in modern (axiomatic) terms by using counting as the basic operation. This counting can be viewed as counting under a concept which can be viewed as an early version of arithmetic of sets. Indeed, this is the route that Cantor would go. The natural numbers can be taken to be the cardinalities for finite sets. The operations on numbers then correspond to operations on sets. Addition corresponds to the disjoint union, while multiplication is achieved by taking the Cartesian product. Finally, and this differs slightly from Hegel's description, powers can be computed as the cardinality of the set of maps.

It is interesting to see that Hegel constructs the natural numbers as one would still do in set theory today.[31] The presented argument is that as soon as we have a unit, we have the set containing this unit $u=\{1\}$. In order to count up one number, we add a disjoint union with this set $u$. The number two is the set $u \sqcup u$, three is $u \sqcup u \sqcup u$, and so on. This is precisely '*Numerieren* (numbering)' as Hegel describes it. This operation turns the unit into multiples of itself. In terms of the equivalence classes of intervals, this corresponds to taking the union of copies of these by putting them next to each other. Interestingly enough, Hegel does not use this geometric version but rather makes the transition from (2) to *Zählzahlen* (counting numbers), or (3′) in our 'U'.[32]

This is then the way to understand a curious feature of Hegel's treatment of quantity noted by Terry Pinkard, which is his use of both *das Eins* and the usual *die Eins* (1981, 456–7).[33] Furthermore, even '*die Eins*' is used in an unusual way, i.e., as a plural rather than a feminine singular. *Das Eins* develops out of Hegel's previous discussion of quality and represents the qualitative standard implicit in quantity that is ultimately made manifest and comprehensible through the idea that quantities are measurements. *Die Eins* is a quantitative unit only in virtue of being the plurality of something repeated, which Hegel emphasizes with the plural verb forms. This is to run in reverse the identity statement Hegel quotes from Zeno in EL§104R: 'it is the same thing to say something *once* and to be saying it *always*.' Thus, Hegel's plural '*die Eins*' names the same concept as the modern mathematician's singular feminine '*die Eins*'; only by saying a quantitative unit always is it even said once. Perhaps, the clearest statement of this conceptual connection comes from the greater *Logic*'s transition between quality and quantity: 'Plurality is not at first posited otherness; limit is only the void, only that in which the ones [*die Eins*] *are not*. But in the limit they also *are*; they are in the void, or their repulsion is their *common connecting reference*' (WL 12:158). Because the plurality of the one is equal to its repetition, saying '*die Eins*' in the plural is equivalent to saying '*die Eins*' in the singular, which is the unit.[34]

More to the current point, the set theoretical interpretation of *Numerieren* brings out the ancient insight that once you have one—even in just a qualitative sense (i.e., *das Eins*)—you are close to two and therefore **N**. So *Numerieren* as the progressive disjoint union of units is the mechanism by which quality turns into quantity. Thus, the fact that the element of discreteness tracks the construction of **R** out of **N** is connected to Hegel's unique way of framing the problem of understanding quantity in terms of the nature of the limit, as was discussed above in §1.3. Counting becomes counting in terms of a measured unit. This is the transition from 2 ➔ 3, i.e., picking a unit from $\mathbf{R}_m$ (measurement numbers or *Maßzahlen*) defines a copy of **N** inside $\mathbf{R}_m$. Here, the copy is just given by $u$, $2\underline{u}$, $3u$, … with $nu + mu = (n + m)u$. For this step, the unit $u$ is arbitrary. It could be 1, or it could be $2\boldsymbol{\pi}$, which would be useful for measuring sin, cosin, and angles. More precisely, the choice of $u$ defines 1, which is made explicit in the move from 3 ➔ 3′. In 3′, whatever unit was picked in 2 ➔ 3 becomes 1, which is used essentially in 3′ ➔ 2′.[35] Hegel is building into his notion of quantity a flexibility of different units for different kinds of investigations that nonetheless still demarcate real limits (e.g., phase transitions). This is connected to Hegel's claim that quantity represents 'the ubiquitous *real possibility* of the one' (WL 21: 177) rather than either a fixed unit or the merely logical possibility of the subjective division of the number line, a point that he emphasizes in his discussion of Kant's second antinomy and Aristotle's superior resolution (WL 21: 179–189).[36]

Often one considers the set $\mathbf{n} = \{1,2,\ldots,n\}$ as a set theoretic incarnation of the number $n$ which is indeed a representation of the above disjoint union. Indeed, the set **n** has the cardinality $n$.[37] The standard construction of set theory then gives the arithmetic operations exactly as Hegel proceeds. Addition is disjoint union ('*Zusammenzählen*'). Here, there is a subtlety in the definition of disjoint union, which Hegel grasps when he writes that 'numbers are *immediately* and *at first* completely undetermined numbers in general, and therefore **unequal** in general' (bold emphasis ours). Mathematically speaking, if we would take the simple union, we would not arrive at counting, since $u \cup u = \{1,1\} = \{1\} = u$. Before putting the two elements 1 into a common set, we have to distinguish them—this is exactly the definition of the disjoint union.

The multiplication of two numbers in this setting is given by the cardinality of Cartesian product: $n \times m = Card(\boldsymbol{n} \times \boldsymbol{m})$. As an example, take $2 \times 3 = \mathrm{Card}(\mathbf{2} \times \mathbf{3}) = \mathrm{Card}(\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}) = 6$. This makes the commutativity of the operation obvious. Hegel realizes that this product can be viewed in two different ways. $Card(\mathbf{2} \times \mathbf{3}) = Card(\{(1,1),(1,2),(1,3)\} \sqcup \{(2,1),(2,2),(2,3)\}) = Card(\mathbf{3} \sqcup \mathbf{3})$, where now we have two as *'Anzahl'* and **3** as *'Einheit'* or $Card(\mathbf{2} \times \mathbf{3}) = Card(\{(1,1),(2,1)\} \cup \{(1,2),(2,2)\} \cup \{(1,3),(2,3)\}) = Card(\mathbf{2} \sqcup \mathbf{2} \sqcup \mathbf{2})$, where now the roles are reversed. This becomes particularly apparent if one uses a matrix array to enumerate the elements,

$$\begin{matrix} (1,,1) & (1,,2) & (1,,3), \\ (2,,1) & (2,,2) & (2,,3), \end{matrix}$$

where now one has the choice to view the rows as '*Anzahl*' and the columns '*Einheit*' or vice versa. In the greater *Logic*, Hegel uses this convertibility to explain the phenomenon we noted just above, i.e., the possibility of different units of counting: 'Since the limit is external, the breaking off point, how much is to be taken, is something contingent [*Zufälliges*], arbitrary [*Beliebiges*]. – The difference between amount [*Anzahl*] and unit [*Einheit*] that emerges in each species of calculation grounds a *system* of numbers (dyadic, decadic, and so forth); any such system rests totally on arbitrariness, on which amount is taken to count as the constant unit' (WL 21, 198).

In order to obtain the squaring operation, Hegel suggests resolving this dichotomy by taking the '*Anzahl*' and '*Einheit*' to be equal, these yielding squares. The further power operations are then derived by iteration. The modern way to realize power operations $n^m$ is to take the cardinality of the set of maps from the finite cardinal **n** to the finite cardinal **m** $Map(\boldsymbol{n}, \boldsymbol{m}) =: \boldsymbol{n}^{\boldsymbol{m}}$ as explained in §3 This does exhibit the equality or sameness of the factors in the power operations, but it also introduces a new '*Anzahl*' and type of '*Einheit*', namely the enumeration of the factors.[38]

The rest of §102 introduces the inverse operations. Here, one should be careful. Inverting addition yields the integers, and then, further inverting the multiplication, one constructs the rational numbers. Now there is a slight deviation, since there is no general inversion of taking roots, and furthermore, taking all possible roots leads to the algebraic closure of **Q** rather than **R**. Nonetheless, this shows that, beginning from **R,** Hegel has produced **N**, and **Q**, obtained by negative operation, by development of aspects of **R** (i.e., 2 ➔ 3). One deficiency at this point is the precise analysis of these inverse operations. The main complication is that for division, unlike for subtraction, there is no way of getting around looking at fractions as equivalence classes (although of course negative numbers are clandestinely also classes). Hegel does take up the complication of equivalence classes in §105–107, which can be used to explain how the result of adding negative operations is a copy of **Q**: since the copy of the natural numbers depended on the choice of a unit inside $\mathbf{R_m}$ so will the copy of the rationals. At this point, Hegel could have alternatively introduced the denominators as new units. That is, ¾ is 3 times the unit ¼, where ¼ is the unit into which 1 decomposes, as is allowed by the formalism. In fact, we still call the pieces of the fraction numerator and denominator. This however would necessitate a further argument about changing units while retaining the underlying quanta. This change may actually not only to be to fractions of the original but to any arbitrary other unit, i.e., any dilatation of **R**. This is postponed, due to its deeper philosophical implications, to the later paragraphs.

Another important aspect for Hegel hidden in these short lines is that he now actually introduces all of what we now think of as **R** by introducing a negative quantum. Up to this point, one could have objected that the result of measurement or the quanta are always positive so that we are only looking at the non-negative reals.[39]

### 2.4.1. Zusatz

The Addition to this section can be read as distinguishing between the affine space given by space or the continuum and its discrete structure as a group. But since number qualifies both given its twin aspect, there is further evidence that what contemporary mathematicians see as two different ways of defining **R** Hegel sees as two different aspects of number, which is necessary to close the U from 3 to 3′. [40]

In the greater *Logic*, he address this by objecting to the idea that geometry and arithmetic deal with fundamentally different objects rather than the same objects in different ways (WL 21, 196–7), and the subsequent development of a *discrete* topology would appear to confirm Hegel's position here.

### 2.4.2. Summary

At this point, let us pause to summarize the development in terms of the U shape that we introduced at the beginning of the paper. In the terms of contemporary mathematics, there are three basic thoughts that run from the initial positing of quantity to the development of **Q**:

(1) Start out with the real line. This Hegel does in §99.
(2) Identify it with **R** by choosing a distance function (measurement in the modern mathematical sense of the term is what Hegel uses in §100) and discover abstract properties from it, most importantly homogeneity and continuity. This Hegel does in §100 through the introduction of the twin aspects of continuous and discrete quantity and in §101 through the excluding limit of quantum. This is to posit measuring numbers or *Maßzahlen*. As will become apparent in §§106–107 (discussed below), these quantities explicitly become *Maß* (measure) in Hegel's sense of the term when regarded as qualitative quanta.
(3) Realize that **Q** is contained in it by picking a unit and applying arithmetic operations to it. This Hegel does in §102 through the development of the arithmetical operations. We arrive here at calculating numbers or *Rechenzahlen*. At this point, Hegel has a unit-based copy of **N** and, by adding negative quanta, a copy of **Z**, and finally a copy of **Q** (by adding the negative operation of division). Mathematically, this is best described as the field of fractions of the ring **Z**—the set of integer numbers with the operations of addition and multiplication (i.e., with the negative quanta and operations formally added to it; see §3 for the mathematical details).

This, we think, Hegel shows quite admirably in pure thoughts without the use of symbolic notation. But Hegel does not yet show in §102 that the **Q** given by adding the negative operations is naturally again a subset of **R**, and thus he has not yet shown that calculating numbers (*Rechenzahlen*, i.e., what comes out of the arithmetic of counting numbers (*Zählzahlen*)) are measuring numbers (*Maßzahlen*) and vice versa. This is accomplished by the arguments in §§105–107.

In parallel but opposite logic, in contemporary mathematics, there is a corresponding *axiomatic* ascent:

(3′) Algebraically construct the rationals **Q**.
(2′) Axiomatically introduce new numbers to complete it to **R**.
(1′) Show that these numbers are complete and homogeneous and hence give a model for the line.

At this point, Hegel diverges fundamentally from the course that was taken by mathematics beginning in the 1860s. Rather than proceeding up the right side of the U to (1′), Hegel returns up the left side of the U, back through (2) to (1). Though a speculative point, obviously, it seems more than likely that Hegel would have had severe doubts about the possibility of success of (1′), i.e., of the ability of axiomatized mathematics to demonstrate that the **R** so constructed is the same measurable continuum with which one began. Even Gauß had his doubts here as evidenced by the quoted text. In Hegel's text, the grounds for this speculative point can be seen in his repeated insistence that the true nature of quantitative variability requires the inclusion within it of a qualitative criterion of significance. To give a brief roadmap of what follows in Hegel's discussion, he attempts to model this qualitative significance first within the notion of degree, then within the notion of relation but finally within the notion of measurement itself. Briefly, in each of these notions, more is allowed to vary and so one has a greater scope of arbitrary differentiation. More importantly, each notion adds new aspects to the mathematical objects already developed in the move from (1) to (3), thus revealing the fine structure of those objects but without the methods of axiomatization, which introduce new objects from (3′) ➔ (1′) which are now separate from the **L** of (1). By such means, Hegel is able to get **R** as the points on **L** in (1), which are naturally cohesive points (this you do not get in (1′), since without additionally adding the metric topology, it is discrete).

Though we will not attempt in any detail to correlate Hegel's moves to the axomatized progression from (3′) to (1′), we will still offer models using symbolic notation for the stages of Hegel's own return journey from (3) to (1). Obviously, Hegel neither had nor wanted the resources of symbolic notation to describe either his own route of development nor that of the (as yet still potential) development to which he would have objected, but we hope nonetheless to make clear the sense in which his rejection of that development would be a principled decision to understand the paradoxical significance of the very arbitrariness of quantitative variation. Nonetheless, what is striking and motivating for us is the degree and depth of the match produced by casting his arguments into the modern symbolic notation, thus showing that some mathematical developments since Hegel's time can be seen as solutions to problems that are nicely articulated in his purely conceptual exposition.

### *2.5. §103*

The differentiation of 'intensive' and 'extensive' magnitude can be interpreted in a mathematical fashion as follows. If we agree that a magnitude, which is the result of a measurement, is a real number, then there are two ways of thinking about it. The real number $r$ is either simply an element of the real line $r \in \mathbf{R}$ (intensive), or it represents the size of the interval $[0,r]$, which in itself is extended—the set as the collection of its members (extensive) and infinitely divisible. The latter conception brings out the discrete character of number as amount, but as Hegel puts it, 'the amount of one and the same unit' (WL 21, 209). The upper boundary of the interval $[0,r]$ is $r$, which as a point is not extended but as a real number nevertheless represents a magnitude ('*Größe*').[41] One member is made the name of the class (*a potiori fit denominatio*). These points of view are dual to each other (i.e., a determinate function connects the two) and indeed are how one today identifies **R** with its affine counterpart, the line (2′ ➔ 1′). Thus, Hegel says in the *Zusatz* that the intensive/extensive distinction characterizes the limit specifically, whereas the continuous/discrete distinction characterizes quantity in general (also WL 21: 208–9).

This idea of limit characterizing the number but not being the actual concept itself can be found in Dedekind cuts, where the real numbers are technically a partition of **Q** pair of subsets which are taken to represent their limit, as explained in the Appendix. [42]

### 2.5.1. Zusatz

In Hegel's time, it is indeed an astute observation that the analog measuring instruments do measure intensive magnitudes with extensive ones.[43] Modern science in the form of quantization has somewhat mediated and confounded Hegel's analysis; the paradigms here are the wave/particle duality and the positioning of momentum. Nonetheless, the example of temperature given here is a good image for degree (*Grad*) and further for measure (*Maß*). The thermometer measures the degrees of temperature, but in order to get a number, we have to fix a measure (say Celsius or Fahrenheit), and then, the measurement transforms an intensive magnitude into an extensive magnitude by using the column of mercury (or gas), for example. To read off, we take the limit of this extensive magnitude, which is the height of the column. In this way, one needs both a degree and a scale (measure), as one can even see in the ordinary notation, e.g., 5°C.[44] As an operational definition, we can say that Dedekind cuts are the limit which is produced by reading off measurements (e.g., the temperature from the thermometer).

## *2.6. §104*

Going back to the degree (*Grad*) being the upper bound of the interval, one could ask whether or not one is discussing the open $(0,r)$ or the closed interval $[0,r]$. The upper bound exists in the latter and not in the former. To reach it or determine it as such, one needs a limiting process.[45] The fact that this limit exists in the reals follows from the fundamental property of the reals to be complete. Thinking of $r$, which is the degree or the intensive representative of the (extensive) quantum $[0,r]$ as the upper bound, the degree is a limit. Indeed, taking the *supremum* means that we need to look at increasing sequences, whence Hegel's claim that the quantum needs to be able to be augmented or decreased. This yields one reading of Hegel's claim that the degree as the concept of quantum already contains the infinite regress, and explains why, in the attempt to understand the way in which the multiplicity of number can be represented in the single limit as simple in the greater *Logic*, intensive magnitude is taken to be a degree (WL 21: 210). In this step, Hegel realizes the discrete aspect of numbers as points but importantly cohesive points.

In order to take the limit to infinity, one would have to invoke further mathematical tools. There is a direct relationship between the interval $(0,r)$ and the real line. These two sets are in one-to-one correspondence or in bijection to each other and moreover are even homeomorphic, which means that there is an invertible map between them and both the map and its inverse are continuous. This preserves both the order and the topology. This means that taking the limit as $t \to r$ in $(0,r)$ and $t \to \infty$ in $\mathbf{R}$ are really the same type of process. This gives a mathematical model for understanding Hegel's claim that degree makes clear the way in which the '*indifference* [Gleichgültigkeit] of the [quantitative] determinateness constitutes its quality, i.e., the determinateness which is in itself as determinateness external to itself [*die an ihr selbst als die sich äusserliche Bestimmtheit ist*]' (WL 21: 211).

The same point can be made in a way that is perhaps a bit closer to the text as follows: the process of adding a quantum to itself repetitively leads to the limit $t \to \infty$ by alternating between taking the limit and adding a new extensive quantum. That is to take the limits $t \to r$ in $[0,r]$ yielding the intensive $r$, adding the quantum to itself by simply repeating it gives rise to the extensive [r,2r] then the limit $t \to 2r$ in $[r,2r]$ and so on. Here, the lower bound becomes the old degree, and to get to the upper bound, we need to use a limit on the extensive quantum. This mathematical model makes good sense of Hegel's otherwise puzzling claim that 'not only *can* every determinateness of magnitude be transcended, not only *can* it be altered: that it *must* alter is now *posited*. The determination of magnitude continues into its otherness in such a way that it has its being only in this continuity with an other; it is not just a limit that *exists* but one that *becomes*' (WL 21: 217). Here, the process of taking limits provides an interpretation of the notion of a becoming or processual limit that is definitional of the magnitude specified by that limit. Here, the variability of this inner processual limit is tied to the possible outer variability of transition to a lower or higher degree.[46] There are two more remarkable features. Hegel realizes that the alternating process is necessary to 'reach' or better define infinity. For a finite multiple, one could first add the interval finitely many times and then take the limit. This does not make sense in the infinite case right away, and this is discussed in the *Zusatz*, in particular, quoting the poem of Haller. The second point is that Hegel articulates the Archimedean property of the reals (i.e., that there is neither an infinitely large nor infinitely small member of $\mathbf{R}$), since this adding process should surpass any given real. It is also interesting to note that in this limit definition indeed the limit does not depend on the original quantum, which is very important.

## *2.7. §105*

Hegel's discussion of ratio is very close to the modern notion of rational numbers but is not quite the same. A mathematical construction of rational numbers defines them as pairs $(r,s)$ modulo the equivalence relation that $(r,s)$ is equivalent to $(p,q)$ if

$rq - ps = 0$. This means that we regard equivalence classes $[r,s]$ which are represented by pairs; this is why 2/4 = 1/2. It is not that the representatives are the same, they are merely equivalent, but the classes are indeed equal. Hegel views $r$ and $s$ as the two quanta, which are posited next to each other or the quantum posited upon itself (‘*an ihm selbst gesetzt.*’). This is the perfect description of the pair. The rational number defined by the ratio then is the *class* given by these two quanta. This is what Hegel calls the exponent. He rightly identifies that in order to give a rational number, one actually uses three quantities, the numerator, the denominator, and the rational number they represent. (This is the insight we referred to in our introduction as Tripartite Relations.) He moreover realizes that the exponent is the equivalence class of all the numbers that are in the same relationship, and indeed an equivalence relation on a set X is given by a relation which is a subset of X x X. In our case, $X = \mathbf{R} \times \mathbf{R}$, and the subset is given by pairs of pairs $((r,s),(p,q))$ such that $rq - ps = 0$, which means that $r{:}s$ and $p{:}q$ are in the same relation or *Verhältnis.*

Hegel’s verbal presentation of this schema is given in the greater *Logic*:

> the two moments *limit* themselves inside the exponent and each is the negative of the other, for the exponent is their determinate unit [*Einheit*]; the one moment becomes as many times smaller as the other becomes greater; each possesses a magnitude of its own to the extent that this magnitude is in it that of their other, that is, is the magnitude that the other lacks. The magnitude of each in this way contributes to the other *negatively*; how much it is in amount [*Anzahl*], that much it supersedes in the other as amount and is what it is only through this negation or limit which is posited in it by the other (WL 21, 315).

In this schema, $r$, $s$, $p$, and $q$ are whole numbers, and the rational numbers are present implicitly in the background as the structure of the relation between the classes of whole numbers. Here, we see already—in the return from (3) to (2) in the ‘U’—a building towards the two levels of ontology of Hegel’s Doctrine of Essence, and so it is not surprising that the variability present in the relation is both wider in scope and more deeply grounded in the nature of quantity than was the case in degree: both terms of the relation can change, but the exponent is itself held constant. Moving from *Verhältnis* to the exponent, Hegel has given the fine structure needed for rational numbers by adding negative operations. The relations are the basic negative operations, the exponent is the number. Up to this point, the exponents are not yet elements of **L**—one needs measure to achieve the latter.[47] The next two paragraphs yield this crucial step.

### *2.8. §106*

Measurements usually occur in units. This means that any magnitude will actually be a multiple of a standard quantity that is a ratio, the denominator being the unit. Fixing this unit, that is, one side of the ratio fixes a measure. This happens for instance when measuring length in feet or meters, but also for the real numbers, one may choose to measure them in units of 1 or $2\pi$.[48] Notice that here there is a difference between pure ratio and measure. Fixing the denominator fixes a unique representative for the ratio. In this sense, the denominator has a new quality to it, besides just being a number. Once we have this unit, we get a copy $\mathbf{N}_u$ of the natural numbers and a copy $\mathbf{Q}_u$ rationals, which are now a subset of **L**. We have used the subscript '*u*' to indicate the embedding of these numbers which depends on the choice of unit.

Thus, Hegel's new argument here is not just that what mathematicians now call the real numbers are best understood against the *background* of our use of them in practices of measurement but rather that they can *themselves* be understood as measures, which is made explicit in the following section. That in terms of which we measure can be thought of as the relevant quality, even if that measure can itself be given a numerical form. Thus in the greater *Logic*, Hegel writes that 'Qualitative quantity … in relating itself to another, becomes a quantitative specifying [*Specificieren*]' (WL 21: 329). In this sense, the quality articulates the aspect of significance of the measured thing that is picked out as motivating the quantitative determination. But Hegel makes the interesting point that only this introduction of the qualitative element first provides the resources to distinguish quantity from quality:

> it can seem at first as if magnitude were merely that which is alterable in general … But if that is the case, then magnitude would not be distinct from *Dasein* (i.e., the second stage of quality) since according to its concept it is equally alterable and the content of that definition would then have to be made complete in such a way that in quantity we have something alterable that despite its alteration remains the same (§106Z).

So the ratio discussed in this section is a first, distinctively quantitative way of framing the relation between quantity and quality as between that which is alterable as related to that which remains the same, where the numerator represents the alterable quantity and the denominator the stable quality or measure. But the true distinction between quantity and quality requires thinking of *quantity itself* as both changing and staying the same, which requires the internalization of the qualitative aspect within the quantitative. Put another way, the true distinction between quantity and quality requires that it be an internal distinction within quantity. So, as Hegel puts it in the next section, 'Measure is the qualitative quantum …'. It is this measure which explicitly lifts the pure numbers to their rightful status as results of measurement and in mathematical terms gives the embedding of the abstract **Q** given by the exponents in the measurement reals $\mathbf{R}_m$: 'By contrast, mere numerical findings as such, apart from the guiding interest which we have discussed here, rightly count as empty curiosities that satisfy neither a theoretical nor a practical concern.'

### *2.9. §107*

Here, Hegel looks at the more abstract structure of the concept of measure. It need not be that the unit or the measure is a natural number. In the physical setup, it would be a real number, so let us stay in this context. One way to look at the entities involved in forming a ratio (now of real numbers) is that this is a function $f$: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ which sends a pair $(r,s)$ to the ratio $r/s$. Fixing one side as the measure $l$, we obtain a function $f_l$: $\mathbf{R} \to \mathbf{R}$ by sending $r$ to $f_l(r){:=}f(r,l) = r/l$.[49]

We wish to use this formalism to express that the measure as 'qualitative quantity' can be viewed as this function $f_l$. Now, the measure is not only the exponent ('*zunächst als* unmittelbares, *ein Quantum*') but also a function indexed by it. The index $l$ is a quality of the function; without it, it ceases to exist. In this fashion, measure attains '*Dasein*' and 'quality,' since for Hegel, the very definition of quality is that its determinacy simply is its being (§90). In the greater *Logic*, Hegel reaches back to the notion of the exponent to articulate this qualitative quantity as a double move: 'In specifying measure … the quantum is taken in one instance in its immediate magnitude, but through the exponent of the ratio is taken in a second instance in another amount [*Anzahl*]' (WL 21:334). In the function formalism, the immediate magnitude is represented by its first manifestation as the variable $r$ and the second instance by its manifestation in the ratio $r/l$.

Besides introducing *Dasein* and quality into the picture, which is very important for the physical and other applications that follow, this theory of changes of measure then completes the return to (1) in the 'U'. Here, for each unit **u,** we get all the exponent's relations with this unit, we get a copy $\mathbf{Q}_u$ of $\mathbf{Q}$, and thinking of the exponent in a real situation as the result of a measurement, we obtain the embedding into the measurement reals. Varying the unit and hence the index of the function above, we obtain the method to go from one copy of the rationals to another which are all equally valid. This establishes the homogeneity of the line. By the use of degree to understand points, the step from (2) back to (1) secures the cohesion of the points in **L** thought of as the degrees of a measurement in units and so making good on the aspect of continuity or attraction found in the notion of pure quantity in §100.

### *2.10. §108*

In the above model, we can change the index $l$ thereby changing $f_l$. If we change $l$ to some other $l'$, this will give a change of scale much like Hegel discusses in the addition to §107. The effect is that $f_l$ is changed to $f_{l'}$, which is a different function thus altering quality. The possibility of changing $l$ is inherent in it being a real number (quantum). Although the two functions do not coincide, they are fabricated by the same rule, which is fixing the second variable of $f$. The subscript notation commonly used in mathematics pays tribute to exactly this fact. The second variable is considered to be a parameter and thus fixed. But of course this parameter can be varied. One would say that the different $f_l$ belong to a family of functions. Each member of this family is formed by the same rule and changing from one to another does not change the general rule. In the greater *Logic*, Hegel uses temperature as an example of such a family in noting that temperature variations have to be understood in terms of the relation between the changes in temperature of different materials in the environment, each of which has its own specifying quantum or rule that defines their 'thermal capacities [*Wärme-Capacitäten*]' (WL 21:335). Thus, e.g., to understand the change in the temperature of the table in relation to the change in the temperature of the air requires grasping the relation as a 'relation of two qualities which are themselves measures' (WL 21:336).

#### 2.10.1. Zusatz

Leaving pure mathematics and passing to physics, one sees that certain phenomena are intimately related to the scale of the system. Here, Hegel makes an astute observation that is still valid today. Fixing a system, one fixes a rough scale. If one goes to the fringes of this scale, the observations lose their validity in as far as one will be observing different phenomena than one originally set out to do. Thus, the notion of scale is the idea that objects dictate their own measure, which thus constitutes a quality of the object. This feature of measure fixes the *asymmetry* required for measurement according to which one quantum counts as the amount that varies and another quantum counts as the unit by which the varying amount is measured, a point Hegel emphasizes in the greater *Logic* by distinguishing between *immediate* and *realized* measure (WL 21:336-9 and 341–4).

An excellent example invoked by Hegel is that by varying some parameters such as pressure or temperature, one can induce phase transitions. There are two types of this transition. In the first-order transition, there is an actual discontinuity in a quantity. In a second order or continuous phase transition, there is no discontinuity in the function measuring a particular quantity but rather in its derivative. It is in these continuous phase transitions that some form of measure breaks down. In modern theory, one likes to describe critical phenomena in terms of certain parameters called correlation lengths. This is the correct scale for the system. When this system becomes critical, this length diverges. A typical example is water vapor near the critical point. Here, the length can be taken as the reciprocal size of the droplets. Increasing pressure, the vapor becomes gas, and as the size of the droplets goes to zero, its reciprocal diverges. It is this type of situation that Hegel analyzes in the next paragraph.

### *2.11. §109*

The next paragraphs move in spirit from the purely mathematical to the physical. This is signaled in the greater *Logic* by Hegel's entitling of the corresponding section 'Real measure' and the extended discussions there of specific physical and chemical relations. Tying together a measure with an entity it measures, one can make quantitative changes which at a certain point may change the quality of the entity. Hegel makes this transition to real objects precisely from the notion captured above by the family of functions: 'Measure is now determined as a connection of measures that make up the quality of distinct self-subsisting somethings, or, in more common language, *things*' (WL 21:345). The strength of

Hegel's claim here should not be missed. He is not merely claiming that physical objects serve as nice examples for the category of real measure nor even that physical objects have a nature which is uniquely though contingently suited for understanding through practices of measuring but rather that having the kind of measure we have designated by the index of a function or the scale of a system in fact *constitutes* the nature of such real objects: 'By measures we no longer mean now merely immediate measures, but measures that are self-subsistent because they become within themselves relations which are specified, and in this being-for-self are thus a something, things that are physical and at first material' (WL 21:346). Here, the sense in which the multiplied quantitative relations build up into something approaching an essence is quite clear.

A good model for this part of Hegel's analysis is a continuous phase transition as mentioned above and used by Hegel himself in the *Zusatz*. Here, the basic underlying question is what exactly happens at the critical point of a continuous transition and how to explain the passage through this point. What happens is a change of measure: at the critical point, the old measure diverges, and this is the measureless ('*das Maßlose*'). Nevertheless, it should be possible to cross the phase line. For this, there should be some change of quantity, which induces this transition. Indeed, a mathematical description of a second-order phase transition is characterized by a discontinuity in the derivative. This means that there is a limit from both sides, but the limits may not match up in all aspects. Hence, as Hegel puts it, there is a limit of an 'infinite process' which leads to the singular discontinuous point ('*das Maßlose*'), but 'on the other side', there is another well-defined phase or measure. If a derivative function, in this case the measure, is discontinuous at a point, then at every other point, there is a little neighborhood where the measure is continuous (basically by definition). This means one can change the quantity without diverging. Nevertheless, one can take a limit to find out that at that particular point, the function is not continuous. As Hegel puts it in the greater *Logic*, 'the preceding quantitative relation, though infinitely near to the succeeding one, is still another qualitative existence [*Dasein*]' (WL 21:365–6).

### *2.12. §110*

At this critical point, something special happens, and here, there is a discussion of nodes and focal points ('*Knoten*'). In fact, in the greater *Logic*, Hegel uses the natural numbers as an example of such a nodal line: 'The system of natural numbers [*Das natürliche Zahlensystem*] already exhibits a *nodal line* of qualitative moments which issue in a merely external progression. In one respect, this progression is a merely quantitative running back and forth, a constant adding and subtracting, each number standing in the same *arithmetical* relation to the one preceding or following it as this last stands to the one preceding or following it in turn, and so on. But the numbers that thus arise also stand to the others that either precede or follow them in some *specific* relation, whether as a multiple of one of them expressed in the form of a whole number, or as a power or root' (WL 21:366).

In an optical node, the parallel rays converge so that the image at that particular point seems to vanish, but the diverging rays on the other side do yield an image. Also given a standing wave, such as in an organ, at the nodes, the waves cancel each other, and there is no movement. Nevertheless, these points are part of the wave. Since these are individual points, there is no intrinsic measure but a certain collapse.[50] This collapse is the unity of quantity and quality.

The node picture also reaffirms that Hegel thinks of the natural numbers as embedded in what mathematicians now call **R** and that their properties as counting numbers (*Zählzahlen*) and calculating numbers (*Rechenzahlen*) are derived from their position in **R**. It is through this embedding that the discrete has continuous properties, and these can be found in any neighborhood. An isolation of the discrete numbers results from focusing only on the critical point, but this is an abstraction from the fuller picture that includes the neighborhood.[51]

### *2.13. §111*

Reanalyzing the critical point, one can say that it contains in its neighborhood several aspects, phases, or qualities and hence is the unit of these. In the simple model of the discontinuous function, one has two limits, which do not coincide. At the limit point, one could take either definition as natural or both. There is another important aspect that Hegel understands and that is that one cannot just restrict oneself to the singular point. One needs to understand a neighborhood of it. There is a branch of mathematics called singularity theory, whose goal it is to analyze the just this type of situation. In Hegel's words, this is the point that the infinite process is in the nature of quantity. Keeping this in mind and focusing on critical points leads to the essence: different phases of matter (e.g., water) are in its essence, since these all would be present at the same time at the triple point. In the greater *Logic*, Hegel emphasizes the way in which this isolation of an essence independent of any exclusive qualitative state continues the independence of quantity from the quantified object discussed above in our §1.1: 'What we have here is … one and the same substantial matter which is posited as the perennial substrate of its differentiations. This detaching of being from its determinateness already begins in quantum in general … This reciprocal transition into the other of the qualitative and the quantitative moments occurs on the basis of their unity, and the meaning of this process is only the *existence* which is the *demonstration* or the *positing* that such a substrate does underlie the process and is the unity of its moments' (WL 21:370).

Here, we have Hegel's return to (1) but a return to a deeper level in which the line **L** looks like an abstraction from a more complexly qualitative continuum, one in which discontinuity is more deeply embedded than was realized at the outset. At this point, the completed detaching of determinateness from being suggests that ontological continuity must be understood as something that holds together quantitative and qualitative discontinuity, which is at first Hegel's notion of essence.

## 3. Philosophical Results

We conclude with a brief summary of the philosophical significance of the two primary themes we identified at the beginning.

### 3.1. *R First*

Hegel's presentation of the discussion of quantity in pure thoughts rather than symbolic notation has some advantages that might be envied by contemporary philosophers of mathematics, particularly when it comes to the real numbers. Consider the following contemporary description of developments since Hegel's time:

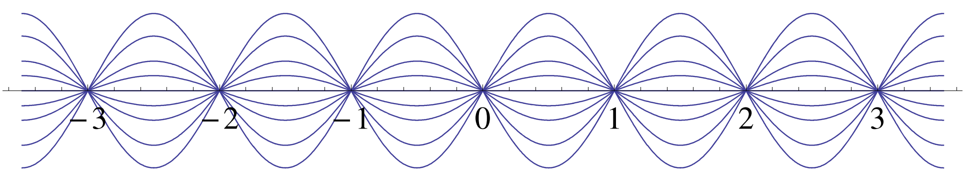
> Perhaps one of the chief items of pride of mathematical philosophy in the last century and a half is the insight that mathematics is the science of formal structures; as opposed to the traditional view, that 'the proper and exclusive subject matter of mathematics is … quantity.' … But the admirable doctrine of freedom of choice constrained only by consistency gives by itself no help in making choices, no hint of which structures, such as the Real numbers, are of central importance; indeed, contemporary mathematical philosophy tends to regard all such choices and distinctions as 'pragmatic,' beyond the pale of serious epistemology (Manders 1986: 253).

For all of the difficulties in its presentation, Hegel's view does have the virtue of explaining the paradigmatic or at any rate, central status of real numbers in a way that is consistent with our use of them to understand material nature through measurement but which nonetheless does not reduce to being a matter of utility or subjective need.[52] Showing that he does so has required us to make use of mathematical concepts and terminology that were not, of course, used by Hegel himself but this is not surprising given that he was writing roughly a half century before the first rigorous definition of **R**; what is more surprising is that he should have anything at all to say about this topic under these circumstances.

Another way of getting at this point is to return to our schema of the descent from (1) to (3) in thoughts and corresponding axiomatic ascent back to (1′). In (1), the line **L** is posited, or **R** lacking certain fundamental mathematical properties; in (1′) is posited **R** in all of its mathematical glory but no longer as the line (§1.4.2). As the intuitive picture of the number line and the presupposition of the continuity of the reals by measurement shows, there is a natural tendency to identify the **L** and **R**; this is precisely what Descartes does in the *Discourse on the Method* (see note 4). and Hegel in §102Z. But it is not obvious that there is any deeply mathematical reason to say that this is so, and thus to say that *Maßzahlen* (**L**) are the same as *Rechenzahlen* (**R** of (1′)), and Dedekind himself wants to deny this (1963: 37). What we see in Hegel's version of the ascent (in thoughts rather than axioms) is a principled reason to say that this is so, though this comes at the cost of both lack of precision and the lack of explicit construction of the completeness of **R** out of

**Q** as it is found in the return to (1′). Neither does he quite get the construction of **R** via Dedekind cuts, though he comes quite close to it in the transition from extensive to intensive magnitudes (as we discuss in our §1.5). In the end, Hegel has presupposed completeness in his beginning with **L** but never quite offered a principled reconstruction of it. The compensating gain for the lack of explicit construction is that he still has the concrete line as a model—thus a route towards the way in which the elaboration of the model generates physical, material things. In contrast, even with Dedekind cuts, real numbers become pairs of subsets of **Q**, not intuitive numbers or points on a real line. Hegel's version comes to fruition precisely in the discussion of measure in §§107–8 that we have interpreted using the modern notation $f_l$ and the related conception of a family of functions.

The connection between the **R** first theory and the divergence of measures is found in the idea of **N** as nodes in the continuum of **R**. This basic idea is fleshed out by the idea that such nodes are not just simple points but critical points at which there is a collapse of measure, such as the points at which standing waves cancel each other out:




So though Hegel has not come to (1′)—**R** as axiomatized—he does come back to (1)—here, **R** as the *x*-axis of the standing wave—with a much-expanded sense of its shape and of the qualitative variation within it. One could of course say that because Hegel does not use the term 'real number' or even get **R** as axiomatized by modern mathematics, he doesn't have a notion of real number at all. But this strikes us as putting too fine a point on the issue, since underneath the terminological absence is a detailed grappling with precisely the characteristics of what we now call the real numbers that make them distinctive and significant. And since precisely that significance is obscured by axiomatization but articulated by Hegel's conceptual thinking, such an interpretive thesis seems doubly unfortunate.

### *3.2. Divergence of Measures*

This function reading of Hegel on this point helps a great deal more than the traditional knotted-rope analogy to reveal the origins of the transition from an ontology of being to an ontology of essence in Hegel's *Logic*. The knotted rope retains the linear and one-dimensional quality that attends to being in Hegel's understanding. But the consideration of the relation between essence and appearance introduces the relevance of many different dimensions of patterns and their relations to each other. In this basic sense, a family of functions could be an essence that appears in particular functions, just as a particular function could itself be an essence relative to specific values it can take on. This insight is connected with the fact that **R** are higher-order operations (sequences or Dedekind cuts) from which **N** are a kind of abstraction, which is why Hegel thinks that **N** can be represented as the set of nodes. Furthermore, the idea of nodes as critical points where measure collapses gives a positive content to that collapse as related to its surrounding neighborhood, which helps to explain how the conceptual resources of the Doctrine of Being are maintained and further developed in the Doctrine of Essence rather than the latter simply starting afresh after the exhaustion of the former.

Thus, even though Hegel claims that logic is wasted on the youth because they have not yet had the requisite experience of the investigation of the world to see the significance of logical categories such as quantity, one needn't appeal to higher order phenomena of semantics or theory construction (both of which more properly belong in Hegel's *Realphilosophie*) to make sense of Hegel's discovery of a quality within quantity.[53] This follows on a point we noted in §2.3, which is the great internal complexity required by the flat ontology developed by Hegel in the Doctrine of Being precisely in virtue of the absence of either an underlying essence or an overlying subject.

Again, this may be seen as a virtue of Hegel's view in comparison with the contemporary approach. To quote Manders following up on the point made in §3.1:

> Closely associated with this insight is the distinction between pure mathematics, the beneficiary of the freedom conferred by the new status, and applied mathematics (in the philosopher's rather than the mathematician's sense of the word), which has been sent into philosophical limbo, supposedly under the care of philosophy of empirical science … As to [this] second 'insight,' the way the distinction between pure and applied math is drawn neglects the fact that 'applications' of mathematics (in the philosopher's sense) are typically to other mathematics, not to empirical science … Taking applications to empirical science as paradigmatic has blocked the idea that theories can be motivated by intended applications in an epistemologically significant way (Manders 1986: 253).

We find a deeper view in this aspect of Hegel's theory, and one can see it embedded in the double meaning of his dominant term for the characterization of quantitative differences, i.e., '*gleichgültig*'. On the one hand, this is usually translated as 'indifferent', and we have used it in this sense earlier. But on the other hand, it also has the etymological structure of 'equally valid', and it is in this sense that the internal complexity of the flat ontology is to be taken. The pragmatic appeals to which Manders objects make the motivation of theory something essentially arbitrary from the epistemological perspective and thus make different theoretical constructs equally *invalid* rather than equally valid. But on Hegel's view, quantifiable being is a structured plurality of *valid* constructs, each of which may be used to define the perspective from which the whole is considered.[54]

To take the example of affine space discussed in our §2.1, it is of course true that there is no single coordinate system that defines it, but it is nonetheless true that the space is defined by the rules of the translation actions that define the non-arbitrary and non-optional relations between different coordinate systems. That the reals are produced by the invariants of these homogeneous actions means that the reals themselves have a foundational role to play precisely as the structure of these relations, i.e., as specifying precisely the sense in which they are equally valid. Thus, there is at least one conception of quantity—what we now call **R**—that is not itself of equal validity as other conceptions of quantity (such as **Q** or **N**) precisely because it articulates the equal validity of different quantitative units of measures. So **R** is first not merely arbitrarily or pragmatically but necessarily and conceptually, precisely because it accounts for the arbitrary or pragmatic choice of units of measure in specific contexts. Arbitrariness itself has a conceptual structure, which is a conceptual point we are used to Hegel making in the philosophy of spirit's discussions of *Willkür*. But here, this is what it means for number to be 'thought as a being that is completely external to itself' (EL§104R) and why mathematics is the most difficult science (EN§259R).

## 4. Mathematical concepts

### *4.1. Real Numbers*

The real numbers **R** can be introduced basically in two fashions: Either as a completion of the rational numbers **Q**, or via an axiomatic system given by Dedekind cuts. In both cases, the starting point is **Q**, which can be algebraically constructed from the natural numbers, **N**. Mathematically, **Q** is the quotient field of the ring of integer numbers **Z**, and **Z** is the group completion of **N**.

There are several facts about the real number **R** that Hegel uses in his text.

(1) The natural numbers **N** are contained in **R**.
(2) **R** is a field that is we have the usual operations of addition and multiplication together with their inverses. There are also two special elements 0 and 1, which are the neutral elements for the addition and the multiplication, respectively.In particular, **R** is an Abelian group under addition, and $R\setminus\{0\}$ is also an Abelian group under multiplication. Here, a group is a set together with an operation (usually called +), which is associative, has a unit, and inverses. If the operation is commutative, the group is called Abelian.Not postulating inverses for a group, one arrives at a monoid.
(3) **N** is the submonoid generated by 1 that is all finite sums $1 + \ldots + 1$.
(4) **R** has an order <. That is for any two elements $a, b \in \mathbf{R}$, $a < b$, $a = b$ or $a > b$ and moreover if $a < b$ then $a + c < b + c$ and $ac < bc$ if $c > 0$.
(5) **R** is homogeneous. This can be viewed on several different levels, the most practical here would be to say that **R** with its additive structure is a Lie group.

This for instance means that the action of **R** on itself by addition is continuous. Here, the action is given by $\lambda(a)(r):=r+a$. Here, one views $\lambda$ as a map $\mathbf{R} \to Map(\mathbf{R}, \mathbf{R})$. This means to any $a \in \mathbf{R}$, one associates a function. This function is $\lambda(a)$, and the value of this function on $r \in \mathbf{R}$ is $r + a$.

(6) **R** has an action of the affine group $\mathrm{Aff}^1 = \mathbf{R}^* \rtimes \mathbf{R}$, which acts by $x \to ax + b$, for invertible $a$ and any $b$. I.e. translation by $b$ and dilatation or scaling by $a$.

(7) **R** has a distance function or metric $d(a,b) = |b - a|$.

(8) **R** is an ordered field, i.e., if $a \leq b$ then $a + c \leq b + c$ and if $0 \leq a$ and $0 \leq b$ then $0 \leq ab$.

(9) **R** is an Archimidian field, i.e., it satisfies that axiom of Archimedes. For any real number $x$, there is a natural number such that $n > x$. This distinguishes it from all the $p$-adic completions.

### 4.1.1. Affine space

The main difference between the affine space $\mathbf{A}^n$ and the real $n$–space $\mathbf{R}^n$ is that one considers $\mathbf{R}^n$ to have a special point **0** and sometimes units in all directions. Technically, one would speak of a vector space or a vector space with a basis. If we forget **0**, we are in the Euclidean geometry situation. We can for instance measure only distances. But we can translate by vectors. In fact, we can translate any point to any other point.

By definition, a principal homogeneous set for a group $G$ is a set $S$ together with an action of $G$ which is a map $t: G \times S \to S$ [55], such that for any two elements $s$ and $s'$ of $S$ there is a unique $G$ such that $t(g,s) = s'$.

If $G$ and $S$ are spaces, one says that one has a homogeneous space if the action is continuous. Now $\mathbf{A}^1$ is a principle homogeneous for **R**. Fix two points O and U on the line $\mathbf{A}^1$. Then, there is a unique one-to-one correspondence preserving distances that sends these points to 0 and 1, respectively. I.e. choose an origin and a unit. Now, **R** acts on itself by translation $t(a)(r):=r+a$. This action can be lifted to $\mathbf{A}^1$ by using the chosen bijection and then it is transitive. That is for any point A of $\mathbf{A}^1$, there is a unique $r$ in **R** such that O gets sent to A by translation by $r$. In fact the whole action of $\mathrm{Aff}^1$ can be lifted.

### 4.1.2. Dedekind Cuts

In the definition of the reals according to Dedekind, a real number is a pair of subsets (L,R) of **Q** such that they are disjoint, their union is **Q**, and every element of L is to the left (i.e., less than) of every element of R. In the original version, the numbers $q$ in **Q** are those partitions where either $q$ is the smallest element of R or the greatest element of L. If there is no such element, then the pair is taken to represent the number in **R** that would be the putative supremum of L. Nowadays, one chooses just one set L that is downward closed and contains no largest element. This has the advantage that the rational numbers are represented by just one cut. Also going along with Hegel, R is determined by L as R = **Q**\L. The arithmetic of these cuts is far from obvious.

#### 4.1.3. Cauchy sequences

The other way to construct the reals is to look at sequences $(r_i)_{i\in N}$ such that for any positive ε in **Q**, there is some natural number $N$ such that for all $n,m > N$: $|r_n - r_m| < \varepsilon$. These sequences are called Cauchy sequences. And these sequences should converge in the to be constructed **R**. Complete means that all Cauchy sequences converge. By adding and multiplying etc. on the elements, one gets the operations on the sequences. Now the reals are not just Cauchy sequences, but classes of Cauchy sequences modulo so-called null sequences. That is, sequences that converge to 0, i.e., for any ε in **Q**, there is some natural number $N$ such that for all $|r_n| < \varepsilon$. The reasoning being that adding such a sequence would not change the putative limit. Formally, two sequences are equivalent if their difference is a null sequence. The usual example is that the constant sequence 1 is **equal as a real number** to the sequence 0, 0.9, 0.99, 0.999 ….

### *4.2. Inverse Operations*

#### 4.2.1. Negatives

The technical term is semi-group completion. Say one has a set with addition +, to be concrete fix the natural numbers. We can then look at pairs $(m,n)$ modulo the equivalence that $(m,n)\sim(k,l)$ if that $m + l = n + k$. If we denote a class by [m,n], we do addition on this set by $[m,n]+[k,l]=[m + k,n + l]$, which does not depend on the choice of representative. Notice that the integers are given by $[m,0]$ and what is usually denoted by $-n$ is just the class $[0,n]$. Hegel's negative quanta are well modeled by this. In this calculus, $m - n$ becomes $[m,0] + [0,m] = [m,n]$. However, if $m > n$, then $[m,n] = [0,m - n]$ and if $m<n$ then $[m,n]=[0,m - \mathrm{n}]$. Note that in the general setting, one should make the equivalence relation read that there exists some $s$ such that $m +l + s = n + k + s$.

#### 4.2.2. Fractions

This is the same procedure now starting with either the natural numbers **N** and multiplication or the integer numbers **Z** with multiplication. Hence, fractions are equivalence classes of pairs $(p,q)$, with $(p,q)\sim(r,t)$ if $pt = rq$.

### *4.3. Cardinalities of sets and arithmetic*

For illustrative purposes, we remain in naïve set theory, this is of course not strictly correct, but serves us well for illustrative purposes. In this setup, a cardinal is an equivalence class of sets under the relation of bijection. That is, two sets have the same cardinality if there is a bijection between them. The natural numbers are the cardinalities of finite sets.

The union of two sets is the set containing elements from both sets $S \cup T = \{x : x \in S \text{ or } x \in T\}$.

Given two sets, their disjoint union roughly is a set, which contains the elements of both sets separately. If two sets $S$ and $T$ are indeed disjoint, their union $S \cup T$ is a good representative. If, however, these sets have common elements, one has to make a slightly more technical definition. One sets $S \sqcup T = \{(s,0) : s \in S\} \cup \{(t,1) : t \in T\}$, which has the effect of first passing to sets which are disjoint but bijective to the original ones and then taking their union. For instance, if $S = \{u\}$, then $S \cup S = S$ and $S \sqcup S = \{(u,0),(u,1)\}$. It is easily seen that the cardinality of the disjoint union is the sum of the cardinalities. Note that the cardinality is actually independent of the particular choice of disjoint sets.

The cross or Cartesian product of two sets is given by the set of pairs of elements. $S \times T = \{(s,t) : s \in S, t \in T\}$. The cardinality of the product is the product of the cardinalities.

Finally, the sets $Map(S,T) = T^S$ is the set of maps $f\colon S \to T$, that is, rules to associate an element $t = f(s)$ of $T$ to any given element $s$ of $S$. The cardinality of $Map(S,T)$ is $|T|^{|S|}$, where $|\;|$ denotes the cardinality.

A relevant example is $|T|^2$ which Hegel characterizes as $|T \times T|$. By the previous arguments, this should also be $|T^2| = |T^{\{0,1\}}|$. The elements in this set are maps $f$ from {0,1} to $T$. Such a map is given by the pair of its values $(f(0),f(1))$. Vice versa and pair $(t_0,t_1) \in T \times T$ defines such a function.

### *4.4. Relations*

A relation on a set $X$ is a subset R of the Catesian product of $X$ with itself $X \times X$. One writes $x \sim y$, if $(x,y)$ is an element of R. An equivalence relation is a relation that is (a) reflexive $x \sim x$ for all $x$ in X, (b) symmetric $x \sim y$ implies $x \sim y$, and (c) transitive $x \sim y$ and $y \sim z$ implies $x \sim z$. Given such an equivalence relation, there on can define the equivalence class of $x$ written [x] as the set of all $y$ such that $(x,y)$ is in R. Any element of this set is called a representative. There is moreover a set X/~ whose elements are the equivalence classes and a map from $X$ to $X/\sim$, which sends $x$ to $[x]$. Any preimage of $[x]$, is a representative.

### *4.5. Topology*

A topology on a set $X$ is the datum of a collection of sets that are called open sets.

This collection has to satisfy (a) that the whole set and the empty set are open, viz. in the collection; (b) that arbitrary unions of open sets are open; and (c) that finite intersection of open sets are open. A topological space is a set with a topology. A function between two topological spaces is continuous if the inverse images of open sets are open.

4.5.1. Metric topology

A metric on a set $X$ is a function $d: X \times X \rightarrow \boldsymbol{R}_{\geq 0}$, i.e., that takes pairs of points to the positive reals, such that (a) $d(x,y) = 0$ is equivalent to $x = y$; (b) $d(x,y) = d(y,x)$; and (c) $d(x,y) + d(y,z) \leq d(x,z)$. If a set has such a metric, for any point $x$, let $B_x(r) = \{y: d(x,y) < r\}$, these are called balls or basic open sets. Then, the set of open sets are all the sets that are arbitrary unions of these basic open sets.

For the reals, $d(a,b) = |a - b|$ and the balls are simply open intervals $(x - r, x + r)$. With this topology, continuous functions are what you would get out of the usual limit or ε-δ criterion.

4.5.2. Discrete toplogy, trivial topology

On any set, one can choose the set of open sets to be all subsets. This is called the discrete topology. In particular, points are open. This is the natural topology for **N** and for **Z**. On any set, the trivial topology is the other extreme, namely only the set itself and the empty set are open. These two topologies are the only ones that one can write down without further input, i.e., selecting open sets by some criterium.

4.5.3. Natural topologies on R and Hegel.

Without a metric on **R**, the discrete topology would be natural, which is the discreteness of the points as elements of a set. Also, the trivial topology could be chosen, in which one could find '*reine Quantität*'. Finally, due to the fact that we want to measure and this is done with the distance function, we get the metric topology which gives the continuous nature of the reals.[56]

*Ralph M. Kaufmann and Christopher Yeomans*
*Department of Mathematics, Purdue University*
*West Lafayette, IN, USA*
*Department of Philosophy, Purdue University*
*West Lafayette, IN, USA*
*cyeomans@purdue.edu*

## NOTES

1 We take the two versions of the Logic to present a single theory, and we will refer to that theory by the capitalized 'Logic' when we are not discussing particular texts. Parenthetical references are as follows: (a) by section number to Hegel's *Enzyklopädie der philosophischen Wissenschaften, Teil I*, Band 8 in *Werke*, ed. E. Moldenhauer and K.M. Michel (Frankfurt: Suhrkamp, 1970)—an 'R' after the section number indicates the published remark to the section and a 'Z' the addition or *Zusatz*; (b) by volume and page to the two editions of Hegel's *Wissenschaft der Logik* in *Gessamelete Werke* (Hamburg: Meiner, 1978 and 1985); (c) by section number to Hegel's *Naturphilosophie* (*Werke*, Band 9), with 'EN' before the § sign.

[2] In physics, a quantity is said to diverge in a given limit, if it tends to infinity. The limit in a phase transition is given by approaching the phase line, say by varying temperature or pressure for example.

[3] But it should be emphasized that mathematics is not logic, in Hegel's sense, and thus the kinds of arguments made by mathematicians cannot be the same as the kinds of arguments made by Hegel, even in his discussion of quantity. Thus, we should not be taken to suggest that Hegel's arguments for specific moves in the *Logic* (i.e., the dialectical development) could be given a formal mathematical translation that would show that contemporary mathematicians and Hegel are thinking in the same mode about real numbers. In fact, the reciprocal insight of mathematics and Hegel's *Logic* that we attempt to demonstrate depends on that not being the case.

[4] One might think that our claim here fails to track Hegel's vocabulary, which puts off until the Philosophy of Nature the introduction of space. Since this is a fundamental terminological question, it may well be worthwhile to say something about it here to clear the ground for the following specific argument. The first and most important thing to point out is Hegel's own explicit connecting of space and quantity. Specifically, Hegel holds that pure quantity gives the general logical structure of space: 'After all, [space] is pure *quantity*, though no longer this same as logical determination but rather as immediately and externally existent' (EN§254R). Thus, it is not surprising that the introduction of space is via the two features that Hegel develops out of pure quantity, viz continuity and discreteness (EN§254 & EL§§99–100). As long we abstract away from the 'immediately and externally existent' nature of space as contemporary geometry does and as is licensed by Hegel's own denial of the Kantian interpretation of space as essentially a form of intuition, there is a formal conception of space that exists in Hegel's text that tracks the modern mathematical understanding and can be legitimately used to interpret the category of pure quantity. Second, since the Philosophy of Nature is post-conceptual, as it were, in the developmental track of Hegel's system, Hegel claims further that it is a conceptual truth about space that it is three dimensional (tracking the three aspects of the concept (*Begriff*)). But this questionable conceptual limitation to three dimensions is absent from the contemporary mathematical understanding of geometric space (and *a fortiori* from that of the line in particular) that we are here using to interpret Hegel's category of pure quantity. Hegel himself provides an attempted conceptual deduction both of the three dimensional nature of space and of the necessary features of point, line, and plane—but he is clear that geometry itself is free from both the ability and obligation to demonstrate these necessities (EN§§255R & 256R). Perhaps the overall point is best put by saying that Hegel begins with space in the contemporary mathematician's sense of 'space', which then serves as the logical core of the intuitive sense of 'space' that is presented in the Philosophy of Nature. This is the sense in which he differs from Gauss as briefly suggested above.

[5] Note that this is subtle, since fractions use pairs and equivalence classes; Hegel realizes this through the notion of *Verhältnis* and *Maß*. This is thus a two step process, first realizing that each unit gives an abstract copy of $\mathbf{Q}$ and then identifying this copy with a subset of the reals to get the measurement copy.

[6] The identification via the realization that there is arithmetic on both sides of 3 and 3′ goes back to Descartes 1902 (*Discourse on the Method*, Part II (AT VI. 20). Hegel deduces his own version of this postulate in EL§202Z.

[7] There are two modern ways of doing this conceived around 1860 by Dedekind using cuts and by Cantor using Cauchy sequences, both of which are given axiomatically. Dedekind's definition however more heavily draws from the geometric intuition of the line. See our §3 for a summary.

[8] For a discussion of Hegel's relation to Gauss, see Beach 2006.

[9] Although different numbers can have different properties (e.g., being prime), as Stekeler-Weithofer correctly sees these are secondary properties (1992: 156 and 2005: 207). In our reconstruction, these are properties that only show up in steps 3, 3′, and 2′ but not in 1 or even in 1′.

[10] Strictly speaking, we always get a positive real number, but translation of the scale might result in negative numbers. A good example here is temperature measured in Kelvin, Centigrade, or Fahrenheit. The negative reals are introduced by Hegel using the negative of quantum in §102Z. This fits with the idea of negative charges.

[11] We do know that at the quantum level, this ceases to be a completely true description. Here a particle, say a fermion, may for instance have the quality of having spin. Measuring its value however does change the particle state fundamentally, although one still has to assume a continuity of the system before and after the measurement. Nonetheless, even in quantum theory, the outcome of measurement is a real number, which is why operators need to be Hermitian. This is a technical condition for operators over the complex numbers that guarantees that one gets real numbers as so-called eigenvalues, which are the possible results of measurements.

[12] Stekeler-Weithofer claims here that Hegel is misquoting mathematicians (1992: 158). But in fact, Hegel has correctly understood mathematical reasoning here. For something to be increased or decreased does not require addition and subtraction but rather ordering (>). Furthermore, with addition and subtraction but without ordering, one cannot get the continuity or completeness of **R** as opposed to **Q**. Order doesn't presuppose addition and subtraction and is more basic. In our reconstruction, there is order in 1 and 1′ but addition and subtraction first in 2 and 2′. This is also apparent in the construction of **R** via Dedekind cuts, where arithmetic operations are introduced after using < to define real numbers.

[13] Actually, in modern terms, Hegel is considering affine space and the affine line for time, which are identifiable with **R** and $\mathbf{R}^3$, but there are several such identifications; see the appendix. This is picked up by Hegel using the notion of quantum.

[14] See also EN§254, R: 'The initial or immediate determination of nature is the abstract *universality of its being external to itself (*Außersichseins*)*, whose immediate indifference is *space* … [Space] is pure *quantity* in general, no longer merely as a logical determination, but as immediately and externally existent.'

[15] In modern terminology, if one has a function of two points that is invariant under the affine action of translation and rotation, then it will be a function of the Euclidean distance between these two points. But there are many such choices which are all obtained from one another by scaling, the last part of the affine action. This choice amounts to picking a quantum which is consequently done by Hegel in the next section.

[16] By Cantor, we know the answer to the first question is $2^{\aleph_0}$, while for the second, it is simply undefined $\int_{-\infty}^{\infty} dx$; for the third, it is (using integration or measure theory) $\int_a^b dx = b - a$ if $a < b$.

[17] Actually, there are many ways to make **Q** complete. There is one for every prime number $p$, the result is called $\mathbf{Q}_p$. These are arithmetically on par with **R**. The only advantage of **R**, the completion with respect to the absolute value, being its connection to geometry—which per Hegel is an input.

[18] This discreteness passes to the rational numbers **Q** if they are viewed as a quotient space of $\mathbf{Z} \times \mathbf{Z}$, that is of pairs $(p,q)$. It is also captured by **Q** being countable.

[19] The interesting thing is that the field axioms are today separate from the topological condition of completeness. This means that the discrete and the continuous

are just aspects. There is the underlying field which is a (discrete) set with arithmetic operations and the continuous nature is an additional structure. To single out the reals however, one should also preserve the Archimidean property of the rationals (see §3).

20 He seems to make the opposite claim at WL 21: 191, but a closer examination shows that he is making the same point about the presence of both aspects in each manifestation of number.

21 Note that 1 is indeed the unit of the multiplication.

22 Mathematically speaking, for instance $1 = N/N$ and we can let $N$ tend to infinity $1 = \lim_{N \to \infty} \frac{\mathrm{N}}{\mathrm{N}}$.

23 It is interesting to note that for this, we actually need to fix an origin first. Without this, we only have an affine space and we can and the lattice of integer points can be moved continuously.

24 Technically, the open and closed intervals are basic to the topology of the reals as explained in §3. They are given by $B_r(p) = \{x: |p - x| < r\}$ for the open interval, or 'less or equal' to $r$ for the closed ones. In fact, these $B_r(p)$ are a basis for the topology of $\mathbf{R}$ which is used for analysis and continuity in the usual sense

25 One might think that the interpretation offered of this brief sentence involves a fallacy of misplaced concreteness, but the context (not quoted here) makes it clear that packed into these three basic characterizations is real structural complexity. So, for example, one might think that '*self-referring*' means nothing more than empty self-relating—but it is actually Hegel's own paraphrase of the feature of quantity that it is 'continuous…a unity' (WL 21.194). Unless 'continuous' is taken to be merely metaphorical (a disastrous interpretive move given its centrality to Hegel's entire discussion of quantity), this kind of self-reference must have an extension of some sort. Similarly, one might think of the '*enclosing*' as simply a collecting of bare ones rather than being contrastive. But Hegel's insight is that the very notion of a 'bare' one and thus a 'natural' number is contrastive (in a sense analogous to that in which indeterminacy is understood by Hegel to be contrasted with determinacy and thus as a kind of determinacy itself). Furthermore, it is of the very nature of this logical category that it is recursive (i.e., that both units and the numbers that are multiples of units can be characterized by its means). These two points are tied together because the internal continuity or unity of a quantum requires assimilating other quanta and thus replacing their units of continuity with that of the first. This connection between internal units and external quanta is required for all arithmetical operations—if each quantum had its own units that could not be transformed into the units of other quanta, the whole system of numbers would fall apart. Nonetheless, this has to be compatible with the breaking of scaling action as suggested in (γ); otherwise, numbers cannot be distinguished from each other.

26 To be precise, the interval plus an orientation, which also explains *negative Größe.*

27 Stekeler-Weithofer 1992 (but cf. Stekeler-Weithofer 2005: 206), Paterson, 1997.

28 Pinkard 1981.

29 It is interesting to note that in the definition of the reals using Cauchy sequences, any number represents a limit of some sequence, also the natural numbers. It is to the credit of Cantor that he realized that limits of arbitrary Cauchy sequences can be used in the definition of the reals, as discussed in the appendix. This is in parallel with Hegel's *intensive Größe*.

30 What a line actually is was of course a historically difficult thing. What we mean is that the real numbers can serve as a model as it satisfies all the necessary axioms. Indeed,

what is meant when we talk about The Real Line or The Number Line is this standard model for the geometric Euclidean line.

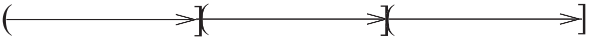
31 In the Zermelo–Frenkel axioms, the natural numbers are basically guaranteed by an axiom.

32 These unit-based natural numbers and the rationals extracted from then are reconciled with the line though *Exponent* (EL§105) and *Maß* (EL§106).

33 There is a misprint on 456, where '*das Eins*' and '*das Eins*' are contrasted, but the context and the following page clarify Pinkard's thought here.

34 There is an inherent analog of Hegel's discussion of *das* and *die Eins* with open intervals, their closure and integers (which is intimately linked by analysis to the wave picture—see our §2.1). Namely, the real number line can be covered by the closed intervals [n, n + 1]. However, this is not a good representation by individuals as the integers each lie in two intervals. Another almost cover is given by the open intervals (n, n + 1):


These are repeated units and a good representation of *die Eins*, but now, the integers do not appear at all. They are, however, just a discrete subset of 'measure zero' (this is a technical term meaning that this is not detected by, say, integrals). This is what one could call Hegel's '*die Grenze ist nur das Leere*'. On the other hand, the closure of the intervals are precisely the limit point, *die Grenze* and *die intensive Größe*. (One can even note that now each integer appears only once, because we take the right limit n + 1 as *die Grenze* of (n, n + 1):



Taking *die intensive Größse* of the open interval in Hegel's sense gives the upper limit. (The process of taking the limit is indicated by the arrow.) The limit point then becomes the identifier. This can be seen as the half open interval (n, n + 1] which is uniquely identified by n + 1. This limit comes from *within* the interval itself.

35 Contra Stekeler-Weithofer (1992: 162), √2 and 2**π** do appear as natural units (√2 for diagonals and 2**π** for trigonometry).

36 For a treatment of this latter discussion, see Sedgwick 1991.

37 This presupposes that the numbers themselves have already been constructed. For the construction of the natural numbers—again as sets—one proceeds as follows: One starts at **0** = ∅ and then iteratively defines **n + 1** = {**1,**…,**n**}. This is a cleaner construction, but a little harder to parse. Indeed, this representation from Neuman is equivalent to taking disjoint unions 0 = ∅, 1 = {∅}, 2 = {∅,{∅}},3 = {∅,{∅},{∅,{∅}}},…

38 One could then iterate this construction equating units and enumerators algebraically yielding power operations of the type $3^{3^3}$, $4^{4^{4^4}}$, and so on. Hegel however does not go down this road, and one usually does stop at the third operation, since this operation is not associative—i.e., $(3^3)^3 \neq 3^{(3^3)}$—and hence, there would be many possible iterations. All the higher operations can be encoded into spaces of maps between sets. Where then one has an iteration of spaces of maps from spaces of maps and so on. One of the interesting things, which comes from this is a construction of the reals as sequences of ***0*** and ***1*** and the famous continuum hypothesis.

39 A purely algebraic version of negative quanta is made explicit in our §3. Another modern or physical way to describe negative quanta is in terms of vectors. In order to give a vector, one usually draws an arrow. The shaft of the arrow is the interval, and the tip points to the right if it is positive and to the left if it is negative. Now, the geometric addition of these vectors corresponds to addition and subtraction in **R**. Even today, one uses the notion of units that is commensurate with the view. By definition, these are all invertible

elements. For instance, the integers have the units 1, −1, and Gauss integers have units 1, −1, i, −i, and **R** has units $\mathbf{R}\setminus\{0\}$. This goes back to Kronecker.

[40] For addition and subtraction, the given arguments fully suffice, especially when interpreted in terms of vectors. For the multiplication, this is a bit more difficult, since one needs some sort of postulate that the unit square is the new unit. Without explicit mention of units, this is technically achieved by Fubini's theorem on integration in higher dimensions. One could argue with Hegel, however, that in the matrix counting argument given above, one is counting entries which do represent squares.

[41] Cf. Stekeler-Weithofer, 1992: 163.

[42] The only difference from Hegel is that the two subsets are 'infinite'. This can be made commensurate with Hegel's vision in two ways. First, by keeping only one set, it determines the second set as the complement, just as in §101. Second, if one wants to have a bounded version, one can first construct positive reals, as partitions of positive rationals and then add the negatives, as Hegel does.

[43] It might be worthwhile adding that quantum phenomena tell us that such continuous nature might not be as clear as we thought. The famous relation $E = h\nu$ show that the energy of electromagnetic waves is indeed counted by quanta.

[44] Cf. Stekeler-Weithofer 1992: 227.

[45] In standard calculus, this particular process is called *supremum* and denoted as such as $r = \sup_{t \in (0,r)} t$.

[46] Particularly in the greater Logic, there is at this point a very extended discussion of differentials, infinitesimals, and approximation. For reasons of space, we cannot here enter into a detailed discussion of this material, but it appears to us that contemporary mathematics largely confirms Hegel's view that 'dx' is not a number.

[47] Cf. Michell 1994.

[48] Hegel insists on the arbitrariness of external standards of measurement at WL 21:330–1 and 333.

[49] Historically, there is a parallel development between our understanding of what **R** is and what a function is. Cf. Paterson 1997.

[50] As an aside, a common aspect is that these phenomena happen at points or at least in positive co-dimension, such as a line in a plane, so that the special points are of measure ***0***. This means that a generic point is not critical.

[51] It is interesting to remark that this realization of the discrete as nodes exactly foreshadows the later development of quantization. Here, the quanta are precisely nodes, say in the quantization of a box potential which is one way to understand Planck's quanta.

[52] For an analogous claim of the advantages of a substantive Hegelian approach in comparison with the mathematical logic of formal systems, see Patterson 1997 and 2010.

[53] Cf. Stekeler-Weithofer 1992, for whom the qualitative element (invariance) is 'a *norm* posited by us or ideal form, admittedly one such that its practical satisfiability depends not only on our techniques but also on our knowledge in terms of content and thereby also on the world' (153). Also Stekeler-Weithofer 2005: 202–4.

[54] See Paterson, 1997: 145 for an analogous criticism of Gödel's realism in comparison with a Hegelian approach.

[55] This action should of course be compatible with the group operation and associative.

[56] The authors would like to thank this journal's referee for extraordinarily detailed and helpful comments on two earlier drafts of this paper.

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