OPERADS, MODULI OF SURFACES AND QUANTUM ALGEBRAS

RALPH M. KAUFMANN*
Oklahoma State University
Stillwater OK, USA
E-mail: kaufmann@math.okstate.edu

We review the relationship of moduli spaces of surfaces to operads and deformations of algebras. We start by recalling basic facts about operads, examples of them, and their relations to algebras. In particular, we regard the Arc operad as well as several suboperads which can be thought of as cacti operads. These play an essential role in the string topology of Chas and Sullivan. It is recalled that spineless cacti and cacti homotopy equivalent to the little discs and framed little discs operads featured prominently in deformation theory. Furthermore, we analyze operations on operads and use the results to relate the spineless cacti operad to the renormalization Hopf algebra of Connes and Kreimer. Finally, we give a cell decomposition for spineless cacti and show that the cellular chains operate on the Hochschild complex of an associative algebra. This gives a solution to Deligne's conjecture about the Hochschild complex and furthermore directly relates the Gerstenhaber structures on the loop space and the Hochschild complex.

1. Introduction

In recent years, there has been a dynamic and vibrant interplay between physics and mathematics which is now spreading to biology. As a result of this exchange the ideas around topological and conformal field theory as well as string theory have found a mathematical manifestation in constructions which use the geometry of moduli spaces to describe deformations of algebras. In this setting, there are two main questions: which geometries govern specific deformations; and given specific geometries, what kind of algebraic objects do they represent? The deformations are usually given in terms of an expansion in one or more deformation parameters and the coefficients of these expansions can be viewed as multilinear functions whose relations are of particular interest. For their study operads provide the

*Partially supported by NSF grant DMS-#0070681

right tool being at once an organizing principle, a theoretical framework and *tertium comparationis*.

For persons not already inclined towards operads one could say operads are very much like Steenrod operations in the following sense. They both have three aspects: 1) There is an algebraic description. Pertaining to the Steenrod operations this is the fact that they form an algebra. For operads the counterpart is the definition of an operad in terms of its constituents and operations. 2) They can be viewed as universal operations, i.e. regarding the Steenrod algebra as operations on cohomology and for operads considering the classes of algebras over this operad. 3) There are realization in concrete models.

In passing from geometry to algebra one usually uses geometric moduli to produce and describe the structure of deformations. For the physically inclined one could state that one starts out with an interpretation of Feynman rules. These can usually be translated into a graph theoretical picture together with a geometrical description such as a stratification of a moduli space indexed by these graphs. This would be a safe place to start for a critical mathematician. Appealing once more to physics; the idea of the path integral, like a usual integral, also encompasses an additivity property in the domain. This translates to glueing or pasting operations which have been known in mathematics as operads. Combining these two ingredients with the idea of strings one is mathematically led to consider operads of surfaces as we discuss below.

The basic example is that of topological field theory where graphs actually represent surfaces with labelled boundaries and the glueing operation is the glueing along the boundaries - this is discussed below in detail. The relation to string theory is the apparent interpretation that the boundaries are strings and as they move they sweep out the surface. To make this interpretation rigorous, one has to introduce orientations and look at the cobordism category which allows to speak of inputs and outputs. With these inputs and outputs a surface looks like a thickened Feynman graph. These surfaces are naturally the geometric part of the picture; the algebra comes in form of a functor to the category of vector spaces as in Atiyah's[1] version of topological field theory.

Here the geometric fact that any surface can be decomposed into pairs of pants, cylinders and discs is translated into the fact that on the vector space there should be an associative commutative multiplication, a non-degenerate bilinear form and an identity for the multiplication.

Starting with this topological picture one can put more structure on the surfaces which will in turn give more operations on the algebraic side. This leads to deformations or augmentation of the basic structure of an associative commutative algebra. One can think for instance of surfaces together with conformal structures. In the algebro–geometric setting one is led in this case either to consider the open moduli spaces $M_{g,n}$ of n–punctured surfaces of genus zero or their Deligne–Mumford compactification $\bar{M}_{g,n}$. In the latter case, one arrives at Gromov–Witten invariants[37] or quantum cohomology which is, as the name states, a deformation of the commutative associative algebra. If one looks at the algebras that are governed by $M_{0,n}$ then one arrives at so–called gravity algebras or Lie_∞ in the genus 0 case which are deformations of Lie algebras. The fact that Lie algebras instead of commutative algebras appear has a very nice interpretation in terms of operads, since the operad $H_*(M_{0,n})$ is Koszul dual to $H_*(\bar{M}_{0,n})$[18,20] just as the operads for commutative and Lie algebras are Koszul duals of each other.

An augmentation can be made by adding for instance G–bundles to the picture for a finite group G. In this case one arrives at G–equivariant theories[27] which are well suited to study so–called "stringy aspects" of global quotients such as symmetric products[28]. Adding conformal structure one arrives at the G-equivariant setting of GW–invariants[26]. For general orbifolds the corresponding invariant structures are contained in[13].

Staying on the differential–topological or analytic side one can keep the boundaries and add punctures for instance. The natural spaces to study are then Teichmüller spaces and their moduli space quotients. One especially promising aspect of this approach is the natural appearance of graphs in this theory. They arise in the theory of Strebel differentials as well as in Penner's work[41,42]. More recently another beautiful view of the topology of movements of strings in terms of operations on loop spaces has been put forth and developed[8,9,45]. This or some aspects of the theory also have an interpretation in terms of surfaces decorated by weighted arcs which again form an operad, the arc operad[34]. In the genus zero case this algebra of string topology is captured by the cacti operad of Voronov[49] which is a suboperad of the arc operad[34,29]. The algebraic structure that appears is that of a Gerstenhaber Batalin–Vilkovisky algebra up to homotopy (basically this is an odd Poisson algebra whose bracket is the odd commutator of a differential that is not a derivation, but a degree two differential operator; see §4.12 for the definitions). This structure is closely related to the gravity algebra structure[16] as is the space defining the arc operad to the open moduli space[42].

For the second type of question – what kind of geometry governs specific deformations – the following two problems are key examples: deformation quantization and Deligne's conjecture about the Hochschild cochain complex. It has surprisingly turned out that they are closely related[32,48]. The solving of the deformation quantization[32] was done through geometry in terms of integrals over configuration spaces. Deligne's conjecture, on the other hand, is directly a question about the geometry governing deformations. It was realized by Gerstenhaber[16] that deformations of the multiplication of the product in an associative algebra into an associative star product are governed by the Hochschild complex of this algebra. He proved that there is a surprising structure, namely that of an odd Poisson algebra, on the homology level. This structure is actually derived from the cochain level. Since the operad governing Gerstenhaber algebras is the homology of a geometric object, that of little discs, it was conceivable that this correspondence could be lifted to the chain level. This is the content of Deligne's conjecture which we prove below. This conjecture has been proven in various ways[32,48,39,50,36] (for a full review see[40]) by basically choosing adequate chain models. The virtue of our approach lies in its directness and that it also establishes the surprising and intriguing fact that the Gerstenhaber structure on the loop spaces and the Gerstenhaber structure on the Hochschild cochains have a common natural interpretation in terms of surfaces. The formal similarities of the Gerstenhaber structure on the arc operad or rather of the suboperad of spineless cacti[29] with the Gerstenhaber structure on the Hochschild cochains were first observed by Gerstenhaber himself and have given rise to our new proof of Deligne's conjecture in terms of spineless cacti[30] which is reviewed below. In fact, spineless cacti are homotopy equivalent to the little discs operad and here is a cell decomposition of the spineless cacti operad indexed by trees which directly give the operation in the Hochschild cochain complex. Furthermore, the spineless cacti can be thought of as surfaces with weighted arcs that satisfy certain natural restrictions and as such can be seen as the substructure of moduli space giving rise to the bracket operations. This fits also well with a path integral description of deformation quantization by[5]. On the other hand, cacti and spineless cacti[49,29] are the operads which correspond to the "string topology"[8] and provide the BV and Gerstenhaber structure, respectively. A realization of the Gerstenhaber structure in this setting is given by the Goldman bracket[8]. In another direction, we show that the top–dimensional cells of the cell decomposition which provides the solution to Deligne's conjecture is related to the pre-Lie operad and to the renormalization Hopf algebra of Connes and Kreimer[11]. It is thus tempting to view the $\mathcal{A}rc$ operad[34] as an underlying "string mechanism" for all of these structures.

The paper is organized as follows:

After fixing some notation in section 2, we introduce and discuss several types of trees in section 3. Trees are the language in which operads are most easily spoken about. In the fourth section, we review the notion of an operad and give many examples of them. These examples include the ones of functions and trees, which are archetype of operads. We also introduce all the operads necessary for our discussion of Deligne's conjecture. Section 5 is devoted to a review of the construction of the $\mathcal{A}rc$ operad[34] which can be seen as a combinatorial version of the operad of moduli spaces of genus g curves with n punctures. In a sense this operad describes hyperbolic field theory. Besides a direct relation discussed in the same volume by Penner, abstractly there is a relation due to the BV nature of the operad which we also review as well as giving a string interpretation for the arc operad. The section 6 deals with the cacti operads[49,29]. We show how they are naturally sub–operads of the arc operad thus relating the arc operad to the string topology[8]. In section 7, we give a natural cell decomposition for spineless cacti which is the basis of our solution of Deligne's conjecture. Paragraph 8 is an intermezzo about universal operations for operads, if one wishes meta-operads. This explains the natural appearance of insertion operads in terms of foliage operators and leads to the definition of pre–Lie, Lie and Hopf algebras for operads. In section 9 we give our proof of Deligne's conjecture based on our cell model of the little discs operad which is given by the cellular chain operad of the natural cell decomposition for the spineless cacti operad or better the chain operad of normalized spineless cacti. We do this by providing two natural ways of letting these chains operate on the Hochschild cochains of an associative algebra. The first operation is of the same type as the operation of the chains of the arc operad on themselves self and is related to string topology. The second realization uses the fact that there is a geometric interpretation of the Hochschild cochains as chains of the spineless cacti operad. This view is implemented by the foliage operator introduced in the previous section. Section 10 contains the application of our results of section 8 and 7 to the operad of pre–Lie algebras. Putting together the previous results we obtain the renormalization Hopf–algebra of Connes and Kreimer as the symmetric group coinvariants of the Hopf algebra of an operad of cells.

The cells are the symmetric top–dimensional cells of our cell decomposition of spineless cacti suitably shifted. Furthermore section 10 contains a proof of a generalization of Deligne's conjecture to operad algebras and mentions the generalization of it to cyclic cohomology of certain types of algebras. In the last section §11 we speculate on the A_∞ generalization of our results as well as on possible relations of the arc formalism to other "quantum phenomena" and additional subjects.

Acknowledgments

I would like to thank Muriel Livernet and Bob Penner for their wonderful collaboration[34]. Furthermore, I would like to thank Nils Tongring for organizing the extremely stimulating Woods Hole meetings. The visits at the Max–Planck Institute for Mathematics and the IHES, were instrumental in the carrying out of the presented research; especially the discussions with Maxim Kontsevich. Also the discussions about cacti with Sasha Voronov were the starting point for our investigations into the several types of cacti. My thanks also goes to Wolfgang Lück and Peter Teichner for their hospitality and interest. In fact the first steps for our solution to Deligne's conjecture were taken during a visit of the SFB in Münster and the final step came to me during a visit to San Diego in response to questions of Peter Teichner. Finally, I wish to thank Jim Stasheff for very enlightening discussions and encouragement as well as Murray Gerstenhaber for pointing out the similarities between his constructions and those of the arc operad[34] which was initial spark for the work on Deligne's conjecture.

The author also acknowledges financial support from the NSF under grant #DMS-0070681.

2. Notation

We denote by $\mathbb{S}_n$ the permutation group on n letters and by C_n the cyclic group of order n. Also, let k be a field.

We will tacitly assume that everything is in the super setting, that is $\mathbb{Z}/2\mathbb{Z}$ graded. For all formulas, unless otherwise indicated, the standard rules of sign[38] apply.

3. Trees

Trees are a very useful organizational tool when dealing with operads. The different types of operads we study give rise to different types of trees. In the following, we introduce the types of trees we will need for our discussion.

3.1. *General Definitions*

Definition 3.1.1. A graph is a one dimensional simplicial–complex. We call the 0–dimensional simplices vertices and denote them by $V(\Gamma)$. The 1–dimensional simplices are called edges and denoted by $E(\Gamma)$.

A tree is an isomorphism class of a connected simply–connected graph.

A rooted tree is a tree with a marked vertex.

We call a rooted tree planted if the root vertex lies on a unique edge. In this case we call this unique edge the "root edge".

We usually depict the root by a small square, denote the root vertex by $root(\tau) \in V(\tau)$ and the root edge by $e_{root(\tau)} \in E(\tau)$.

Notice that an edge e of a graph or a tree is gives rise to a set of vertices $\partial(e) = \{v_1, v_2\}$. In a tree the set $\partial e = \{v_1, v_2\}$ uniquely determines the edge e. An ordered edge is an edge together with an orientation of that edge. On a tree to give an orientation to the edge e given by the boundary vertices $\{v_1, v_2\}$ is equivalent to giving an order (v_1, v_2). If we are dealing with trees, we will denote the edge corresponding to $\{v_1, v_2\}$ just by $\{v_1, v_2\}$ and likewise the ordered edge corresponding to (v_1, v_2) just by (v_1, v_2).

An edge that has v as a vertex is called an adjacent edge to v.

An edge path on a graph Γ is an alternating sequence of vertices and edges $v_1, e_1, v_2, e_2, v_3, \ldots$ with $v_i \in V(\Gamma), e_i \in E(\Gamma)$, s.t. $\partial(e_i) = \{v_i, v_{i+1}\}$.

Definition 3.1.2. Given a tree τ and an edge $e \in E(\tau)$ one obtains a new tree by contracting the edge e. We denote this tree by $con(\tau, e)$.

More formally let $e = \{v_1, v_2\}$, and consider the equivalence relation $\sim$ on the set of vertices which is given by $\forall w \in V(\tau) : w \sim w$ and $v_1 \sim v_2$. Then $con(\tau, e)$ is the tree whose vertices are $V(\tau)/\sim$ and whose edges are $E(\tau)\setminus\{e\}/\sim$ where $\sim'$ denotes the induced equivalence relation $\{w_1, w_2\} \sim' \{w'_1, w'_2\}$ if $w_1 \sim' w'_1$ and $w_2 \sim w'_2$ or $w_2 \sim w'_1$ and $w_1 \sim w'_2$.

3.2. *Structures on Rooted Trees*

A rooted tree has a natural orientation, toward the root. In fact, for each vertex there is a unique shortest edge path to the root and thus for a rooted tree τ with root vertex $root \in V(\tau)$ we can define the function $N : V(\tau) \setminus \{root\} \to V(\tau)$ by the rule that

$$N(v) = \text{the next vertex on unique path to the root starting at } v$$

This gives each edge $\{v_1, v_2\}$ with $v_2 = N(v_1)$ the orientation $(v_1, N(v_1))$.

We call the set $\{(w,v)|w \in N^{-1}(v)\}$ the set of incoming edges of v and denote it by $\in (v)$ and call the edge $(v, N(v))$ the outgoing edge of v.

Definition 3.2.1. We define the valence of v to be $|v| := |N^{-1}(v)|$. The set of leaves V_{leaf} of a tree is defined to be the set of vertices which have valence zero, i.e. a vertex is a leaf if the number of incoming edges is zero. We also call the outgoing edges of the leaves the leaf edges and denote the collection of all leaf edges by E_{leaf}.

CAVEAT: Our $|v|$ is the number of incoming edges, which is the number of adjacent edges minus one for all edges except the root edge where $|v|$ is indeed the number of adjacent edges.

Remark 3.2.2. For a rooted tree there is also a bijection which we denote by $out : V(\tau) \setminus \{root\} \to E(\tau)$. It associates to each vertex except the root its unique outgoing edge $v \mapsto (v, N(v))$.

Definition 3.2.3. An edge e' is said to be above e if e lies on the edge path to the root starting at the vertex of e' which is farther from the root. The branch corresponding to an edge e is subtree of made out of the all edges which lie above e (this includes e) and their vertices. We denote the resulting tree by $br(e)$.

3.3. *Planar Trees*

Definition 3.3.1. A planar tree is a pair (τ, p) of a tree τ together with a so–called pinning p which is a cyclic ordering of each of the sets given by the adjacent edges to a fixed vertex.

3.4. *Structures on Planar Trees*

A planar tree can be embedded in the plane in such a way that the induced cyclic order from the natural orientation of the plane and the cyclic order of the pinning coincide.

The set of all pinnings of a fixed tree is finite and is a principal homogeneous set for the group

$$\mathbb{S}(\tau) := \times_{v \in V(\tau)} \mathbb{S}_{|v|}$$

where each factor acts $\mathbb{S}_{|v|}$ by permutations on the set of cyclic orders of the edges adjacent to v. This action is given by symmetric group action permuting the $|v|+1$ edges of v modded out by the subgroup of cyclic permutations which act trivially on the cyclic orders $\mathbb{S}_{|v|} \simeq \mathbb{S}_{|v|+1}/C_{|v|+1}$.

3.5. *Planted Planar Trees*

Given a rooted planar tree there is a linear order at each vertex except for the root. This order is given by the cyclic order and designating the outgoing edge as the smallest element. The root vertex has only a cyclic order, though.

On a planar planted tree there is a linear order at all of the vertices, since the root now has only one incoming edge and no outgoing one.

Furthermore, on such a tree there is a path which passes through all the edges exactly twice -once in each direction- by starting at the root going along the root edge and at each vertex continuing on the next edge in the cyclic order and finally terminating in the root vertex. We call this path the outside path.

By omitting recurring elements this yields a linear order $\prec^{(\tau,p)}$ starting with the root edge on the set $V(\tau) \amalg E(\tau)$. This order induces an order on the set of vertices $V(\tau)$, on the set of all edges $E(\tau)$, as well as a linear order for all the vertices incident to the vertex $v \prec_v^{(\tau,p)}$ whose smallest element is the outgoing edge. We omit the superscript for $\prec^{(\tau,p)}$ if it is clear from the context.

3.6. *Labelled Trees*

Definition 3.6.1. For a finite set S an S labelling for a tree is an injective map $L : S \to V(\tau)$. An S labelling of a tree yields a decomposition into disjoint subsets of $V(\tau) = V_l \amalg V_u$ with $V_l = L(S)$. For a planted rooted tree, we demand that the root is not labelled: $root \in V_u$.

An n-labelled tree is a tree labelled by $\bar{n} := \{1, \ldots, n\}$. For such a tree we call $v_i := L(i)$.

A fully labelled tree τ is a tree such that $V_l = V(\tau)$.

A leaf labelled tree τ is a labelled tree in which exactly the leaves are labelled $V_l = V_{leaf}$.

3.7. *Black and White Trees*

Definition 3.7.1. A black and white graph (b/w graph) τ is a graph together with a function color : $V(\tau) \to \{0, 1\}$.

We call the set $V_w(\tau) := \text{color}^{-1}(0)$ the set of white vertices and call the set $V_b(\tau) := \text{color}^{-1}(1)$ the set of black vertices.

By a bipartite b/w tree we understand a b/w tree whose edges only connect vertices of different colors.

An S labelled b/w tree is a b/w tree in which exactly the white vertices are labelled, i.e. $V_l = V_w$ and $V_u = V_b$.

For a rooted tree we call the set of black leaves the tails.

A rooted b/w tree is said to be without tails, if all the leaves are white.

A rooted b/w tree is said to be stable if there are no black vertices of valence 1.

A rooted b/w tree is said to be fully labelled, if all vertices except for the root and the tails are white and labelled.

Definition 3.7.2. For a black and white bi–partite tree, we define the set of white edges $E_w(\tau)$ to be the edges $(v_1, N(v_1))$ with $N(v_1) \in V_w$ and call the elements white edges and likewise we define E_b with elements called black edges, so that there is a partition $E(\tau) = E_w(\tau) \amalg E_b(\tau)$.

Notation 3.7.3. For a planar planted b/w tree, we understand the adjective bipartite to signify the following attributes: both of the vertices of the root edge are black, i.e. *root* and the vertex $N^{-1}(root)$ is black and the tree after iteratively contracting tails is bipartite otherwise. By iterative contraction of tails, we mean that the operation of contracting the tail edges is repeated until there are no tail edges left. The root edge is considered to be a black edge. Also in the presence of tail, all non–white tail edges are considered to be black.

Definition 3.7.4. For a planar planted τ b/w bi–partite tree we understand by the branch of $e = (b, N(b)$, $b \in V_b(\tau)$ to be the planar planted bi–partite rooted tree which given by the branch of e where the color of $N(b)$ is changed to black and this black vertex is the root. In the case that the tree to which e is labelled the branch of e is the tree which is labelled by the set of labels of its white edges – we stress that this does not include the root $N(b)$. If the tree τ is labelled then the tree $br(e)$ for any $e \in E(\tau)$ is a labelled tree with the labelling induced by that of τ. Notice that by definition the root of $br(e)$ will be unlabelled.

3.8. *Notation I*

N.B. A tree can have several of the attributes mentioned above; for instance, we will look at bipartite planar planted rooted trees. To fix the set of trees, we will consider the following notation. We denote by $\mathcal{T}$ the set of all trees and use sub and superscripts to indicate the restrictions. The superscript r, pp, nt will mean rooted and planar planted, without tails while the subscripts $b/w, bp, st$ will mean black and white, bi–partite, and stable, where bi–partite and stable insinuate that the tree is also b/w. E.g.

$\mathcal{T}^r$ The set of all rooted trees

$\mathcal{T}^{pp}_{b/w}$ The set of planar planted b/w trees

$\mathcal{T}^{pp}_{bp}$ The set of planar planted bipartite trees

Furthermore we use the superscripts fl and ll for fully labelled and leaf labelled trees. E.g.

$\mathcal{T}^{ll}$ The set of all rooted leaf labelled trees

We furthermore use the notation that $\mathcal{T}(n)$ denotes the n–labelled trees and adding the sub and superscripts denotes the n–labelled trees of that particular type conforming with the restrictions above for the labelling. Likewise $\mathcal{T}(S)$ for a set S are the S labelled trees conforming with the restrictions above for the labelling. E.g.

$\mathcal{T}^{pp}_{b/w}(n)$ The set of planar planted b/w trees with n white vertices which are labelled by the set $\{1, \ldots, n\}$.

3.9. *Notation II*

Often we wish to look at the free Abelian groups or free vector spaces generated by the sets of trees. We could introduce the notation $Free(\mathcal{T}, \mathbb{Z})$ and $Free(\mathcal{T}, k)$ with suitable super and subscripts, for the free Abelian groups or vector spaces generated by the appropriate trees. In the case that there is no risk of confusion, we will just denote these freely generated objects again by $\mathcal{T}$ with suitable sub and superscripts to avoid cluttered notation. If we define a map on the level of trees it induces a map on the level of free Abelian groups and also on the k vector spaces. Likewise by tensoring with k a map on the level of free Abelian groups induces a map on the level of vector spaces. Again we will mostly denote these maps in the same way.

3.10. *Notation III*

If we will be dealing with operads of trees, we will consider the collection of the $\mathcal{T}(n)$ with the appropriate sub and superscripts. Again to avoid cluttered notation when dealing with operads, we also denote the whole collection of the $\mathcal{T}(n)$ just by $\mathcal{T}$ with the appropriate sub and superscripts.

3.11. *The Map cppin* : $\mathcal{T}^r \to \mathcal{T}^{pp}_{bp}$

There is a map from planted trees to rooted trees given by contracting the root edge. This map actually is a bijection between planted and rooted trees. The inverse map is given by adding one additional vertex which is designated to be the new root and introducing an edge from the new root to the old root. We call this map *plant*.

Also, there is a map *pin* from the free Abelian group of planted trees to that of planted planar trees given by.

$$pin(\tau) = \sum_{p \in \text{Pinnings}(\tau)} (\tau, p)$$

Finally there is a map from planted planar trees to planted–planar bi–partite trees. We call this map *bp*. It is given as follows. First color all vertices white except for the root vertex which is colored black, then insert a black vertex into every edge.

In total we obtain a map

$$cppin := bp \circ pin \circ plant : \mathcal{T}^r \to \mathcal{T}_{bp}^{pp}$$

that plants, pins and colors and expands the tree in a bipartite way.

Using the map *cppin*, we will view $\mathcal{T}^r$ as a subgroup of $\mathcal{T}_{bp}^{pp}$. The image of $\mathcal{T}^r$ coincides with the set of invariants of the actions $\mathbb{S}(\tau)$. We will call such an invariant combination a symmetric tree.

Remark 3.11.1. The inclusion above extends to an inclusion of the free Abelian group of fully labelled rooted trees to labelled bi–partite planted planar trees: $cppin : \mathcal{T}^{r,fl}(n) \to \mathcal{T}_{bp}^{pp}(n)$.

3.12. *The Map* $st_\infty : \mathcal{T}_{st}^{pp} \to \mathcal{T}_{bp}^{pp}$

We define a map from the free groups of stable b/w planted planar trees to the free group of bi–partite b/w planted planar trees in the following way: First, we set to zero any tree which has black vertices whose valence is greater than two. Then, we contract all edges which join two black vertices. And lastly, we insert a black vertex into each edge joining two white vertices. We call this map st_∞.

Notice that st_∞ preserves the condition of having no tails and induces a map on the level of labelled trees.

This nomenclature is chosen since this map in a certain precise sense forgets the trivial A_∞ structure of an associative algebra in which all higher multiplications are zero.

4. Operads

In this section we briefly review the notion of an operad and give the main examples of operads for the algebras we will be considering.

First let us fix a symmetric monoidal category $(\mathcal{C}, \otimes)$. We will use following candidates $\mathcal{S}et, \mathcal{T}op, \mathcal{C}hain, \mathcal{V}ect_k$ — the categories of finite sets with disjoint union, topological spaces with Cartesian products, the category of chain complexes of Abelian groups with tensor product over $\mathbb{Z}$ and the category of chain complexes of vector spaces over a field k together with tensor product.

Operads have been defined in many places[40]. The pure definition seems at first difficult, but the example below 4.4 is paradigmatic for the definition, and the uninitiated reader might want to start there. The cognizant reader, however, might want to skip ahead.

4.1. *Operads*

Definition 4.1.1. An operad in $\mathcal{C}$ is a collection of objects $\mathcal{O} := \{O(n) : O(n) \in \mathcal{C}, n \geq 1\}$ together with an $\mathbb{S}_n$ action on $O(n)$ and maps

$$\circ_i : O(m) \otimes O(n) \to O(m+n-1), i \in \{1, \dots m\} \tag{4.1}$$

which are associative and $\mathbb{S}_n$–equivariant and an element $id \in O(1)$ that satisfies for all $op_n \in O(n), i \in \{1, \dots, n\}$

$$\circ_i(op_n, id) = \circ_1(id, op_n) = op_n$$

i) Associativity: for $op_k \in O(k), op'_l \in O(l)$ and $op''_m \in O(m)$

$$(op_k \circ_i op'_l) \circ_j op''_m = \begin{cases} (op_k \circ_j op''_m) \circ_{i+m-1} op'_l) & \text{if } 1 \leq j < i \\ op_k \circ_i (op'_l \circ_{j-i+1} op''_m) & \text{if } i \leq j < i+l \\ (op_k \circ_{i-l+1} op'_l) \circ_j op''_m & \text{if } i+l \leq j \end{cases}$$

ii) Equivariance: $op_m \in O(m)$ and $op_n \in O(n)$

$$\sigma_m(op_m) \circ_i \sigma'_n(op_n) = \sigma_m \circ_i \sigma'_n(op_m \circ_{\sigma_m(i)} op_n)$$

where $\sigma_m \circ_i \sigma'_n \in \mathbb{S}_{m+n-1}$ is the block or iterated permutation

$$(1, 2, \dots, i-1, (1', \dots, m'), i+1 \dots, n) \mapsto \\ \sigma_n(1, 2, \dots, i-1, \sigma'_m(1', \dots, m'), i+1 \dots, n) \tag{4.2}$$

induces on $(1'', \dots, (m+n-1)'')$ where

$$j'' = \begin{cases} j & 1 \leq j \leq i-1 \\ (j-i+1)' & i \leq j \leq i+n-1 \\ j-n & i+n \leq j \leq m+n-1 \end{cases}$$

That is the permutation which permutes the j'' with $i \leq j \leq i+n-1$ according to σ_n and then permutes the all of the j'' according to σ_m treating the previous j'' as a block in the position i.

As an example of a block permutation let's regard $(123) \circ_2 (12)$ this is the permutation

$$\begin{pmatrix} 1324 \\ 3241 \end{pmatrix} \circ \begin{pmatrix} 1234 \\ 1324 \end{pmatrix} = \begin{pmatrix} 1234 \\ 3241 \end{pmatrix}$$

We call the operads in $\mathcal{S}et, \mathcal{T}op, \mathcal{C}hain, \mathcal{V}ect_k$ combinatorial, topological operads, chain operads and linear operads, respectively. We also call $O(n)$ the n–th component of $\mathcal{O}$.

Remark 4.1.2. Both the three different cases for associativity and the block permutation can be naturally understood in the examples of functions and trees.

Definition 4.1.3. A morphism of two operads $\mathcal{O}, \mathcal{O}'$ in the same monoidal category is a collection of morphisms from $O(n) \to O'(n)$ which respect all the structures, i.e. respect the glueings and are $\mathbb{S}_n$ *equivariant*. Such a morphism is also called an operadic morphism.

A suboperad is an injective morphism of operads. In this case, we call $\mathcal{O}$ a suboperad of $\mathcal{O}'$.

4.2. *Induced Operads*

As we mentioned before, the categories we are interested in are $\mathcal{S}et, \mathcal{T}op, \mathcal{C}hain$ and $\mathcal{V}ect_k$. Now there is the singular homology functor $H_* : \mathcal{T}op \to \mathcal{V}ect_k$ which given a topological operad yields a linear operad. Using the functor of singular chains $C_* : \mathcal{T}op \to \mathcal{C}hain$ gives a map from topological operads to chain complexes and finally taking homology $H_* : \mathcal{C}hain \to \mathcal{V}ect_k$ of a chain complex yields a functor from chain operads to linear operads.

For the intermediate level, the chain level, there might be other operad structures depending on the choice of model for the chains which is compatible with the operadic compositions.

Given a CW complex one can try to define an operad structure on the level of CC_* of cellular chains, and given a triangulated space one can consider the simplicial chains C_*^Δ. In these cases one has to additionally check that the compositions on the chains are indeed (a sum) of chains to obtain an acceptable model.

There is also a functor of $\mathcal{F} : \mathcal{S}et \mapsto \mathcal{C}hain$ which associates the free Abelian group to a set, however, usually there are some signs that appear in geometric situations coming from different orientations. In a given case, there might also be a candidate of a possible non–trivial differential.

4.3. *The Fundamental Examples*

There are two fundamental examples which help to explain the notion of an operad. If one wishes, an operad is the abstraction of the algebraic properties of these examples. They are the operad of functions and the operad of leaf labelled trees.

4.4. *The Operad of Functions*

Fix a set X and regard $Funct(n) := Map(X^n, X)$. The $\mathbb{S}_n$ action is the action induced by permutation of the variables:

$$\sigma(f)(x_1, \dots, x_n) := f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

and the maps $\circ_i$ are defined to be the substitution maps

$$(f \circ_i g)(z_1, \dots z_{m+n-1}) := f(z_1, \dots . z_{i-1}, g(z_i, \dots, z_{i+n-1}), z_{i+n}, \dots, z_{m+n-1})$$

This map is the following substitution: say f is a function of the variables x_i and g is a function of the variables y_j then setting $x_i = g(y_1, \dots, y_n)$ yields

$$\begin{aligned} f(x_1, \dots, x_m) \circ_i g(y_1, \dots, y_n) &= f(x_1, \dots . x_{i-1}, g(y_1, \dots, y_n), x_{i+1}, \dots, x_m) \\ &= (f \circ_i g)(z_1, \dots, z_{m+n-1}) \end{aligned}$$

with

$$z_j = \begin{cases} x_j & 1 \leq j \leq i-1 \\ y_{j-i+1} & i \leq j \leq i+n-1 \\ x_{j-n} & i+n \leq j \leq m+n-1 \end{cases} \tag{4.3}$$

The $\mathbb{S}_n$ equivariance means that when first applying a fixed permutation $\sigma_x \in \mathbb{S}_n$ to the x_j and a fixed permutation $\sigma_y \in \mathbb{S}_m$ to the y_k and then composing, there is a permutation σ_z of the z_l which has the same effect, i.e.

$$\sigma_x(f) \circ_i \sigma_y(g) = \sigma_z(f \circ_{\sigma_x(i)} g)$$

The permutation which can be constructed from this condition is in fact unique and is exactly the block permutation $\sigma_x \circ_i \sigma_y$ for the general case.

The associativity translates to the fact that if one does two substitutions the order in which the substitutions are performed does not matter. Notice that this gives the three cases in the general definition. The first and the third case correspond to the substitutions in which two of the variable x, say x_i and x_j of the function f are substituted by functions g_1 and g_2 respectively. The first case corresponds to a substitution in which where $i < j$ and the third to the case $i > j$. The second case corresponds to a nested substitution for three functions f, g, h. The two sides of the equation being either first substituting g into f and then substituting h into the outcome of this substitution or first substituting h into g and then substituting the outcome of this substitution into f.

4.5. *Rooted Leaf Labelled Trees*

Another useful primordial example is that of rooted trees with labelled leaves $\mathcal{T}^{ll}$.

The n–th component of this operad are rooted trees with n labelled leaves. The $\mathbb{S}_n$ action on the n–th component is given by permuting the labels.

The operation of grafting defines the compositions. The composition $\tau \circ_i \tau'$ is defined to be the rooted tree obtained by grafting the tree τ' to the vertex v_i of τ.

To graft two rooted leaf labelled trees τ and τ' at the vertex v_i of τ identify the root of τ' with the vertex v_i. The root of the tree is the image of the root of τ. We label the leaves of the composition analogously to the example 4.4 in equation (4.3). The $\mathbb{S}_n$ equivariance then follows naturally.

An example of grafting is as depicted in figure 1.

The condition of associativity which is met can be rephrased as stating, that the result of several graftings does not depend on the order in which they are performed. There are two basic situations for associativity which are depicted in figure 2. They correspond to the first and third case, and the second case.

Remark 4.5.1. The grafting procedure still makes sense for planar trees with labelled leaves, by keeping the linear order at the grafted vertex.

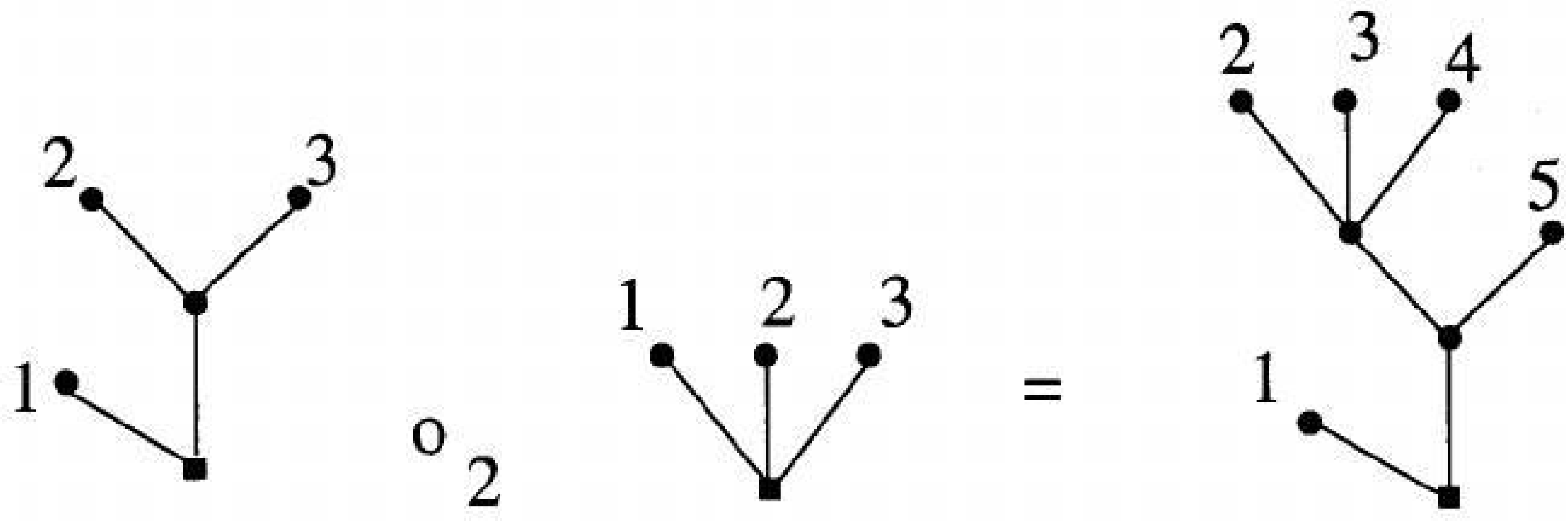


Figure 1. Example of grafting two trees

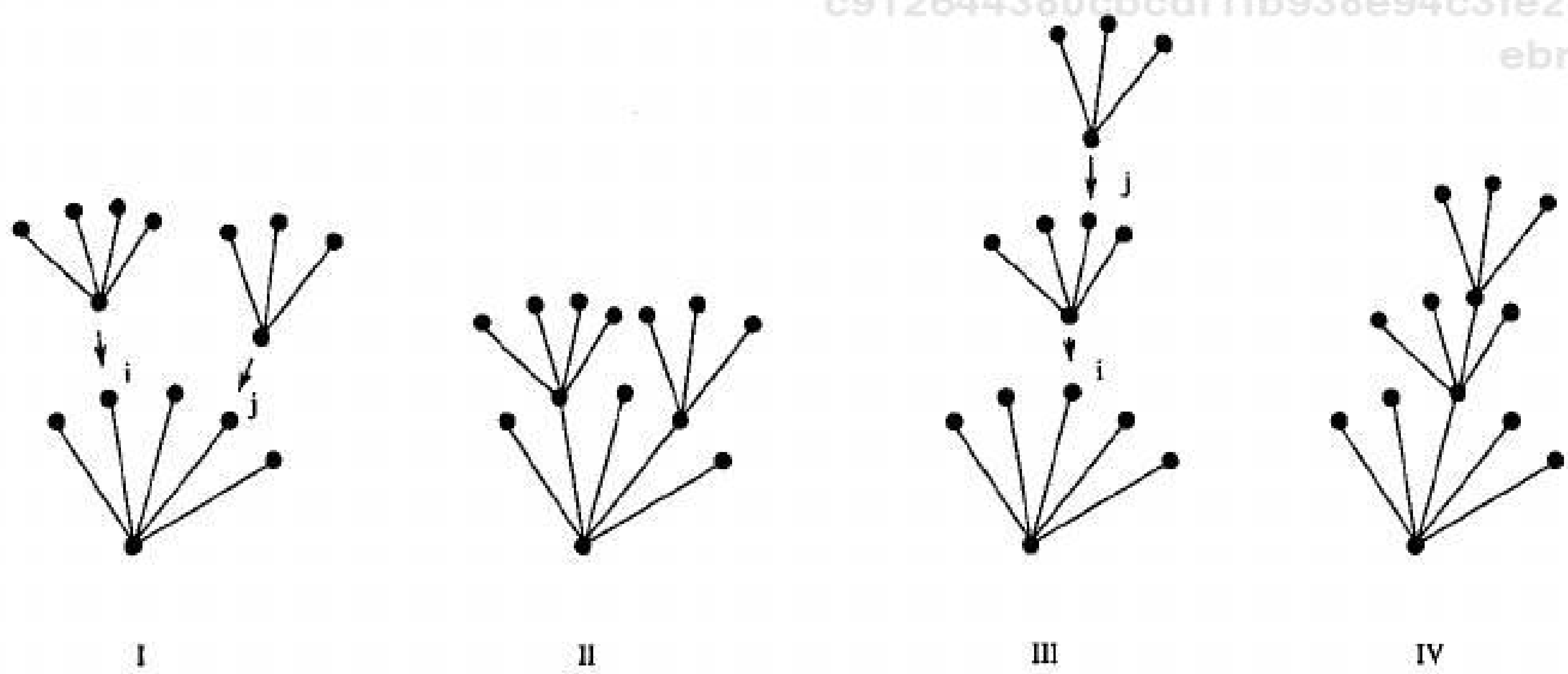


Figure 2. The first type of associativity for gluing is depicted in I. II shows the result of either order of gluing. The second type of associativity is depicted in III. The result of either order of gluing for this case is depicted in IV.

Definition 4.5.2. In the case of planar planted trees, we fix by convention that for planted planar trees $\tau \circ_i \tau'$ denotes the planted planar tree in which after the grafting the image of the root edge of τ' is contracted.

An example of his procedure is as depicted in figure 3.

Definition 4.5.3. In the case of planar planted bi-partite trees, the grafting operation $\circ_i$ for v_i a leaf is defined to be the grafting operation for planar planted trees followed by the contraction of the image of the outgoing edge of v_i.

An example of his procedure is as depicted in figure 4. Notice that at the moment, since we only defined how to glue for leaves, this is only a partial operad structure (cf. §4.10.5); the full operad structure is explained in §4.7.

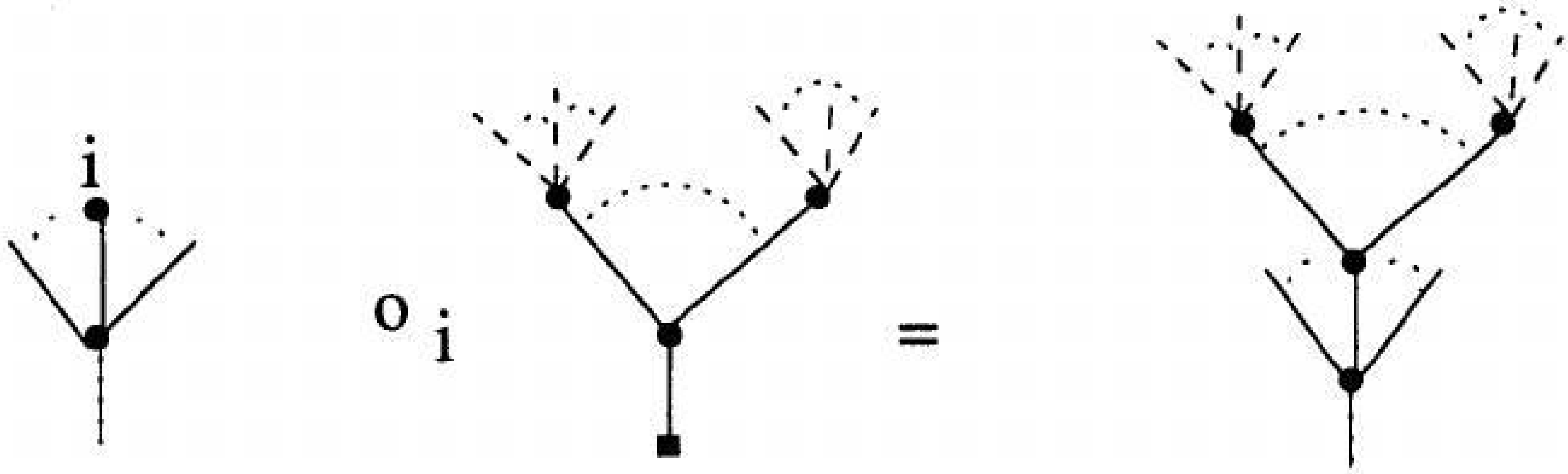


Figure 3. Grafting two planted planar trees

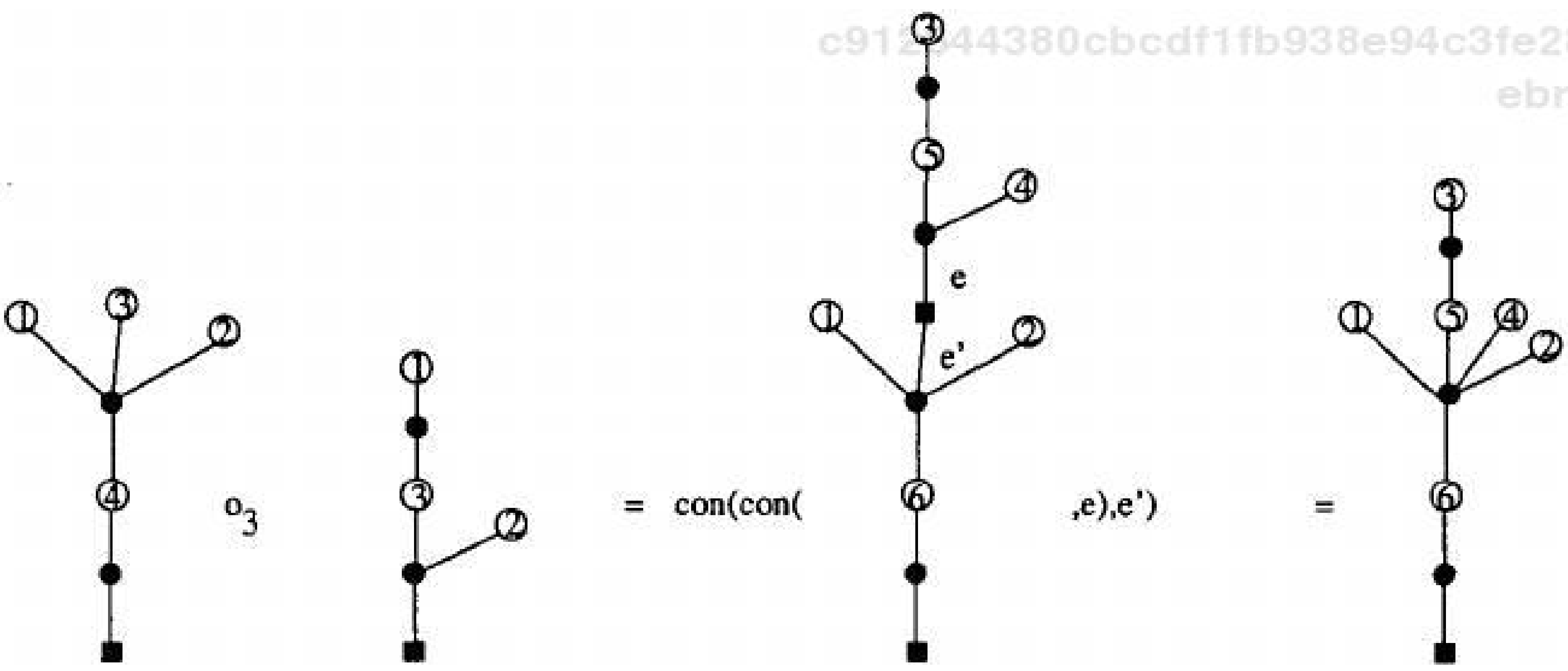


Figure 4. Grafting two bi-partite planted planar trees

4.6. *Bordered Surfaces and Corollas*

We can also consider bordered topological surfaces with or without punctures and genus. Here the space $\Sigma(n)$ is the space of homeomorphism classes of bordered surfaces $(\Sigma, \partial\Sigma)$ whose boundary is homeomorphic to the disjoint union of $n+1$ circles which are labelled by $0, \ldots, n$: $\partial\Sigma \sim_{homeo} \amalg_{i=0,\ldots,n} S^1_i$.

The operad structure is defined via glueing of surfaces with boundary along their boundaries. I.e. $\Sigma \circ_i \Sigma'$ is the surface obtained by glueing the boundary 0 of Σ' to the boundary i of Σ as depicted in Figure 5. The $\mathbb{S}_n$ action is given by permuting the labels, and the labelling after the grafting is again defined analogously to equation (4.3). This guarantees the $\mathbb{S}_n$ equivariance and the associativity.

Another way to graphically encode the same operad is to represent each surface by a tree which has a root vertex and n labelled leaves. Such a tree

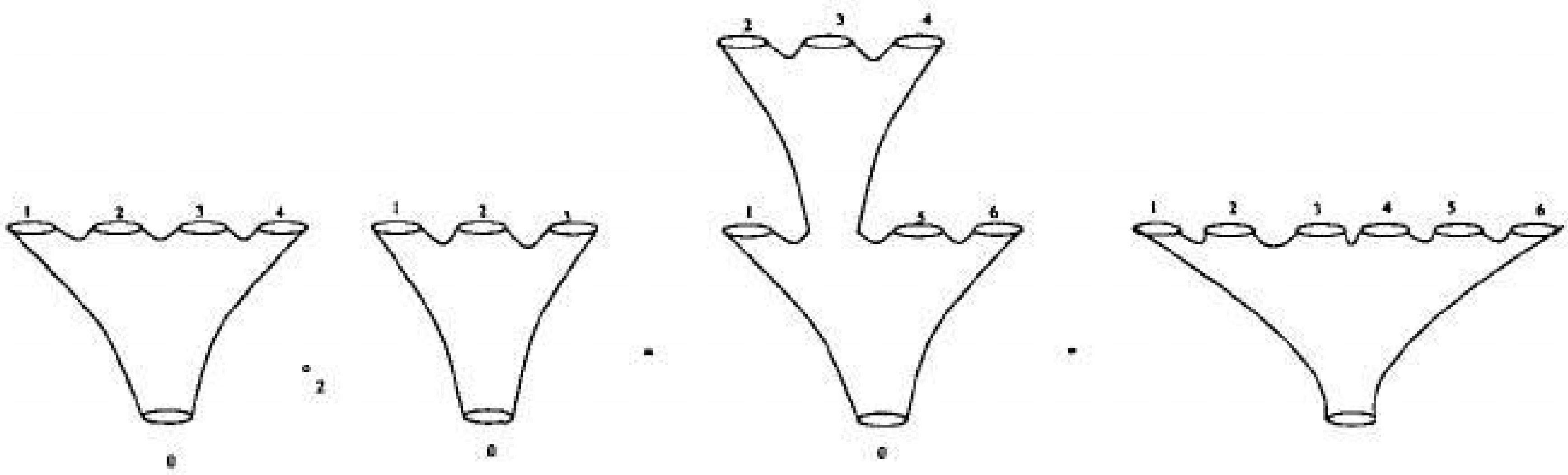


Figure 5. Example of grafting two surfaces

is called a (labelled) corolla.

The operad structure comes from glueing corollas τ, τ' and then contracting the edge, $(root(\tau'), v_i)$. The $\mathbb{S}_n$ action is by permutation the labels and the labelling of the composition is again defined according to (4.3).

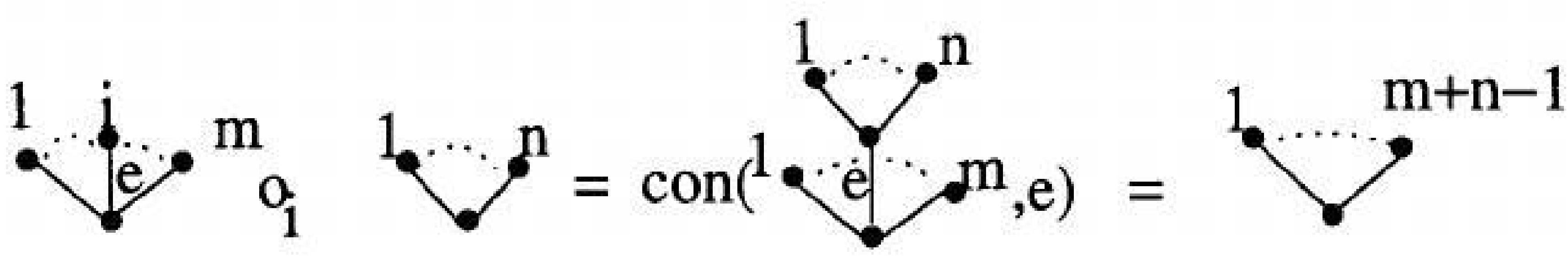


Figure 6. Example of grafting two corollas

In order to make the two pictures meet, we plant the corolla. This planted corolla then has one internal vertex representing the surface, a root which represents the boundary 0 and leaves representing the other boundaries. In this picture, we can view the surface as a thickening of the (planted) corolla.

4.7. *Tree Insertion Operads*

Lately, another type of operad of trees has appeared[36,7] where the glueings are not restricted to the leaves. This means that there is a grafting procedure into inner vertices as well. The natural origin of these operads is discussed in detail section 8.

We will need the following variant of an insertion operad structure[30]. It is defined on the collection of free Abelian groups $\mathcal{T}_{bp}^{pp}(n)$ that is n–labelled rooted planar planted bipartite trees. Recall that this means that there are n white vertices which are labelled from 1 to n.

The description in words is as follows: there are three steps. First cut off the branches corresponding to the incoming edges of v_i. Notice that these branches have a linear order according to the linear order of the edges adjacent to v_i. Second graft τ' as a planar planted tree onto the remainder of τ at the vertex v_i which is now a leaf. This grafting is in the sense of bi–partite planar planted trees. Lastly sum over the possibilities to graft the cut off branches onto the white vertices of the resulting planar tree which before the grafting belonged to τ'. Since we are dealing with planted planar trees, the grafting entails a choice of the linear order at the vertices at which we graft after the grafting. We only sum over those choices in which the order of the branches given by the linear order at v_i is respected by the grafting procedure. I.e. the branches after grafting appear in the same order on the grafted tree as they did in τ.

An example of such an insertion is depicted in figure 7

To make this definition precise:

Definition 4.7.1. Given $\tau \in \mathcal{T}_{bp}^{pp}(m)$ and $\tau' \in \mathcal{T}_{bp}^{pp}(n)$, we define the tree

$$\tau \circ_i \tau' := \sum_{(gr:N^{-1}(v_i)\to V_w(\tau'),p(gr))}{}' (\tau_{gr}, p(gr)) \in \mathcal{T}_{bp}^{pp}(m+n-1)$$

where

(1) gr is a bijection.

(2) τ_{gr} is the tree whose vertices are $V(\tau \circ_i \tau') := (V_\tau \setminus \{v_i\}) \amalg (V_{t'}) \setminus \{root(\tau'), N^{-1}(root(\tau'))\})$ and the following edges: the root edge of τ' and the outgoing edge of v_i are deleted, all edges not incident to v_i or $N^{-1}(root(\tau'))$ are kept, the edges $(w, N^{-1}(root(\tau')))$ are replaced by edges $(w, N(v_i))$. The order of the edges at $N(v_i)$ is given by first enumerating the edges incident to $N(v_i)$ which came before the edge $(v_i, N(v_i))$, then the edges $(w, N(v_i))$ according to the order of incidence of $(w, N^{-1}(root(\tau')))$ at $N^{-1}(root(\tau')$ and lastly the rest of the edges incident to $N(v_i)$ which came after the edge $((v_i), N(v_i))$. This is precisely the grafting of τ' onto the tree τ with cutoff branches at v_i. Finally the incoming edges of v_i are connected according to gr, i.e. give rise to edges $(v, gr(v))$ for $v \in N^{-1}(v_i)$.

(3) $p(gr)$ is a pinning of τ_{gr} and

(4) the $'$ on $\sum'$ indicates that the sum is over all compatible pairs of a bijection gr and a pinning $p(gr)$ that preserve the linear order. Here compatibility is the following: Let $\prec^{p(gr)}$ be the linear order induced by the pinning $p(gr)$. We call gr and $p(gr)$ compatible if

the linear order on the edges τ' as well as for the edges of τ that are kept is respected and $e_l = (v_l, v_i) \prec e_k = (v_k, v_i)$ in τ implies that $(v_l, gr(v_l)) \prec^{p(gr)} (v_k, gr(v_k))$ in the linear order $\prec^{p(gr)}$ of τ_{gr}.

4.8. *Signs*

As discussed later on, if the trees have a geometrical interpretation, in terms of cells of a complex, then it is necessary to introduce signs

$$\tau \circ_i \tau' := \sum_{(gr:N^{-1}(v_i)\to V_w(\tau'),p(gr))}' \pm(\tau_{gr}, p(gr))$$

into the concatenations, which are dictated by the orientations of the cells. One such consistent choice of sign is provided as follows: we order the white edges of τ and t' according to the linear order of the respective trees. Now the white edges of $\tau \circ_i \tau'$ are correspond exactly to the union of white edges of τ and τ'. We now define the sign to be the sign of the shuffle which shuffles the edges of τ' into their position, i.e. the shuffle from $E_w(\tau) \amalg E_w(\tau)$ to $E_w(\tau \circ_i \tau')$ where we regard E_w as ordered. There are also other natural choices as discussed in §7.7.

Remark 4.8.1. With the above compositions $\mathcal{T}_{bp}^{pp,nt}$ is a suboperad of $\mathcal{T}_{bp}^{pp}$.

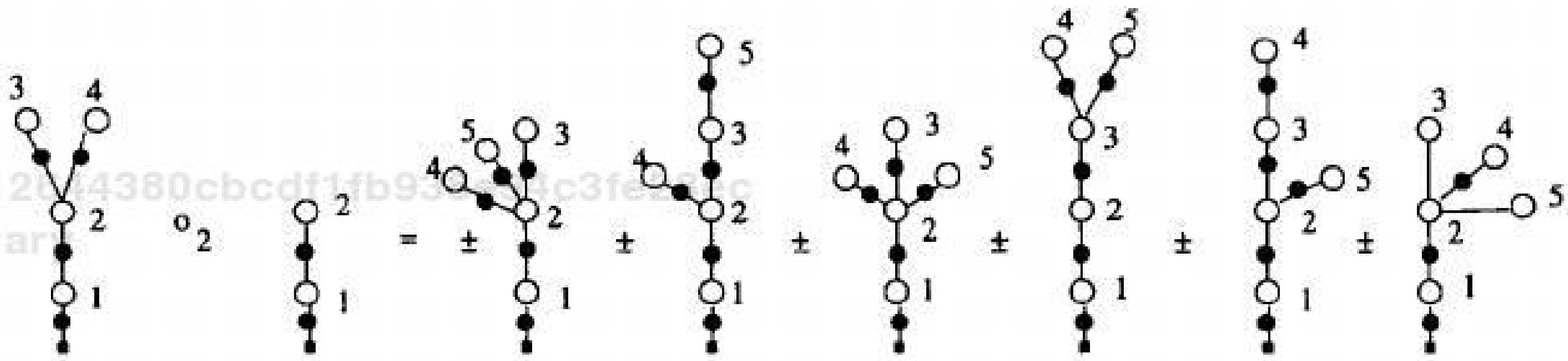


Figure 7. Example of the insertion of a bi-partite planted planar tree

4.9. *Other Tree Insertion Operads and Compatibilities*

There are two tree insertion operads structures already present in the literature on rooted trees $\mathcal{T}^{r,fl}$ [7] and, historically the first, on planar planted stable b/w trees without tails $\mathcal{T}_{b/w}^{pp,st,nt}$[36].

In the gluing for $\mathcal{T}^r$ one omits mention to the order and in the case of $\mathcal{T}_{b/w}^{pp,st,nt}$ one also allows glueing to the images of the black vertices of τ'.

Also in the first case the basic grafting of trees is used (no contractions) while in the second case the grafting for planted trees is used, i.e. the image of the root edge is contracted, but not the outgoing edge of v_i.

The signs for the first gluing are all plus[7] and in the second gluing are dictated by a chain interpretation[36] see also below.

Remark 4.9.1. The operad structure defined above restricted to $\mathcal{T}_{bp}^{pp,nt}$ is compatible with those of $\mathcal{T}^{r,fl}$ and $\mathcal{T}_{b/w}^{pp,st,nt}$ under the maps *cppin* and st_∞.

In 7.7, signs for the operad are defined by giving a cell interpretation. The result is that there is an orientation for top–dimensional cells corresponding to a choice of signs which makes *cppin* into an operadic map and an orientation of cells which fixes the signs in such a way that st_∞ is an operadic map, see §10.3 and §9.9.

There are topological versions of this type of insertion glueing[49,34,29] as explained in §5 and §6.

4.10. *Variations of Operads*

There are some variations of the structure of operads which we will need.

Remark 4.10.1. One can also consider operads as collections $\mathcal{O}(n)$ starting at $n = 0$.

Definition 4.10.2. If the requirement of having an identity id omitted the resulting structure is called a non–unital operad. Omitting the $\mathbb{S}_n$ action and the $\mathbb{S}_n$ equivariance yields the structure of a non–Σ operad.

It is clear that when considering operads one can consider indexing by arbitrary sets instead.

We will also need the following weakening of the structure of an operad:

Definition 4.10.3. [29] A quasi–operad is an operad where associativity need not hold.

Remark 4.10.4. If a quasi–operad in the topological category is homotopy associative then its homology has the structure of an operad. In certain cases, like the ones we will consider, the structure of an operad already exists on the level of a particular chain model.

Definition 4.10.5. Lastly in a partial operad the concatenations need not all be defined on the whole components $\mathcal{O}(n)$, but when they are defined the axioms hold.

Definition 4.10.6. A cyclic operad is an operad together with an action of $\mathbb{S}_{n+1}$ on each $\mathcal{O}(n)$ where $\mathbb{S}_{n+1}$ acts on the set $\{0, 1, \ldots, n\}$ extending the action of $\mathbb{S}_n$ on $\{1, \ldots, n\}$ and additionally satisfying the following compatibility. Let * denote the action of the cycle $(012\ldots n) \in \mathbb{S}_{n+1}$ then for $a \in O(n)$

$$(a \circ_n b)^* = (-1)^{|a||b|} b^* \circ_1 a^* \tag{4.4}$$

where we denoted the possible $\mathbb{Z}/2\mathbb{Z}$ degree of an element a by $|a|$.

The additional axiom means that the root is not distinguished any more.

Examples are the $\Sigma(n)$ and the operad of rooted trees with grafting on leaves and also the operads $\bar{M}_{g,n}$, $M_{g,n}$ and $\mathcal{A}rc$ (see below). In the operad $\Sigma(n)$ the cyclic nature is inherent since there is no topological distinction between the boundary components.

An example illustrating the equation (4.4) using leaf labelled trees together with a labelling of the root by 0 is given in figure 8

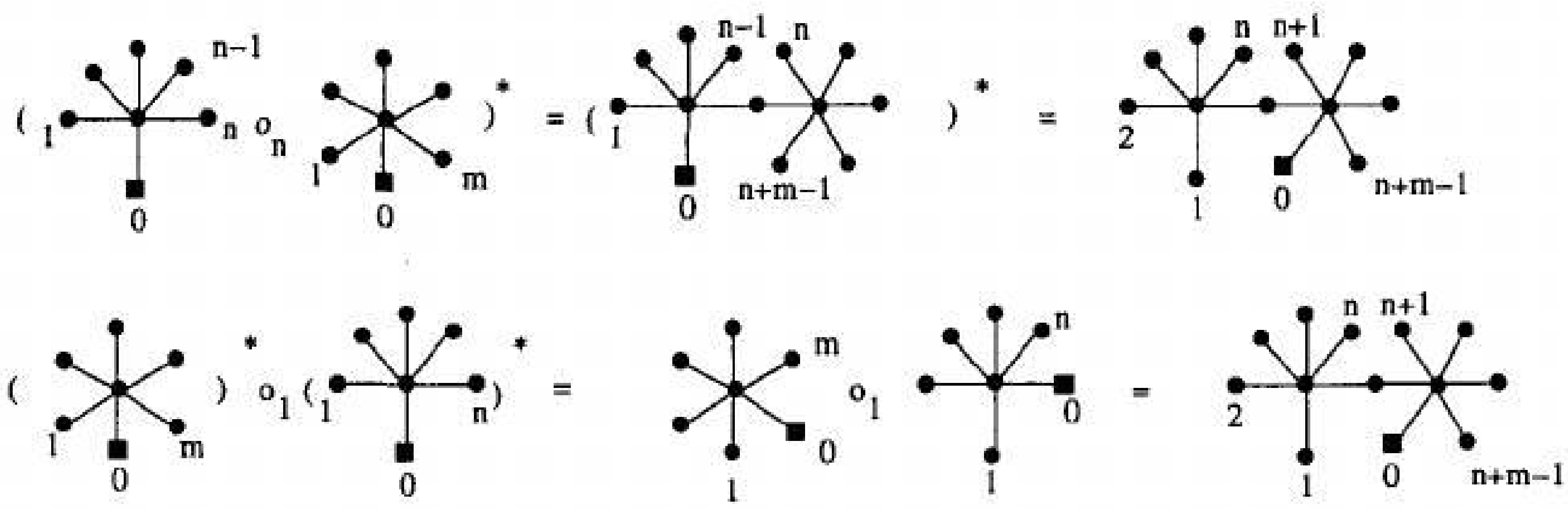


Figure 8. An illustration of the equation (4.4)

4.11. *Algebras Over Operads*

If we take the example 4.4 to be linear functions of a fixed vector space V onto itself, we obtain the linear operad $\mathcal{H}om_V$ with $Hom_V(n) := Hom_{k-lin}(V^{\otimes n}, V)$

Definition 4.11.1. An algebra V over a linear operad $\mathcal{O}$ is a vector space V over k together with an operadic map $\mathcal{O} \to \mathcal{H}om_V$.

The idea of the definition is that each element of $O(n)$ defines a n–ary multilinear operation on V. The *Ur*–operation one has in mind is an algebra multiplication $V^{\otimes 2} \to V$.

Definition 4.11.2. An algebra over a cyclic operad $\mathcal{O}$ is vector space V together with a symmetric non–degenerate bilinear form $\langle\, ,\, \rangle$ and a map of operads $\phi : \mathcal{O} \to \mathcal{H}om_V$ s.t.

$$\langle v_0, \phi(op_n)(v_1, \ldots, v_n)\rangle = \langle v_n, \phi(op_n^*)(v_0, v_1, \ldots, v_{n-1})\rangle$$

for $op_n \in O(n)$.

Remark 4.11.3. The concept of an algebra over a cyclic operad illustrates the idea of a cyclic operad. The prime example being $\mathcal{H}om_V$. Now if V has a non–degenerate bi–linear form then $\mathcal{H}om_v(n) = V^{\otimes n} \otimes V^* \cong V^{\otimes n+1}$. This isomorphism makes the action of $\mathbb{S}_{n+1}$ perspicuous. Sometimes this is stated as that $\mathbb{S}_{n+1}$ also permutes the symbol of the function.

As an example Frobenius algebras –that is associative, commutative algebras with a non–degenerate bi–linear pairing, which is invariant with respect to the pairing $\langle a, bc\rangle = \langle ab, c\rangle$– are algebras over the cyclic operad $\Sigma(n)$ where the $\mathbb{S}_{n+1}$ action permutes all labels $0, 1, \ldots, n$.

4.12. *Operads Classifying Algebras*

More examples of operads come from operads classifying algebras; i.e. operads s.t. each algebra over them is of a certain type –like associative commutative etc– and vice–versa each algebra of the given type is indeed an algebra over this operad.

4.13. *The Operad for Commutative Algebras*

The operad $\mathcal{COM}$ of associative, commutative algebras over a field k is given by

$$Com(n) := k \text{ as the trivial } \mathbb{S}_n \text{ module}$$

With glueing maps given by the identification $k \otimes_k k \simeq k$.

It is easily seen that this operad coincides with the operad $\Sigma(n)$ or that of corollas if one takes their k–linear span.

For an algebra over this operad the operation for $1 \in k = Com(2)$ defines a map $V^{\otimes 2} \to V : v_1 \otimes v_2 \mapsto v_1 \cdot v_2$ which has to be commutative since it is $\mathbb{S}_2$ invariant. Furthermore, it is associative from glueing two two–corollas in the two possible different ways to obtain the three–corolla.

The element $1 \in k = Com(n)$ represented by the n–corolla is necessarily the map $V^{\otimes n} \mapsto V : v_1 \otimes \cdots \otimes v_n \mapsto v_1 \cdot \cdots \cdot v_n$ which is uniquely defined since the algebra is associative and the multiplication is independent of the order due to the $\mathbb{S}_n$ invariance.

4.14. *The Operad for Associative Algebras*

The operad $\mathcal{ASSOC}$ of associative algebras over a field k is given by

$$Assoc(n) := \text{The regular representation of } \mathbb{S}_n$$

with glueing maps given by composition of permutations as in the axiom of $\mathbb{S}_n$ equivariance of 4.1.1.

It is easily seen that this operad coincides with planar corollas.

An algebra over this operad has two multiplications $\cdot$ and $\cdot^{op}$ coming from $\mathcal{ASSOC}(2)$. The multiplication $\cdot$ is associative and the planar n–corollas corresponding to the basis element e_σ of the regular representation of $\mathbb{S}_n$ represent the maps $v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \cdot \ldots \cdot v_{\sigma(n)}$

This operad is the same as the natural one for planar planted corollas.

4.15. *The Operad for Gerstenhaber Algebras*

Definition 4.15.1. A Gerstenhaber algebra A is a graded vector space together with two operations, a graded commutative and associative product $\cdot$ of degree 0, a bracket $\{\bullet\}$ of degree 1 (which is a Lie graded bracket on ΣA, the suspension of A sometimes also called an odd Lie–bracket) such that the bracket is odd Poisson for the multiplication $\cdot$.

More precisely: if we denote the degree of $x \in A$ by $|x|$ and by $|sx| = |x| + 1$ (the degree of x in ΣA), then the following equations hold:

$$\begin{aligned}
(x \cdot y) &= (-1)^{|x||y|} y \cdot x \\
x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\
\{x \bullet y\} &= -(-1)^{|sx||sy|}\{y \bullet x\} \\
\{x \bullet \{y \bullet z\}\} &= \{\{x \bullet y\} \bullet z\} + (-1)^{|sx||sy|}\{y \bullet \{x \bullet z\}\} \\
\{x \bullet y \cdot z\} &= \{x \bullet y\} \cdot z + (-1)^{|sx||y|} y \cdot \{x \bullet z\}.
\end{aligned}$$

The penultimate equation is sometimes called odd Jacobi and the last equation is the odd Poisson property. The idea is that $\bullet$ is of degree 1. We will actually see later on that $\bullet$ can be interpreted geometrically as a one dimensional simplex.

Definition 4.15.2. The little discs operad D_2 is the operad given by $D_2(n) :=$ {configurations of n discs labelled from 1 to n which are embedded into the unit disc of $\mathbb{R}^2$} together with the glueing described below and the $\mathbb{S}_n$ action by permuting the labels.

Let us call the unit disc the outer disc. The glueing $\circ_i$ is defined by scaling the second configuration by a homothety so that the diameter of the outer disc coincides with the diameter of the i–th disc of the first configuration and then glueing in the scaled second configuration into the i-th disc of the first configuration. This is done by identifying the scaled outer disc of the second configuration and the i–th disc of the first configuration and erasing the identified boundary.

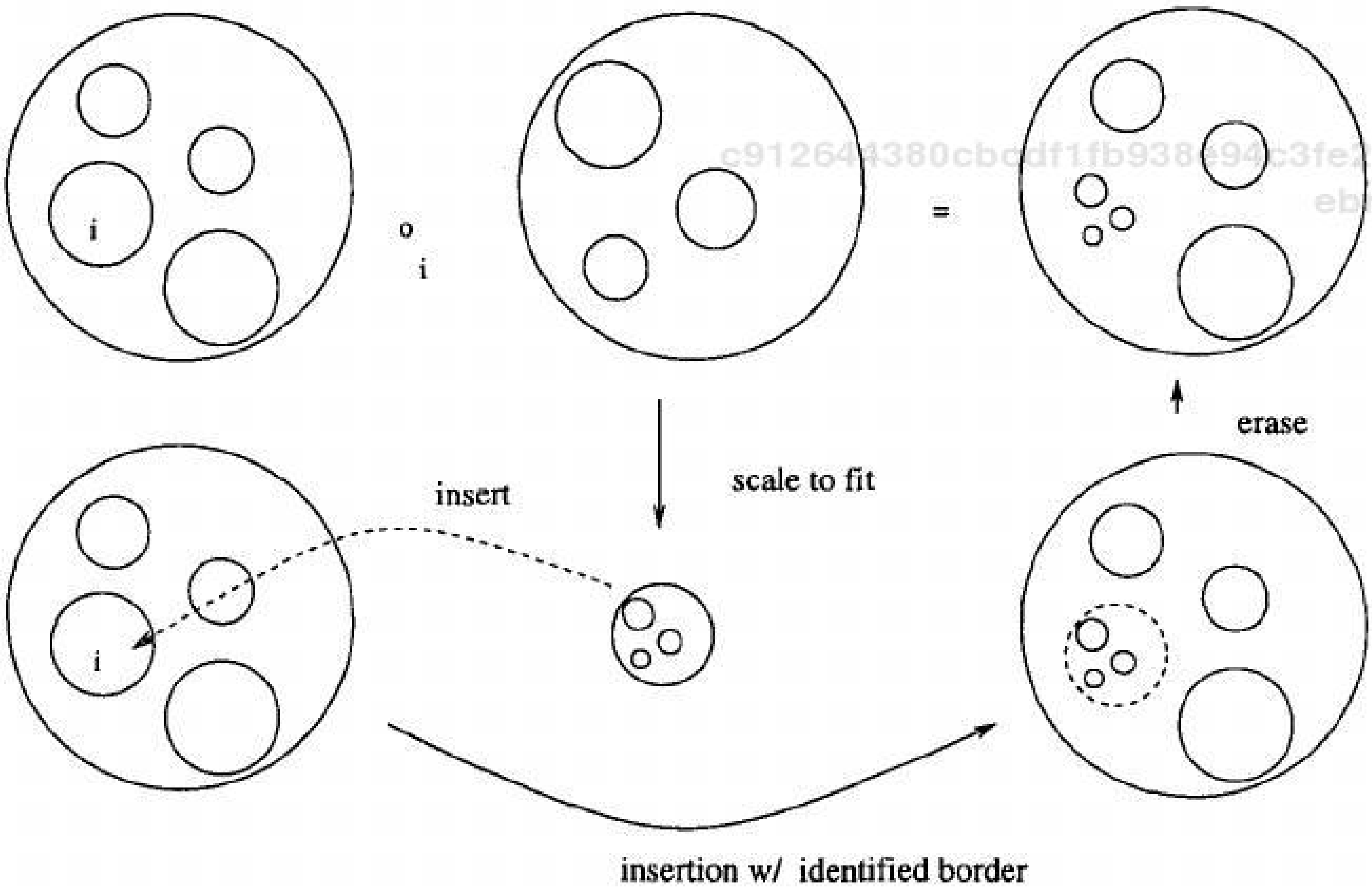


Figure 9. The composition maps for little discs

The following proposition is well known[3,4].

Proposition 4.15.3. *(Cohen)Any Gerstenhaber algebra is an algebra over the homology of the little discs operad and vice versa.*

4.16. *The Operad for Batalin–Vilkovisky (BV) Algebras*

Definition 4.16.1. A Batalin–Vilkovisky (BV) algebra is an associative super–commutative algebra A together with an operator Δ of degree 1 that satisfies

$$\Delta^2 = 0$$
$$\Delta(abc) = \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{|sa||b|}b\Delta(ac) - \Delta(a)bc$$
$$-(-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c)$$

Proposition 4.16.2. *For any BV–algebra (A, Δ) define*

$$\{a \bullet b\} := (-1)^{|a|}\Delta(ab) - (-1)^{|a|}(\Delta(a))b - a\Delta(b) \tag{4.5}$$

Then $(A, \{\bullet\})$ is a Gerstenhaber algebra.

We call a triple $(A, \{\bullet\}, \Delta)$ a GBV–algebra if (A, Δ) is a BV algebra and "$\{\bullet\} : A \otimes A \to A$ satisfies the equation (4.5)". By the above proposition $(A, \{\bullet\})$ is a Gerstenhaber algebra.

Definition 4.16.3. The framed little discs operad fD_2 is the operad given by $fD_2(n) :=$ {configurations of n discs labelled from 1 to n which are embedded into the unit disc of $\mathbb{R}^2$ together with an orientation (i.e. an angle $\theta \in [0, 2\pi]$) of each of the n discs} together with the permutation action of $\mathbb{S}_n$ on the labels and the glueing $\circ_i$ which is given by first rotating the second configuration by the angle θ_i, then scaling the configuration and finally inserting it.

Proposition 4.16.4. *(Getzler) Any BV algebra is an algebra over the homology of the framed little discs operad and vice versa.*

4.17. *The Pre–Lie Operad*

Definition 4.17.1. (Gerstenhaber) A pre–Lie algebra is a graded vector space V together with a bilinear operation $*$ that satisfies

$$(x * y) * z - x * (y * z) = (-1)^{|y||x|}((x * z) * y - x * (z * y))$$

Proposition 4.17.2. *(Gerstenhaber) Define*

$$\{a \bullet b\} := a * b - (-1)^{(|a|+1)(|b|+1)} b * a \tag{4.6}$$

Then $(V, \{\bullet\})$ is an odd Lie algebra.

In the case of no signs the operad which defines pre–Lie algebras is actually isomorphic to an insertion operad of trees. It is the operad structure on $\mathcal{T}^{r,fl}$ discussed in 4.7.

Proposition 4.17.3. *(Chapoton–Livernet)*[7] *Any pre–Lie algebra is an algebra over the operad of labelled rooted trees $\mathcal{T}^{r,fl}$ and vice versa.*

To keep the signs is more tricky, but by using a cell interpretation in terms of the symmetric top–dimensional cells of spineless cacti $(CC_*^{top}(Cact^1))^{\mathbb{S}}$ we are able to identify the operad for graded pre–Lie algebras as an operad of trees see Theorem 10.2.3.

Proposition 4.17.4. *Any graded pre–Lie algebra is an algebra over the operad of rooted trees* $\mathcal{T}^{r,fl}$ *with a grading the choice of signs* $\overline{Nat}$ *(see 7.7) and vice versa. More precisely any graded pre–Lie algebra is an algebra over the operad* $(CC_*^{top}(Cact^1))^{\mathbb{S}}$

Remark 4.17.5. We summed up the results of the previous sections in the table 4.17

Operad	Algebras
$\mathcal{COM}$	commutative algebras
$\mathcal{ASSOC}$	associative algebras
$\mathcal{T}^{r,fl}$	pre–Lie algebras
$(CC_*^{top}(Cact^1))^{\mathbb{S}}$	graded pre–Lie algebras
H_*(little discs)	Gerstenhaber algebras
H_*(framed little discs)	BV algebras

Remark 4.17.6. It is interesting to point out, that the pre–Lie structure essentially lives on the chain level, while the Gerstenhaber structure only lives on the homology level. This is the case, since the relation for the associator does hold on the chain level, while the derivation properties involve homotopies.

4.18. *Operads of Surfaces with Extra Structure*

Starting with the operad of topological surfaces with boundary $\Sigma(n)$ one can endow the surfaces with extra structures and then take care to glue these extra structures. Essentially this means that one studies moduli spaces of surfaces.

4.19. *Operads of Moduli Spaces of Curves*

Going over to an algebraic point of view, one replaces boundaries by punctures and considers the following moduli spaces.

The Deligne–Mumford compactifications $\bar{M}_{g,n}$ of the moduli spaces of curves of genus g with n marked points form an operad. The glueing is essentially glueing of two curves at two chosen marked points. If one allows self– glueing one obtains a so–called modular operad[22]. The theory of Gromov–Witten invariants yields the statement that the cohomology of a smooth variety is an algebra over the modular operad $H_*(\bar{M}_{g,n})$.

The spaces $\bar{M}_{0,n}$ form a suboperad (not modular) and the cohomology of a smooth variety as an algebra over $H_*(\bar{M}_{0,n})$ is usually called quantum cohomology.

One can also consider the open moduli spaces $M_{g,n}$ and these also form a modular operad[22]. The algebras over the suboperad $H_*(M_{0,n})$ are called gravity algebras. The relationship between $H_*(M_{0,n})$ and $H_*(\bar{M}_{0,n})$ is that they are Koszul dual to each other as quadratic operads[18].

4.20. *Arc Operads*

Staying in the topological realm of surfaces with borders, we will be interested in the extra structure of adding arcs to the surface, which can be viewed as a version of hyperbolic field theory[34] and the next section.

5. The Arc Operad

5.1. *The space*

There is an operad based on bordered surfaces with arcs projectively weighted by non–negative real numbers[34], which is an extension of the bordered surface operad. This operad is called the Arc operad and we will briefly recall its definition[34] here.

We fix a surface of genus g with r punctures and $n+1$ boundary components that are labelled from 0 to n and call it $F = F^s_{g,n+1}$. We also fix a window which is a closed proper subset $W_i \subsetneq \partial_i F$, for each boundary component $\partial_i F$.

Definition 5.1.1. An essential arc in F is an embedded path a in F whose endpoints lie among the windows, where we demand that a is not isotopic rel endpoints to a path lying in ∂F. Two arcs are said to be parallel if there is an isotopy between them which fixes each $\partial_i - W_i$ pointwise (and fixes each W_i setwise) for $i = 1, 2, \ldots, r$. An arc family in F is the isotopy class of an unordered collection of disjointly embedded essential arcs in F, no two of which are parallel. Thus, there is a well-defined action of the pure mapping class group on arc families. Where the pure mapping class group $PMC = PMC(F)$ is the group of isotopy classes of all orientation-preserving homeomorphisms of F which fix each $\partial_i - W_i$ pointwise (and fix each W_i setwise), for each $i = 1, 2, \ldots, r$.

The arc families have a natural partial order which is given by inclusion. This allows to build a simplicial cell complex whose k skeleton is composed of simplices indexed by arc families with k+1 arcs. These are attached to the k-1 skeleton by the face maps given by deleting one arc from the collection.

Definition 5.1.2. We define $Arc'(F)$ to be the complex obtained in the above manner and $Arc(F)$ to be the topological space obtained as quotient of $Arc'(F)$ by the action of PMC. We also define $DArc(F) := Arc'(F) \times \mathbb{R}_{>0}$.

We would like to remark that the points of $Arc(F)$ can and should be thought of as a mapping class group orbit of a projectively weighted (by positive real numbers) arc family on F and points of $DArc(F)$ as mapping class group orbits of weighted (by positive real numbers) arc families.

Any point of $Arc(F)$ lies in the interior of some cell of minimal dimension or on a vertex of the complex. If the vertex lies inside a cell, then assigning barycentric coordinates. We can now look at the point as being given by positive real co–ordinates w_j assigned to the arcs which make up the family; we call the w_j weights. If we drop the condition that the sum of the coordinates is one, then we obtain $DArc(F)$ by picking the coordinate on $\mathbb{R}_{>0}$ to be given by the sum and the coordinates on $Arc(F)$ to be given by the normalization. Finally, viewing $Arc(F)$ as the quotient of $DArc(F)$ by the action of $\mathbb{R}_{>0}$ scaling all coordinates at the same time, we obtain projective weights. If it lies on a vertex the point corresponds to a single arc which we can think of having any non–zero weight.

We will represent such a point by choosing a representative family and representative weights.

The underlying topological space for the n–th component our operad is an open subset of $\bigcup_{g,s} Arc(F^s_{g,n+1})$.

Definition 5.1.3. An arc family is said to be exhaustive if for each boundary component ∂_i, for $i = 1, 2, \ldots, r$, there is at least one component arc in α with its endpoints in the window W_i. Likewise, a PMC-orbit of arc families is said to be exhaustive if some (that is, any) representing arc family is so. Define the topological spaces

$$Arc^s_g(n) = \{[\alpha] \in Arc(F^s_{g,n+1}) : \alpha \text{ is exhaustive}\} \tag{5.1}$$

5.2. *The Operad Structure alias the Glueing Maps.*

The definition of the glueing maps is best and most naturally done in the setting of partial measured foliations[34]. The basic idea however is the following: First we glue two surfaces in the standard operadic fashion 4.5, i.e. boundary 0 of the second surface to the boundary of the first surface. Secondly, we have to give a procedure, how to glue together the bands. To this end we would like to think of the weighted arcs as bands of width given by the weights; this can be done by thickening the arcs into a partially measured foliation – which we view as a collection of bands. Now we arrange the bands in the window, such that in a neighborhood of the window the bands looks is depicted in figure 10 IV. In this way, the bands or the partially measured foliation can equivalently be thought of as a partition on an interval as depicted in figure 11 I.

Now if the sum of the weights of the arcs hitting the two boundary components of the two surfaces that are to be glued happen to coincide, we can "splice" the bands according to the largest common refinement of the two partitions; this is depicted in figure 12 I.

5.3. *Several Models for Arcs*

To elucidate the role of the windows, we would like to briefly recall[34] several geometric models for the common underlying combinatorics of arc families.

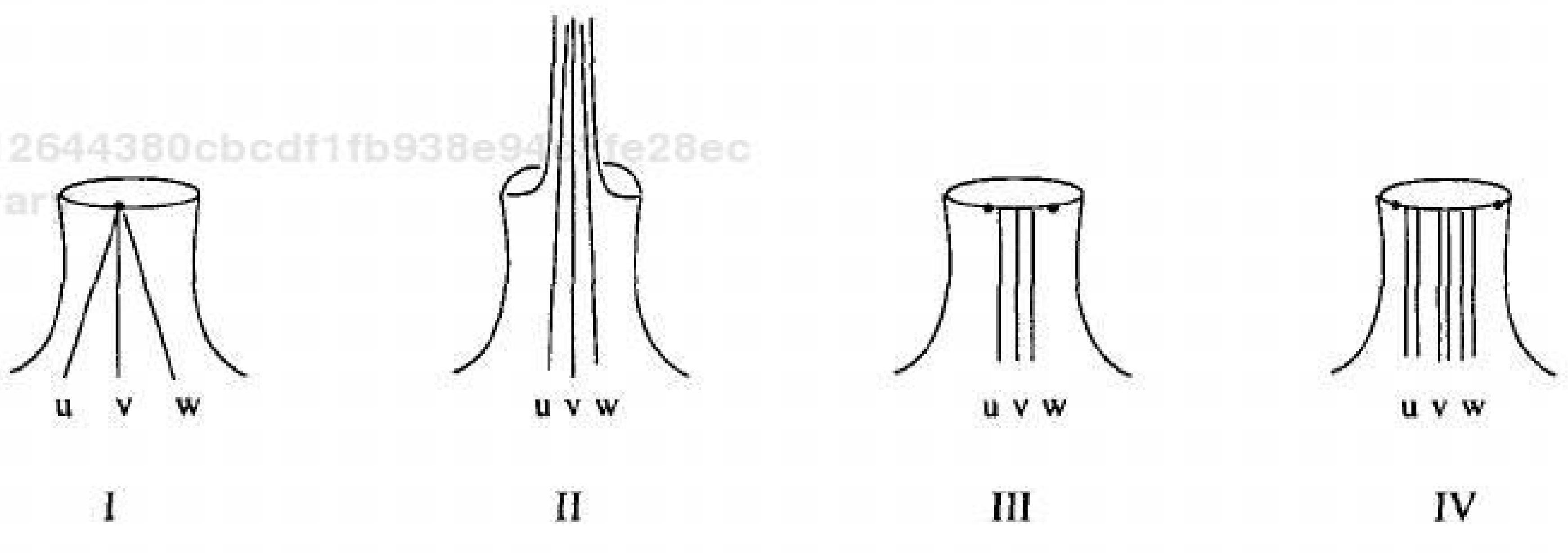


Figure 10. I. Arcs running to a point on the boundary; II. Arcs running to a point at infinity; III. Arcs in a window; IV. Bands in a window

For the first such model, let us choose a distinguished point $d_i \in \partial_i$, for $i = 1, 2, \ldots, r$, and consider the space of all complete finite-area metrics on F of constant Gauss curvature -1 (so-called "hyperbolic metrics") so that each $\partial_i^\times = \partial_i - \{d_i\}$ is totally geodesic (so-called "quasi hyperbolic metrics") on F. To explain this, consider a hyperbolic metric with geodesic boundary on a once-punctured annulus A and the simple geodesic arc a in it asymptotic in both directions to the puncture; the induced metric on a component of $A - a$ gives a model for the quasi hyperbolic structure on $F^\times = F - \{d_i\}_1^r$ near $\partial_i^\times$. The first geometric model for an arc family α in F is as a set of disjointly embedded geodesics in $F^\times$, each component of which is asymptotic in both directions to some distinguished point d_i; see part II of figure 10.

In the homotopy class of each ∂_i, there is a unique geodesic $\partial_i^* \subset F^\times$. Excising from $F - \cup\{\partial_i^\times\}_1^r$ any component which contains a point of $\partial^\times$, we obtain a hyperbolic structure on the surface $F^* \subseteq F^\times$ with geodesic boundary (where in the special case of an annulus, F^* collapses to a circle). Taking $\alpha \cap F^*$, we find a collection of geodesic arcs connecting boundary components (where in the special case of the annulus, we find two points in the circle).

This is our second geometric model for arc families. We may furthermore choose a distinguished point $p_i \in \partial_i^*$ and a regular neighborhood U_i of p_i in ∂_i^*, for $i = 1, 2, \ldots, r$. Provided $p_i \notin \alpha$, we may take U_i sufficiently small that $U_i \cap \alpha = \emptyset$, so the arc $V_i = \partial_i^* - U_i$ forms a natural "window" containing $\alpha \cap \partial_i^*$. There is then an ambient isotopy of F^* which shrinks each window V_i down to a small arc $W_i \subseteq \partial_i^*$, under which α is transported to a family of (non-geodesic) arcs with endpoints in the windows W_i. In case p_i does lie in α, then let us simply move p_i a small amount in the direction of the natural orientation (as a boundary component of F^*) along ∂_i^* and perform the same construction; see part III of figure 10.

This leads to our final geometric model of arc families, namely, the model we used to define our spaces: arcs in a bounded surface with endpoints in windows. This third model is in the spirit of train tracks and measured foliations[44] as we shall see and is most convenient for describing the operadic structure.

In this picture there is also then a unique orientation-preserving mapping[34]

$$c_i^{(\alpha)} : \partial_i(\alpha) \to S^1 \tag{5.2}$$

which maps the (class of the) first point of $\partial_i(\alpha)$ in the orientation of the window W_i to $0 \in S^1$, contracts the parts of the boundary outside the window and not hit by the bands, and maps the boundary points which are part of the bands to S^1 according to the normalized partial measure on the bands, which in our rudimentary discussion is just given by the weight and the normalizations means that the total weight is one. If we do not normalize, the map will take us to a circle of radius given by the total weight.

5.4. *Pictorial Representations of Arc Families*

As explained before, there are several ways in which to imagine weighted arc families near the boundary. They are illustrated in figure 10. It is also convenient to view arcs near a boundary component as coalesced into a single wide band by collapsing to a point each interval in the window complementary to the bands; this *interval model* is illustrated in figure 11, part I. It is also sometimes convenient to further take the image under the maps 5.2 to produce the *circle model* as is depicted in figure 11, part II.

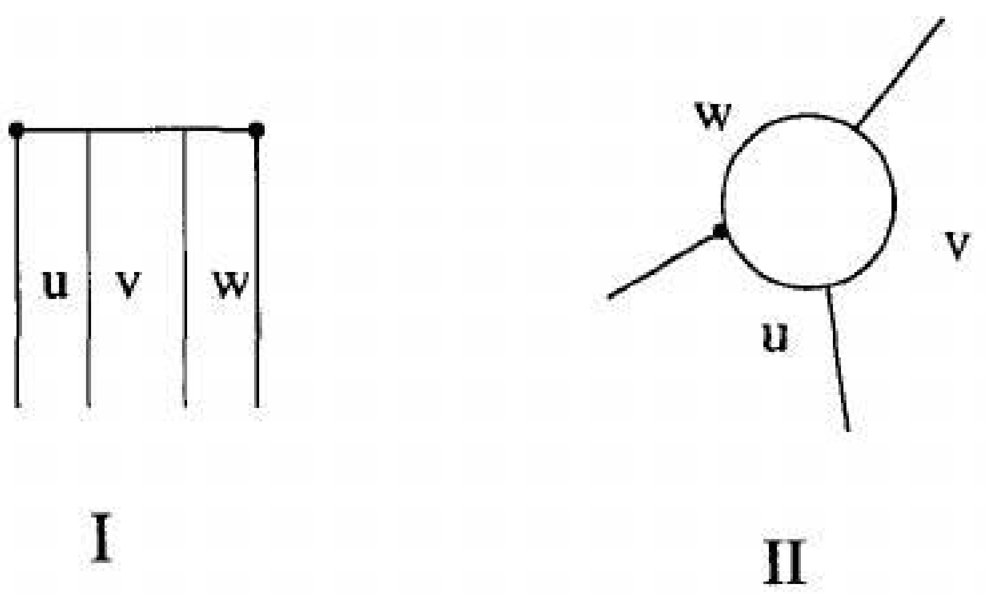


Figure 11. I. Bands ending on an interval; II. Bands ending on a circle

5.5. *Glueing Weighted Arc Families.*

Given two weighted arc families (α') in $F^s_{g,m+1}$ and (β') in $F^t_{h,n+1}$ so that $\mu_i(\partial_i(\alpha')) = \mu_0(\partial_0(\beta'))$, for some $1 \leq i \leq m$, we shall next make choices to define a weighted arc family in $F^{s+t}_{g+h,m+n}$ as follows.

First of all, let $\partial_0, \partial_1, \ldots, \partial_m$ denote the boundary components of $F^s_{g,m+1}$, let $\partial'_0, \partial'_1, \ldots, \partial'_n$ denote the boundary components of $F^t_{h,n+1}$, and fix some index $1 \leq i \leq m$. Each boundary component inherits an orientation in the standard manner from the orientations of the surfaces, and we may choose any orientation-preserving homeomorphisms $\xi : \partial_i \to S^1$ and $\eta : \partial'_0 \to S^1$ each of which maps the initial point of the respective window to the base-point $0 \in S^1$. Glueing together ∂_i and ∂'_0 by identifying $x \in S^1$ with $y \in S^1$ if $\xi(x) = \eta(y)$ produces a space X homeomorphic to

$F^{s+t}_{g+h,m+n}$, where the two curves ∂_i and ∂'_0 are thus identified to a single separating curve in X. There is no natural choice of homeomorphism of X with $F^{s+t}_{g+h,m+n}$, but there are canonical inclusions $j : F^s_{g,m+1} \to X$ and $k : F^t_{h,n+1} \to X$.

We enumerate the boundary components of X in the order

$$\partial_0, \partial_1, \ldots, \partial_{i-1}, \partial'_1, \partial'_2, \ldots \partial'_n, \partial_{i+1}, \partial_{i+2}, \ldots \partial_m$$

and re-index letting ∂_j, for $j = 0, 1, \ldots, m+n-1$, denote the boundary components of X in this order. Likewise, first enumerate the punctures of $F^s_{g,m+1}$ in order and then those of $F^t_{h,n+1}$ to determine an enumeration of those of X, if any. Let us choose an orientation-preserving homeomorphism $H : X \to F^{s+t}_{g+h,m+n}$ which preserves the labelling of the boundary components as well as those of the punctures, if any.

In order to define the required weighted arc family, consider the partial measured foliations $\mathcal{G}$ in $F^s_{g,m+1}$ and $\mathcal{H}$ in $F^t_{h,n+1}$ corresponding respectively to (α') and (β'). By our assumption that $\mu_i(\partial_i(\alpha')) = \mu_0(\partial_0(\beta'))$, we may produce a corresponding partial measured foliation $\mathcal{F}$ in X by identifying the points $x \in \partial_i(\alpha')$ and $y \in \partial_0(\beta')$ if $c_i^{(\alpha)}(x) = c_0^{(\beta)}(y)$. The resulting partial measured foliation $\mathcal{F}$ may have simple closed curve leaves which we must simply discard to produce yet another partial measured foliation $\mathcal{F}'$ in X. The leaves of $\mathcal{F}'$ thus run between boundary components of X and therefore, as in the previous section, decompose into a collection of bands B_i of some widths w_i, for $i = 1, 2, \ldots I$. Choose a leaf of $\mathcal{F}'$ in each such band B_i and associate to it the weight w_i given by the width of B_i to determine a weighted arc family (δ') in X which is evidently exhaustive. Let $(\gamma') = H(\delta')$ denote the image in $F^{s+t}_{g+h,m+n}$ under H of this weighted arc family.

It is a fact that this is well defined also on PMC orbits[34].

Definition 5.5.1. Given $[\alpha] \in Arc^s_g(m)$ and $[\beta] \in Arc^t_h(n)$ and an index $1 \le i \le m$, let us choose respective deprojectivizations (α') and (β') and write the weights

$$w(\alpha') = (u_0, u_1, \ldots, u_m),$$
$$w(\beta') = (v_0, v_1, \ldots, v_n).$$

Define

$$\rho_0 = \sum_{\{b \in \beta : \partial b \cap \partial_0 \neq \emptyset\}} v_i,$$
$$\rho_i = \sum_{\{a \in \alpha : \partial a \cap \partial_i \neq \emptyset\}} u_i,$$

where in each sum the weights are taken with multiplicity, e.g., if a has both endpoints at ∂_0, then there are two corresponding terms in ρ_0.

Since both arc families are exhaustive, $\rho_i \neq 0 \neq \rho_0$, and we may re-scale

$$\rho_0 w(\alpha') = (\rho_0 u_0, \rho_0 u_1, \ldots, \rho_0 u_m),$$
$$\rho_i w(\beta') = (\rho_i v_0, \rho_i v_1, \ldots, \rho_i v_n),$$

so that the 0^{th} entry of $\rho_i w(\beta')$ agrees with the i^{th} entry of $\rho_0 w(\alpha')$.

Thus, we may apply the composition of 5.5 to the re-scaled arc families to produce a corresponding weighted arc family (γ') in $F^{s+t}_{g+h,m+n}$, whose projective class is denoted $[\gamma] \in Arc^{s+t}_{g+h}(m+n-1)$. We let

$$[\alpha] \circ_i [\beta] = [\gamma],$$

in order to define the composition

$$\circ_i : Arc^s_g(m) \times Arc^t_h(n) \to Arc^{s+t}_{g+h}(m+n-1), \text{ for any } i = 1, 2, \ldots, m.$$

5.6. *A Pictorial Representation of the Glueing*

A graphical representation of the glueing can be found in figure 12, where we present the glueing in three of the different models.

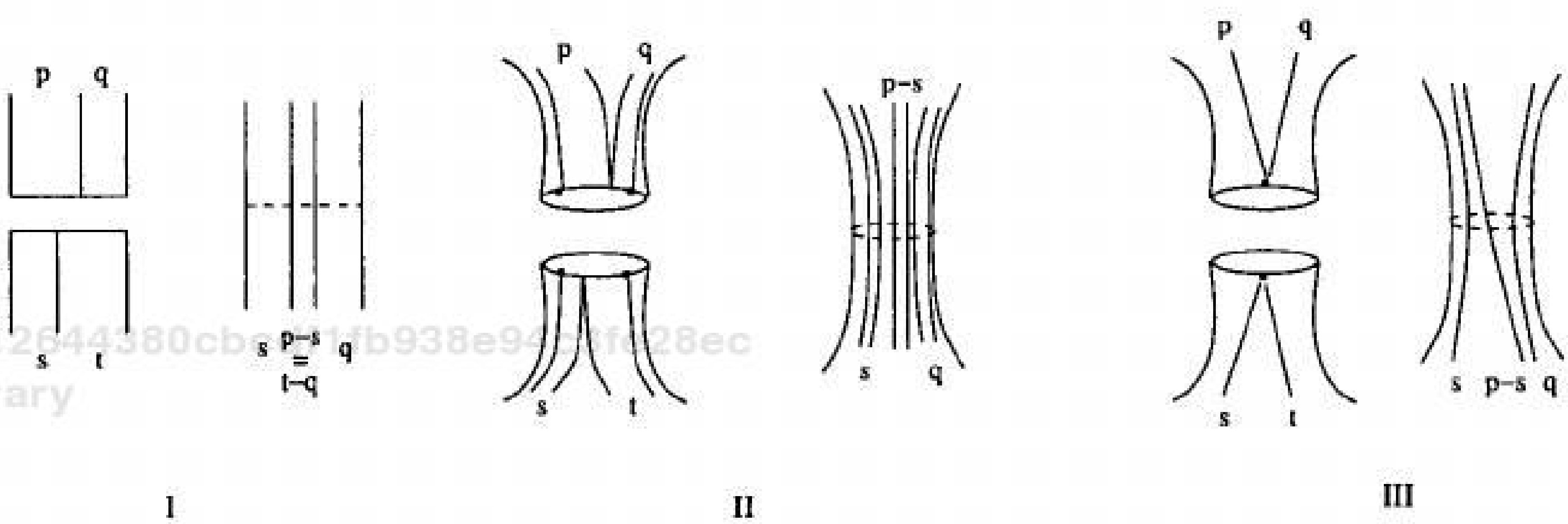


Figure 12. The glueing: I. in the interval picture, II. in the windows with bands picture and III. in the arcs running to a marked point version.

Definition 5.6.1. For each $n \geq 0$, let $\mathcal{A}rc_{cp}(n) = Arc^0_0(n)$ (where the "*cp*" stands for compact planar), and furthermore, define $\mathcal{A}rc(n)$ of the union over all $Arc^s_g(n)$, which we give the direct limit topology as $g, s \to \infty$.

Theorem 5.6.2. *The compositions $\circ_i$ of Definition 5.5.1 imbue the collection of spaces $\mathcal{A}rc(n)$ with the structure of a topological operad under the natural $\mathbb{S}_n$–action on labels on the boundary components. The operad has* *a unit* $1 \in \mathcal{A}rc(1)$ *given by the class of an arc in the cylinder meeting both boundary components, and the operad is cyclic for the natural* $\mathbb{S}_{n+1}$*–action permuting the labels of the boundary components. Furthermore the spaces* $\mathcal{A}rc_{cp}$ *form a cyclic suboperad.*

5.7. *The Deprojectivized Spaces* $\mathcal{DA}rc$

For the following it is convenient to introduce deprojectivized arc families. This amounts to adding a factor $\mathbb{R}_{>0}$ for the overall scale.

Let $\mathcal{DA}rc(n) = \mathcal{A}rc(n) \times \mathbb{R}_{>0}$ be the space of weighted arc families; it is clear that $\mathcal{DA}rc(n)$ is homotopy equivalent to $\mathcal{A}rc(n)$.

As the definition 5.5.1 of gluing was obtained by lifting to weighted arc families and then projecting back, we can promote the compositions to the level of the spaces $\mathcal{DA}rc(n)$. This endows the spaces $\mathcal{DA}rc(n)$ with a structure of a cyclic operad as well. Moreover, by construction the two operadic structures are compatible. This type of composition can be compared to the composition of loops, where such a rescaling is also inherent. In our case, however, the scaling is performed on both sides which renders the operad cyclic.

In this context, the total weight at a given boundary component given by the sum of the individual weights w_t of incident arcs makes sense, and thus the map 5.2 can be naturally viewed as map to a circle of radius $\sum_t w_t$.

5.8. *Notation*

We denote the operad on the collection of spaces
$\mathcal{A}rc(n)$ by $\mathcal{A}rc$ and the operad on the collection of spaces $\mathcal{DA}rc(n)$ by $\mathcal{DA}rc$. By an "Arc algebra", we mean an algebra over the homology operad of $\mathcal{A}rc$. Likewise, $\mathcal{A}rc_{cp}$ and $\mathcal{DA}rc_{cp}$ are comprised of the spaces $\mathcal{A}rc_{cp}(n)$ and $\mathcal{DA}rc_{cp}(n)$ respectively, and an "Arc$_{cp}$ algebra" is an algebra over the homology of $\mathcal{A}rc_{cp}$

5.9. *Suboperads and PROPS*

There are several natural suboperads for the arc operad, given by imposing certain conditions on the arcs.

For example, one may specify a symmetric $(n+1)$-by-$(n+1)$ matrix $A^{(n)}$ as well as an $(n+1)$-vector $R^{(n)}$ of zeroes and ones over $\mathbb{Z}/2\mathbb{Z}$ and consider the subspace of $Arc(F^s_{g,n+1})$ where arcs are allowed to run between boundary components i and j if and only if $A^{(n)}_{ij} \neq 0$ and are required to meet the boundary component k if and only if $R_k^{(n)} \neq 0$. For instance, the case of interest for cacti corresponds to $A^{(n)}$ the matrix whose entries are all one, and $R^{(n)}$ the vector whose entries are also all one.

For a class of examples of PROPS (the generalization of operads with arbitrary inputs and outputs[40]), consider a partition of $\{0, 1, \dots n\} = I^{(n)} \sqcup O^{(n)}$, into "inputs" and "outputs", where $A_{ij}^{(n)} = 1$ if and only if $\{i, j\} \cap I^{(n)}$ and $\{i, j\} \cap O^{(n)}$ are each singletons, and $R^{(n)}$ is the vector whose entries are all one.

This are the PROPS or di-operads which are of interest in string topology. In words these are the arc families in which the boundaries can be and are partitioned into incoming and outgoing in such a way that there are only arcs running between incoming and outgoing boundaries.

Definition 5.9.1. The trees suboperad is defined for arc families in surfaces with $g = s = 0$ in the notation explained above by the allowed incidence matrix $A^{(n)}$, whose non–zero entries are $a_{0i} = 1 = a_{i0}$, for $i = 1, \dots, n$, and required incidence relations $R^{(n)}$, whose entries are all equal to one.

This is a suboperad of $\mathcal{A}rc_{cp}$, and it has a representation in terms of labelled trees[29].

Dropping the requirement that $g = s = 0$, we obtain a suboperad of $\mathcal{A}rc$ called the *rooted graphs* or *Chinese trees* suboperad.

5.10. *Linearity Condition*

We say that an element of the (Chinese) trees suboperad satisfies the Linearity Condition if the linear orders match, i.e., the bands hitting each boundary component in their linear order are a subchain of all the bands in their linear order derived from the 0–th boundary.

It is straightforward to check that this condition is stable under composition.

We call the suboperad of elements satisfying the Linearity Condition of the (Chinese) trees operad the *(Chinese) linear trees operad.*

The following proposition[34] clarifies the role of this condition.

Proposition 5.10.1. *The suboperad generated by (Chinese) linear trees and $\mathcal{A}rc_{cp}(1)$ inside $\mathcal{A}rc$ coincides with (cyclic Chinese) trees, where cyclic Chinese trees are those arc families in Chinese trees in which the cyclic orders match.*

5.11. *String Interpretation*

One way that one can view the $\mathcal{A}rc$ operad is as tracing n closed strings as they move, split and recombine to form one loop. If one traces fixed base points of the strings and keeps track of the length of the strings that the closed strings may break into, one arrives exactly at the arc picture. Here one arc of weight w or better band of width w depicts the movement of a piece of string of length w. Starting with n strings that may move, break and recombine to become one string, one obtains the trees suboperad.

If the strings have in this way swept over a genus zero surface (with no punctures) this information is enough to recover the surface, since one only needs to know the number of boundary components. If there is non-trivial topology, however, this information may not be enough, but one can consider all graphs of the traces of the base point on a given fixed topological surface. This is what the arc operad does, cf. Figure 13.

If one wishes to see n strings move and recombine into m strings one arrives at the notion of the props discussed above. This corresponds to having a kind of Morse function for the surface which gives the state of the strings at a given time. In this view, the singular level sets are the important ingredients and a careful analysis of this picture leads to the $\mathcal{C}acti$ and the considerations relevant to string topology[8].

On the other hand, from the point of view of closed string field theory[51] there is actually no big distinction of incoming and outgoing circles in the sense of our prop definitions, as strings may also annihilate.

Thus even in genus zero, we do not need to restrict to the trees in the arc operad, but see all of $\mathcal{A}rc_{cp}$ and of course in general all of $\mathcal{A}rc$. The depiction of a pair of pants according to closed string field theory is not a figure eight as before, but a theta shape, as indicated in figure 13 and figure 28. This type of shape has recently also played a role in the geometry of the so-called stringor bundles[46].

5.12. *Relation to Moduli Spaces*

In this section, we would like to very briefly digress on the relation of the spaces $Arc(F)$ and the $\mathcal{A}rc$ operad to moduli spaces, due to Penner[42,43].

Assume that $r \neq 0$ (and allow both cases $s = 0$ or $s > 0$). Enumerate the (smooth) boundary components of F as ∂_i, where $i = 1, \ldots, r$ and set $\partial = \cup\{\partial_i\}_1^r$. Let $Hyp(F)$ be the space of all hyperbolic metrics with

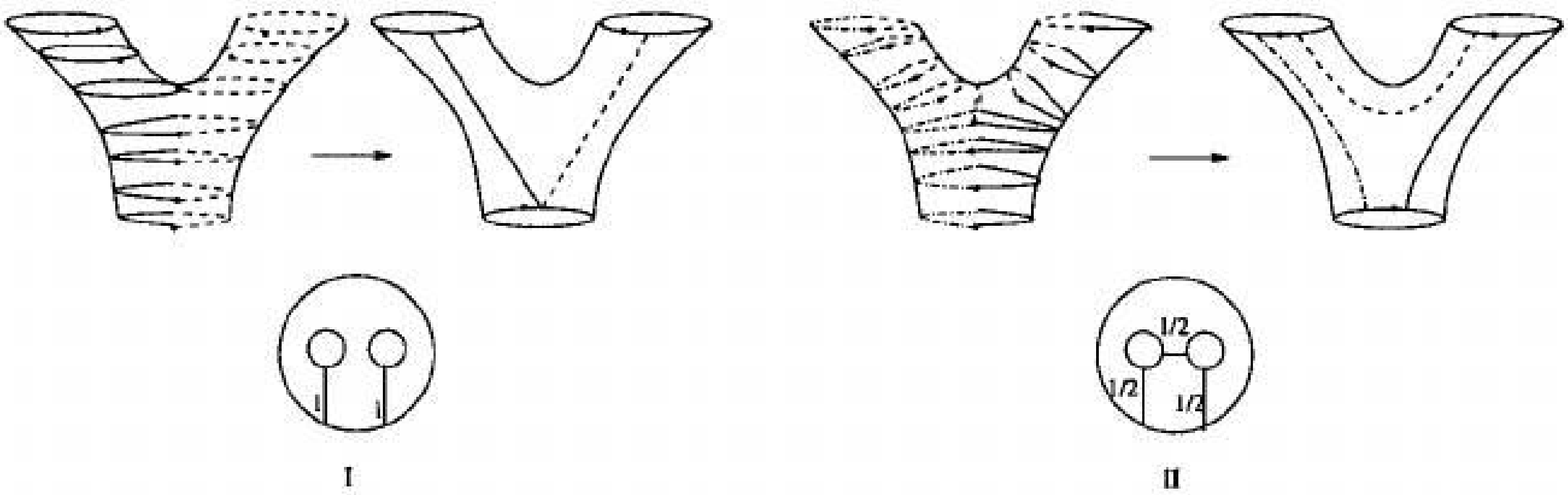


Figure 13. Movements of strings combining I and breaking, annihilating and combining II

geodesic boundary on F. Define the moduli space to be

$$M = M(F) = [Hyp(F)(\coprod_1^r \partial_i)]/\sim$$

where $\sim$ is the equivalence relation generated by push-forward of metric under orientation- preserving diffeomorphism

$$f_*(\Gamma, (\xi_i)_1^r) = (f_*(\Gamma), f(\xi_i)_1^r)$$

where $\Gamma \mapsto f_*(\Gamma)$ is the usual push-forward of metric on $Hyp(F)$.

Now let $M(F)/\mathbb{R}_{>0}$ denote the quotient of the moduli space of the bordered surface F by the action of $\mathbb{R}_{>0}$ by homothety on the tuple of hyperbolic lengths of the geodesic boundary components. Denote by $Arc_\#(g,r)^s$ the space of quasi-filling arc families, which means that complementary regions are either a polygon or a once punctured polygon.

Notice that for $g = s = r = 0$ $Arc_\#(0,0)^0 = Arc_{cp}$ and in general $Arc_\#(g,r)^s \subset Arc^s_{g,r}$.

Theorem 5.12.1. [42] *For any bordered surface $F \neq F^0_{0,2}$, $Arc_\#(F)$ is proper homotopy equivalent to $M(F)/\mathbb{R}_{>0}$.*

Furthermore, each of $M(F)/\mathbb{R}_{>0}$ and $Arc_\#(F)$ admit natural $(S^1)^r$–actions (moving distinguished points in the boundaries for the former and "twisting" arc families around the boundary components for the latter); the proper homotopy equivalence in the Theorem above is in fact a map of $(S^1)^r$–spaces.

There is also an interpretation in terms of ribbon graphs and Strebel differentials, see §6.15.

It is astonishing that in the case of genus zero with no punctures, there is another relationship on the algebraic level. It was shown by Getzler[17] that the homology operad of the spaces $M_{0,n}$ yields the notion of a gravity or L_∞ algebra. The corresponding cohomology classes are also present in our $\mathcal{A}rc$ picture through the BV structure discussed below, since this structure allows to define the higher brackets[17] as well. Actually there are good candidates for arc families leading to these brackets which we will discuss elsewhere[35].

In a sense, what the $\mathcal{A}rc$ operad does is to replace the conformal field theory given by the operads $H_*(M_{g,n})$ by its hyperbolic counterpart. The benefit, as always when dealing with hyperbolic aspects of moduli theory, the formulas become discrete and all spaces tend to have a PL–structure which makes everything very manageable.

The BV structure below legitimizes this point of view, since it is what one would expect from the non–compactified moduli. These are in turn related to the operad constituted by the Deligne–Mumford compactified spaces by Koszul duality for quadratic operads.

It is also interesting that the cell decomposition of trees we present in §7.6.5 is also related to Strebel differentials via the indexing set of trees[36].

5.13. *Arc Families and their Induced Operations.*

The points in $\mathcal{A}rc_{cp}(1)$ are parameterized by the circle, which is identified with $[0,1]$, where 0 is identified to 1. To describe a parameterized family of weighted arcs, we shall specify weights that depend upon the parameter $s \in [0,1]$. Thus, by taking $s \in [0,1]$ figure 14 describes a cycle $\delta \in C_1(1)$ that spans $H_1(\mathcal{A}rc_{cp}(1))$.

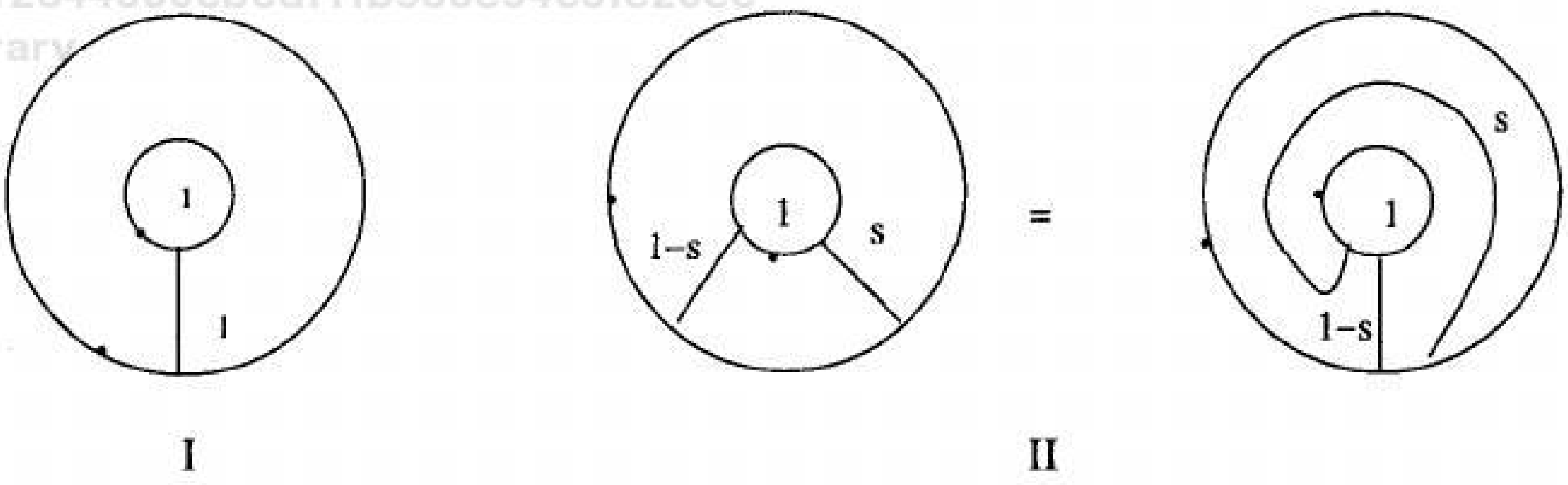


Figure 14. I. The identity and II. the arc family δ yielding the BV operator

As stated above, there is an operation associated to the family δ. For instance, if F_1 is any arc family $F_1 : k_1 \to \mathcal{A}rc_{cp}$, δF_1 is the family parameterized by $I \times k_1 \to \mathcal{A}rc_{cp}$ with the map given by the picture by inserting

F_1 into the position 1. By definition,

$$\Delta = -\delta \in C_1(1).$$

In $C_*(2)$ we have the basic families depicted in figure 15 which in turn yield operations on C_*.

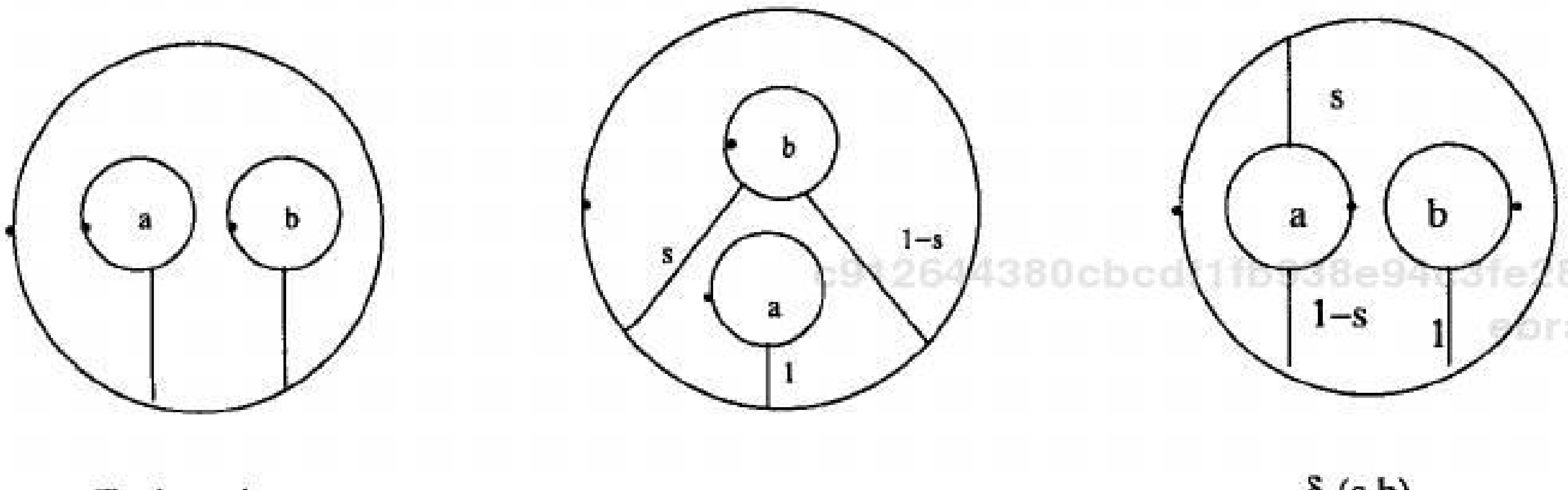


Figure 15. The binary operations

To fix the signs, we fix the parameterizations we will use for the glued families as follows: say the families F_1, F_2 are parameterized by $F_1 : k_1 \to \mathcal{A}rc_{cp}$ and $F_2 : k_2 \to \mathcal{A}rc_{cp}$ and $I = [0,1]$. Then $F_1 \cdot F_2$ is the family parameterized by $k_1 \times k_2 \to \mathcal{A}rc_{cp}$ as defined by figure 15 (i.e., the arc family F_1 inserted in boundary a and the arc family F_2 inserted in boundary b).

Interchanging labels 1 and 2 and using $*$ as a chain homotopy as in figure 16 yields the commutativity of $\cdot$ up to chain homotopy

$$d(F_1 * F_2) = (-1)^{|F_1||F_2|} F_2 \cdot F_1 - F_1 \cdot F_2 \tag{5.3}$$

Notice that the product $\cdot$ is associative up to chain homotopy.

Likewise $F_1 * F_2$ is defined to be the operation given by the second family of figure 15 with $s \in I = [0,1]$ parameterized over $k_1 \times I \times k_2 \to \mathcal{A}rc_{cp}$.

By interchanging the labels, we can produce a cycle $\{F_1, F_2\}$ as shown in figure 16 where now the whole family is parameterized by $k_1 \times I \times k_2 \to \mathcal{A}rc_{cp}$.

$$\{F_1, F_2\} := F_1 * F_2 - (-1)^{(|F_1|+1)(|F_2|+1)} F_2 * F_1.$$

Remark 5.13.1. We have defined the following elements in C_*:

δ and $\Delta = -\delta$ in $C_1(1)$;

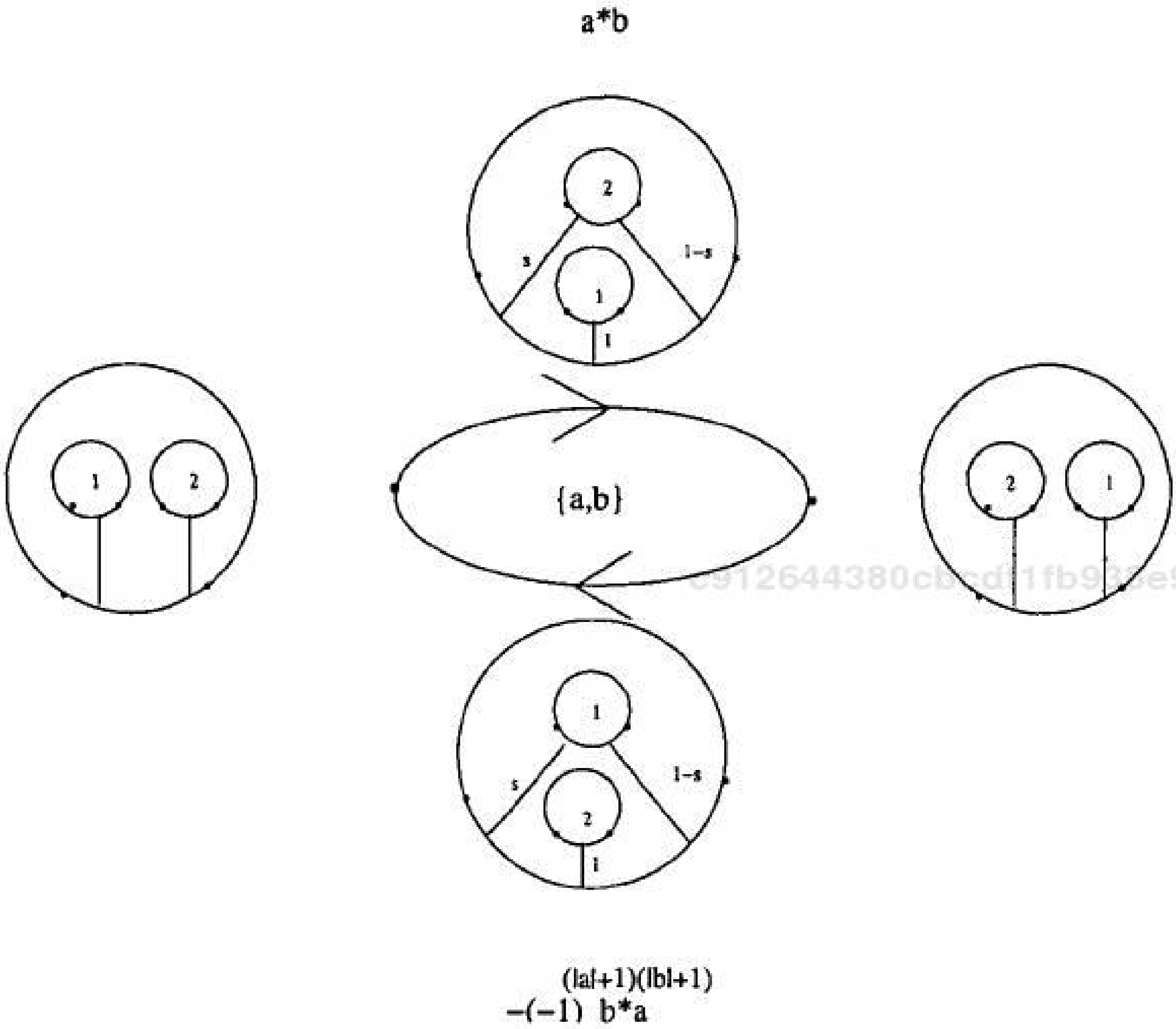


Figure 16. The definition of the Gerstenhaber bracket

$\cdot$ in $C_0(2)$, which is commutative and associative up to a boundary.
$*$ and $\{-,-\}$ in $C_1(2)$ with $d(*) = \tau \cdot - \cdot$ and $\{-,-\} = * - \tau *$.
Note that $\delta, \cdot$ and $\{-,-\}$ are cycles, whereas $*$ is not.

5.14. *The BV Operator*

The operation corresponding to the arc family δ is easily seen to square to zero in homology. It is therefore a differential and a natural candidate for a derivation or a higher order differential operator. It is easily checked that it is *not a derivation*, but it *is a BV operator*.

Proposition 5.14.1. *The operator Δ satisfies the relation of a BV operator up to chain homotopy.*

$$\Delta^2 \sim 0$$

$$\Delta(abc) \sim \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{|sa||b|}b\Delta(ac) - \Delta(a)bc \\ -(-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c) \tag{5.4}$$

Thus, any Arc algebra and any Arc_{cp} algebra is a BV algebra.

Lemma 5.14.2.

$$\delta(a,b,c) \sim (-1)^{(|a|+1)|b|} b\delta(a,c) + \delta(a,b)c - \delta(a)bc \tag{5.5}$$

Proof. The proof is contained in figure 17. Let $a : k_a \to \mathcal{A}rc_{cp}$, $b : k_b \to \mathcal{A}rc_{cp}$ and $c : k_c \to \mathcal{A}rc_{cp}$, be arc families then the two parameter family filling the square is parameterized over $I \times I \times k_a \times k_b \times k_c$. This family gives us the desired chain homotopy. □

Given arc families $a : k_a \to \mathcal{A}rc_{cp}$, $b : k_b \to \mathcal{A}rc_{cp}$ and $c : k_c \to \mathcal{A}rc_{cp}$, we define the two parameter family defined by the figure 18 where the families in the rectangles are the depicted two parameter families parameterized over $I \times I \times k_a \times k_b \times k_c$ and the triangle is not filled, but rather its boundary is the operation $\delta(abc)$.

From the diagram we get the chain homotopy consisting of three, and respectively twelve, terms.

Remark 5.14.3. The fact that the chain operads of $\mathcal{A}rc$ and as we show below $\mathcal{C}act(i)$ or $\mathcal{C}act^1(i)$ all possess the structure of a G(BV) algebra up to homotopy means that for any algebra V over them the algebra as well as $\mathcal{H}om_V$ have the structure of G(BV). If one is in the situation that one can lift the algebra to the chain level, then the G(BV) will exist on the chain level up to homotopy.

Remark 5.14.4. We would like to point out that the symbol $\bullet$ in the standard super notation of odd Lie–brackets $\{a \bullet b\}$, which is assigned to have an intrinsic degree of 1, corresponds geometrically in our situation to the one–dimensional interval I.

5.15. *The Associator*

It is instructive to do the calculation in the arc family picture with the operadic notation. For the glueing $* \circ_1 *$ we obtain the elements in $C_2(2)$ presented in figure 19 to which we apply the homotopy of changing the weight on the boundary 3 from 2 to 1 while keeping everything else fixed. We call this normalization.

Unravelling the definitions for the normalized version yields figure 20, where in the different cases the glueing of the bands is shown in figure 21.

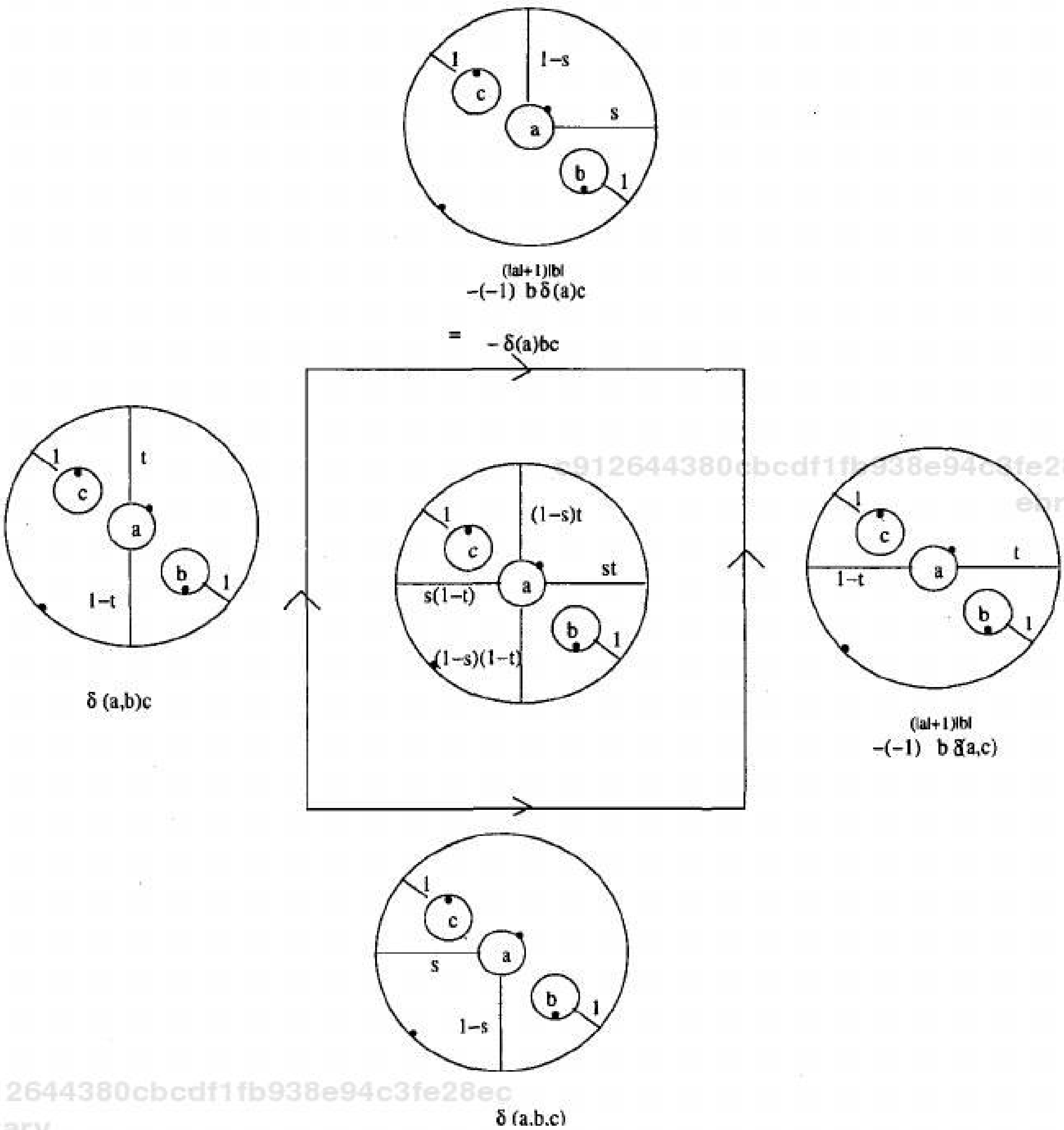


Figure 17. The basic chain homotopy responsible for BV

The glueing $* \circ_2 *$ in arc families is simpler and yields the glueing depicted in figure 22 to which we apply a normalizing homotopy — by changing the weights on the bands emanating from boundary 1 from the pair $(2s, 2(1-s))$ to $(s, 1-s)$ using pointwise the homotopy $(\frac{1+t}{2}2s, \frac{1+t}{2}(1-s))$ for $t \in [0,1]$:

Combining figures 20 and 22 while keeping in mind the parameteriza-

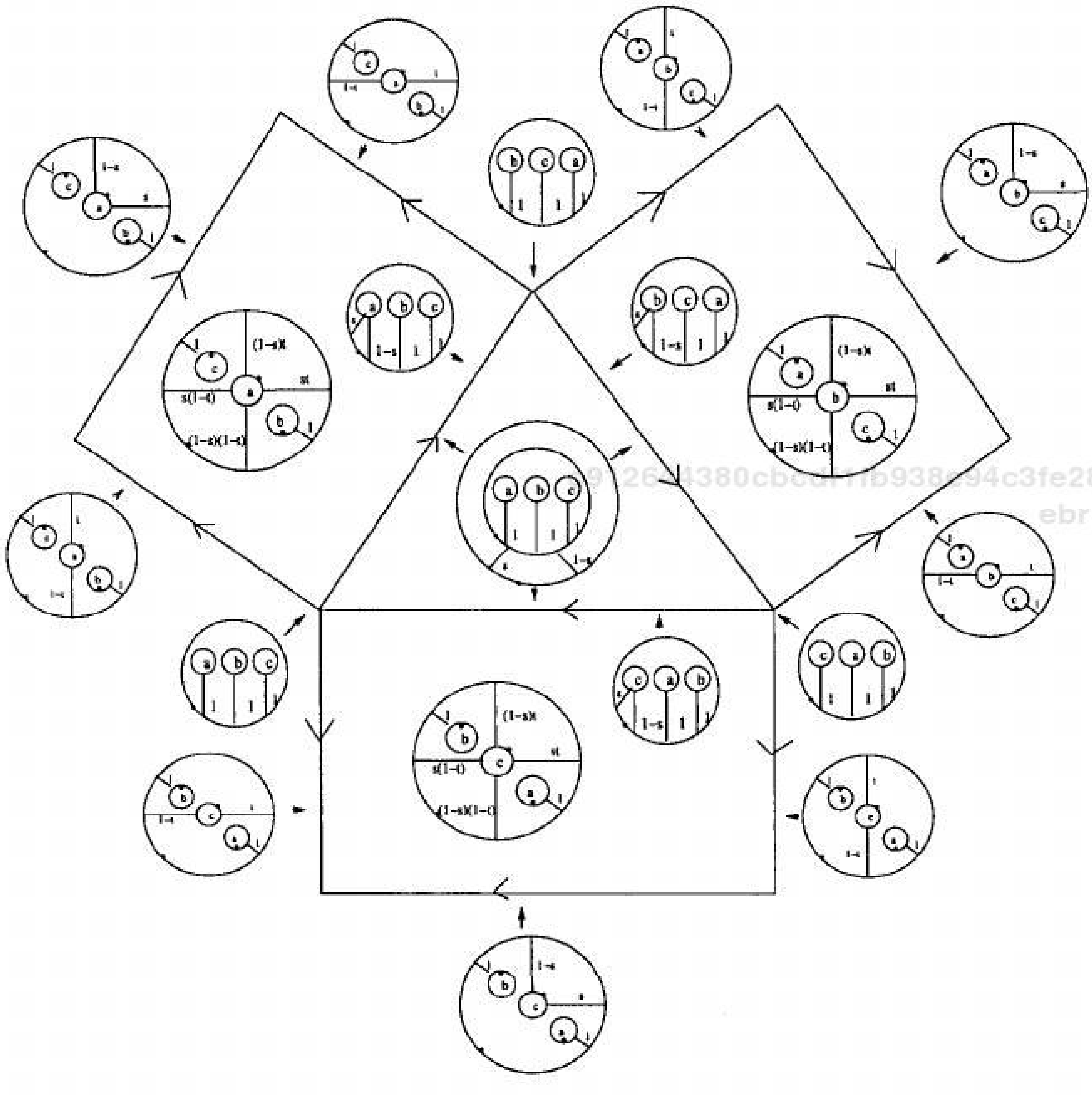


Figure 18. The homotopy BV equation

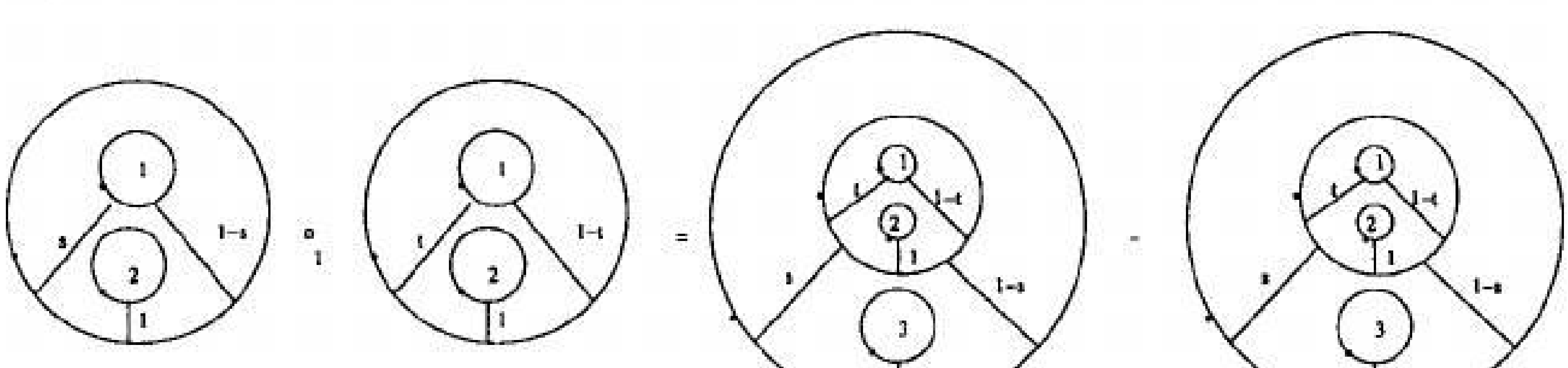


Figure 19. The first iterated glueing of $*$

tions we can read off the pre–Lie relation:

$$F_1 * (F_2 * F_3) - (F_1 * F_2) * F_3 \sim (-1)^{(|F_1|+1)(|F_2|+1)} (F_2 * (F_1 * F_3) - (F_2 * F_1) * F_3) \quad (5.6)$$

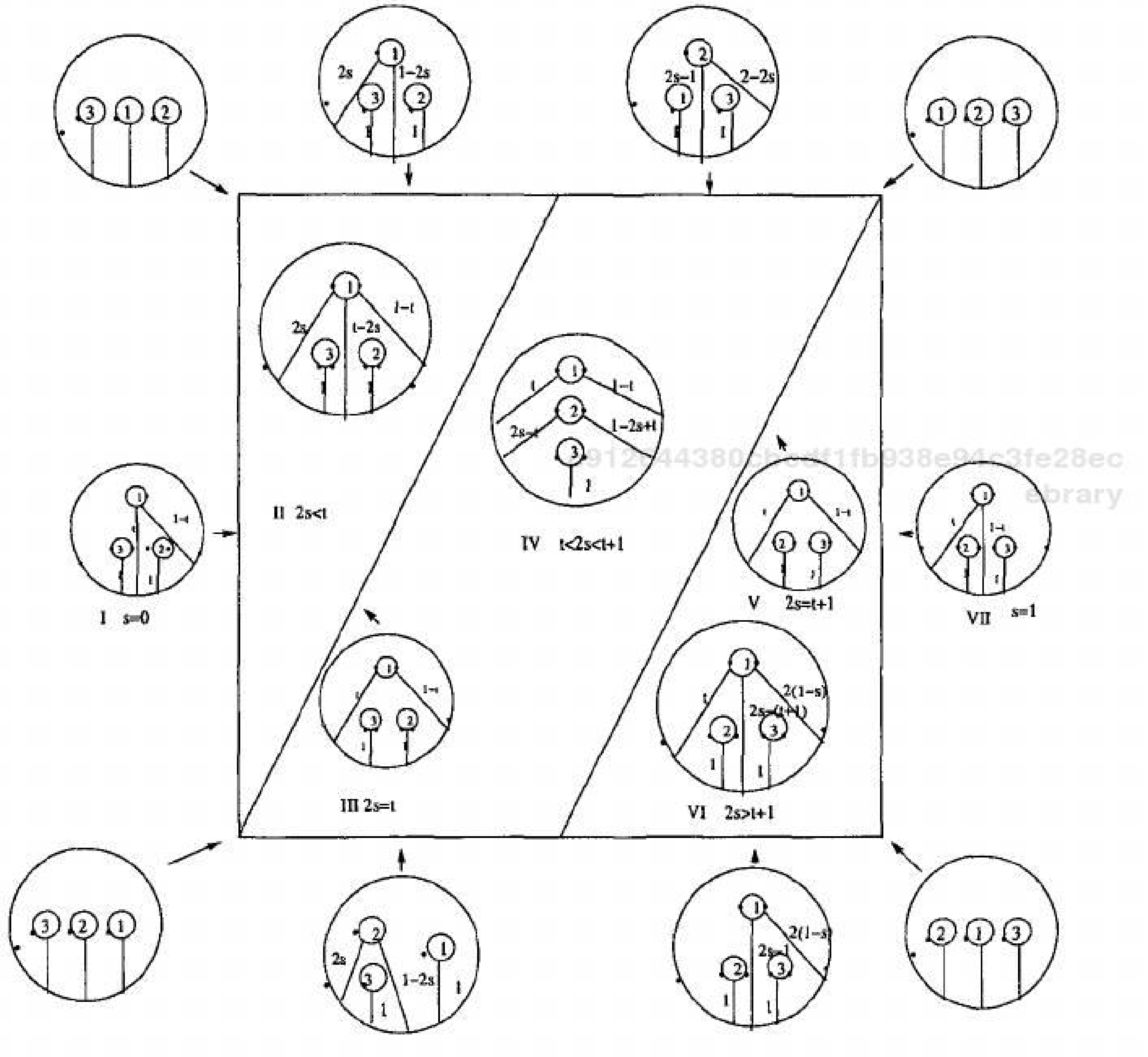


Figure 20. The glued family after normalization

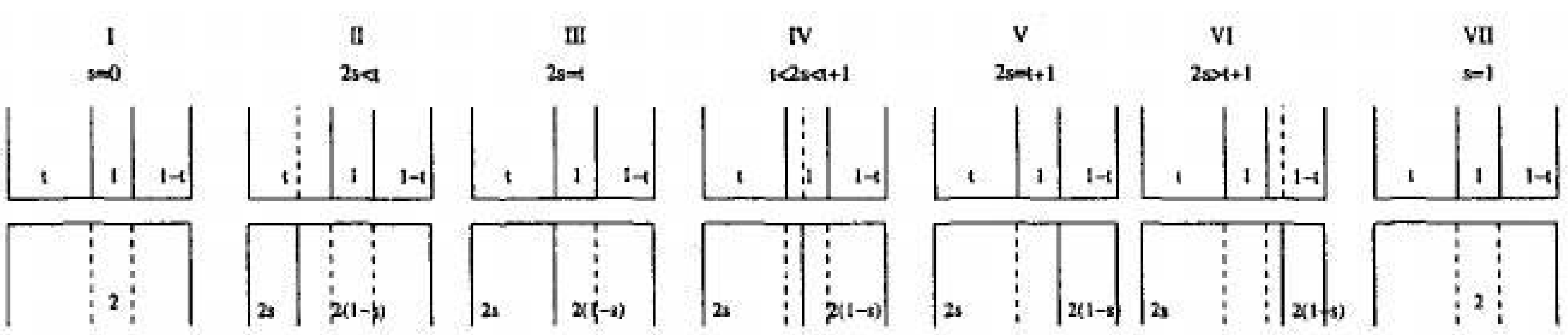


Figure 21. The different cases of glueing the bands

which shows that the associator is symmetric in the first two variables and thus following Gerstenhaber [G] we obtain:

Corollary 5.15.1. $\{F_1, F_2\}$ *satisfies the odd Jacobi identity.*

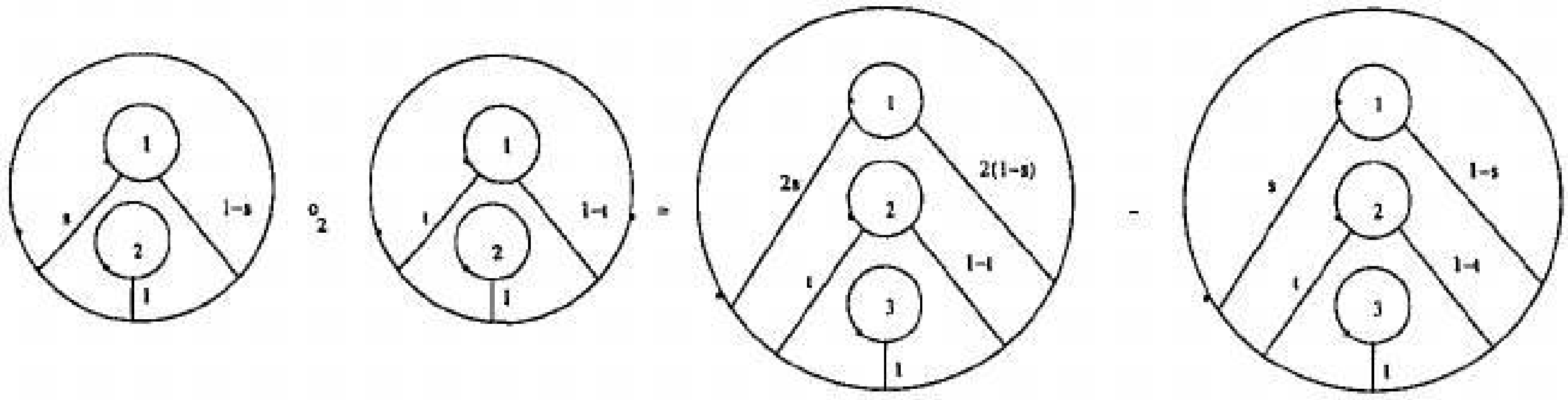


Figure 22. The other iteration of $*$

6. Species of Cacti and their Relations to other Operads

6.1. *Configurations of Loops and their Graphs*

There are several species of cacti[29] to which we refer the reader for details. We briefly recall the main definitions here. In words, the cacti operad which was introduced by Voronov[49] has as its n–th component defined in connected, planar tree-like configurations of parameterized loops (of possibly different circumference), together with a marked point on the configuration. The spineless cacti[29] are the suboperad where the zero of the parametrization corresponds to the lowest intersection point. There are also normalized versions of these configurations[29] where the circumference of each loop is fixed to be one.

To give a more precise definition we need the following definitions.

We denote the standard circle of radius r by $S^1_r := \{(x,y) \subset \mathbb{R}^2 : x^2 + y^2 = r^2\}$.

Definition 6.1.1. A configuration of n parameterized loops is a collection $(l_1, \dots, l_n)$ of n orientation preserving continuous injections —called loops— $l_i : S^1_{r_i} \to \mathbb{R}^2$ considered up to isotopy. Where the isotopy is required to fix the incidence conditions, that is if $h^i_t : S^1_{r_i} \times I$ are the isotopies and $h^i_0(p) = l_i(p) = l_j(q) = h^j_0(q)$ then for all t: $h^i_t(p) = h^j_t(q)$ and vice–versa if $h^i_0(p) = l_i(p) \neq l_j(q) = h^j_0(q)$ then for all t: $h^i_t(p) \neq h^j_t(q)$. A pointed configuration is a configuration together with a marked component l_i and marked point on this component $* \in S^1_{r_i}$.

Definition 6.1.2. For a configuration of n parameterized loops, with only finitely many intersection points, we can define a bipartite b/w graph as follows: There is one white vertex for each loop and one black vertex for each intersection point. We join a white vertex and a black vertex by an edge, if the intersection point corresponding to the black vertex lies on the loop corresponding to the white vertex. We call this black and white graph the graph of the configuration. We also endow each vertex with the cyclic order coming from the orientation in the plane.

For a pointed configuration, we include one more black vertex for the marked point and a second black vertex, the root and draw edges from the vertex for the marked point to each white vertex of components the marked point lies on. We also endow the vertex for the marked point by the linear order given by the cyclic order induced from the plane and the choice of the smallest element being the root edge followed by the edge for the marked component.

6.2. *Cacti and Spineless Cacti*

Definition 6.2.1. The cacti operad which was introduced by Voronov[49] has as its n–th component pointed configurations of n parameterized loops whose image is connected and whose graph is a tree. The space $Cact(n)$ is endowed with the action of $\mathbb{S}_n$ by permuting the labels.

Definition 6.2.2. Notice that the tree (graph) of a cactus is actually a bi–partite planar planted tree without tails which thus has a linear order on all of the vertices. We choose to plant the tree to reflect the linear order at the root.

We will allow ourselves to talk about the image of a cactus in $\mathbb{R}^2$ by picking a representative $(l_1, \dots, l_n)$ and considering $\bigcup_i l(S^1_{r_i})$ keeping in mind that this is only defined up to isotopy.

The loops, and also the inside of these loops, are sometimes called lobes — again the above remark applies.

Definition 6.2.3. Given a cactus $(l_1, \dots, l_n)$ whose loops have radii r_i there is a surjective orientation preserving map from $S^1_{R=\sum r_i} \mapsto \bigcup_i l(S^1_{r_i})$, whose only multiple points are the intersection points of the loop. This map is defined as follows. Start at the marked point (the global zero) and go around the marked loop counterclockwise; if a double point is hit continue on the next loop to the right (i.e. the next in the cyclic order) and continue in this manner until one returns to the marked point. We will call this map the "outside circle" and sometimes refer to the marked point as the "global zero", since it is the image of $0 \in S^1_R$.

6.3. *Glueing for Cacti*

We define the following operations

$$\circ_i : Cacti(n) \times Cacti(m) \to Cacti(n+m-1) \tag{6.1}$$

by the following procedure: given two cacti without spines we re-parameterize the outside circle of the second cactus to have length r_i which is the length of the i-th circle of the first cactus. Then glue in the second cactus by identifying the outside circle of the second cactus with the i–th circle of the first cactus.

Proposition 6.3.1. *The glueings above together with the $\mathbb{S}_n$ action on $Cacti(n)$ by permuting the labels imbue the collection Cacti of the $Cacti(n)$ with the structure of an operad. Endowing the spaces $Cacti(n)$ with the topology as subspaces of $\mathcal{D}Arc$, see below, turns the operad Cacti into an operad of topological spaces.*

Definition 6.3.2. The *spineless* variety of cacti is obtained by postulating that the local zeros defined by the parameterizations of the loops coincide with the first intersection point of the perimeter with a loop (sometimes called a "lobe") of the cactus. Here the first intersection point is the point given by the black vertex which lies on the outgoing edge of the white vertex representing the parameterized loop under consideration. This suboperad inherits the permutation action of $\mathbb{S}_n$ on the labels.

Proposition 6.3.3. *The symmetric group actions permuting the label together with the restriction of the glueing for Cacti*

$$\circ_i : Cact(n) \times Cact(m) \to Cact(n+m-1) \tag{6.2}$$

makes Cact into a topological operad which is a suboperad of Cacti.

6.4. *The Chord Diagram and Planar Planted Tree of a Cactus*

There is another representation of a cactus. If one regards the outside loop, then this can be viewed as a collection of points on an S^1 with an identification of these points, plus a marked point corresponding to the global zero. We can represent this identification scheme by drawing one chord for each two points being identified as the beginning and end of a circle. This chord diagram comes equipped with a decoration of its arcs by their length or alternatively can be thought of as embedded in $\mathbb{R}^2$. To obtain a cactus from such a diagram, one simply has to collapse the chords. This type of chord diagram of the outside circle is explicit in the embedding of cacti into the arc operad where the perimeter is indeed the outside circle, as explained below. The local zeros will then be extra points on the outside circle, which coincide with the beginnings of the chords in the spineless case. There is a special case for the chord diagram which is given if there is a closed cycle of chords. This happens only if three or more lobes intersect at the global zero. Here one can delete the first chord, if so desired. It does play a role however in the completed chord diagram[29] which parameterizes the fiber of the map forgetting the n–th lobe[29].

This kind of representation is reminiscent of Kontsevich's formalism of chord diagrams[2] as well as the shuffle algebras and diagrams of Goncharov[23]. We wish to point out that although the multiplication is similar to Kontsevich's and also could be interpreted as cutting the circle at the global zero resp. the local zero, it is not quite the same. However, the exact relationship and the co–product deserve further study.

Lastly, we can recover a planar tree as the dual tree of the chord diagram. This is the dual tree on the surface with boundary the outside circle, i.e. one vertex for each chamber inside the circle and an edge for chambers separated by a chord. This planar tree is the tree obtained from the tree describing the cactus by contracting all black edges. The special case corresponds again to the case where three or more lobes intersect at the global zero. If one chooses to keep the whole cycle of chords the dual tree the rooted tree which is obtained by contracting the root edge of the planted tree. If one also removes the first chord in the cycle, then the tree is the rooted tree which is obtained by contracting the root edge of the planted tree *and* the next edge which appears in the outside path.

A representation of a cactus without spines in all possible ways including its image in the $\mathcal{A}rc$ operad can be found in figure 23.

6.5. *Normalized Cacti and Normalized Spineless Cacti*

Definition 6.5.1. The spaces $\mathcal{C}acti^1(n)$ are the subspaces of $\mathcal{C}acti(n)$ with the restriction that all the radii of the lobes are fixed to one. The elements are called normalized spineless cacti.

6.6. *Gluing for Normalized Cacti*

We define the following operations

$$\circ_i : Cacti^1(n) \times Cacti^1(m) \to Cacti^1(n+m-1) \tag{6.3}$$

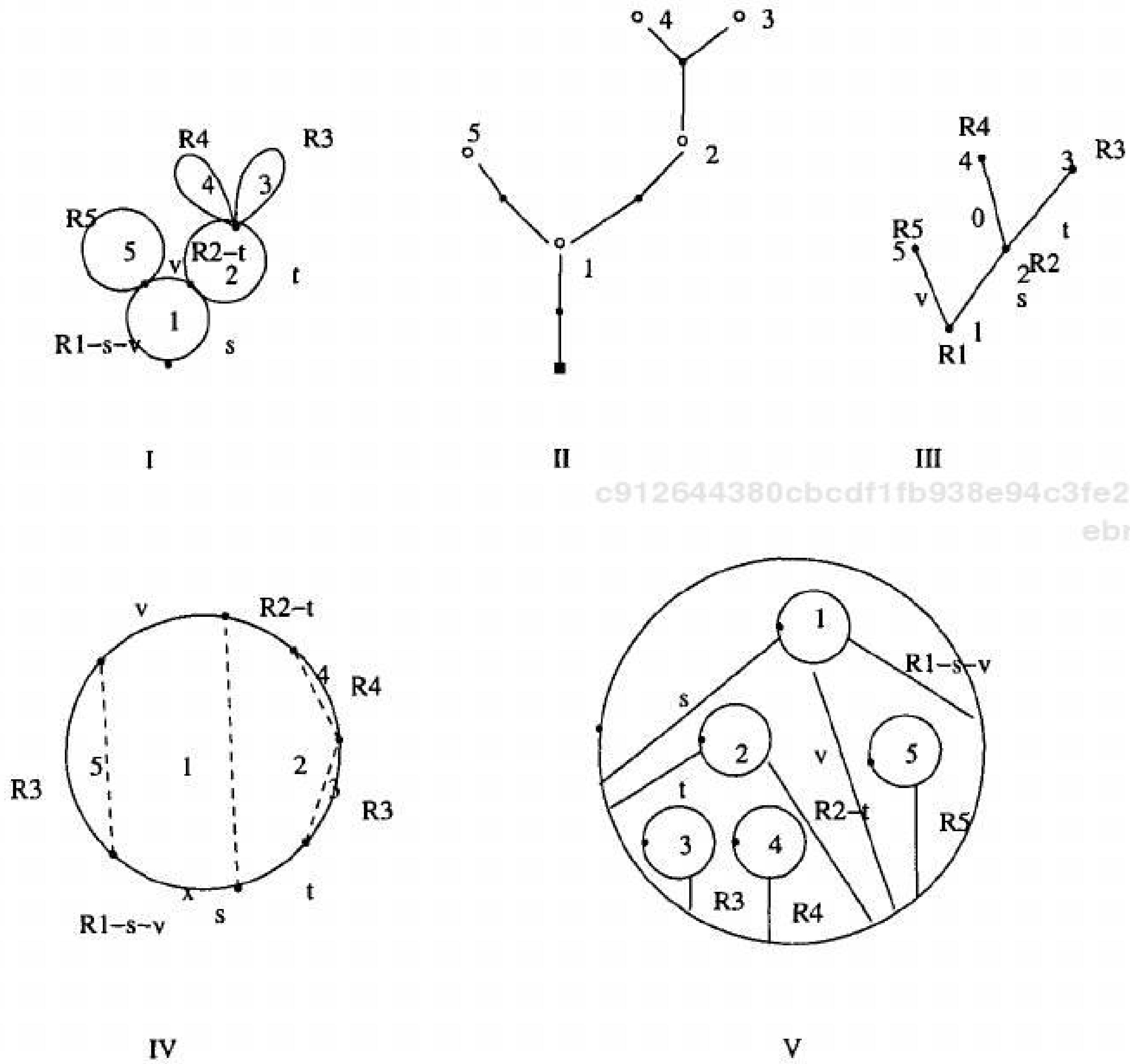


Figure 23. I. A cactus without spines; II Its planted planar bi–partite tree; III Its dual tree; IV Its chord diagram; V Its image in $\mathcal{A}rc$

by the following procedure: given two normalized cacti we reparameterize the i–th component circle of the first cactus to have length m and glue in the second cactus by identifying the outside circle of the second cactus with the i–th circle of the first cactus. Here we match the global zero of the second cactus to the local zero of the i–th lobe.

These glueings do not endow the normalized spineless cacti with the structure of an operad, but with the slightly weaker structure of a quasi–operad defined in Definition 4.10.3.

Theorem 6.6.1. [29] *The glueings above together with the* $\mathbb{S}_n$ *action on* $Cacti^1(n)$ *by permuting the labels imbue the collection of* $Cacti^1(n)$ *with the structure of a quasi–operad.*

Furthermore the quasi–operad of normalized cacti is homotopy associative. It is homotopic as a quasi–operad to the operad of cacti and is quasi–isomorphic to the operad of cacti. I.e. the homology operads of normalized cacti and cacti are isomorphic.

Definition 6.6.2. The spaces $Cact^1(n)$ are the subspaces of $Cact(n)$ with the restriction that all the radii of the lobes are fixed to one. The elements are called normalized spineless cacti.

Theorem 6.6.3. [29] *The symmetric group actions permuting the label together with the restriction of the glueing for* $Cacti^1$

$$\circ_i : Cact^1(n) \times Cact^1(m) \to Cact^1(n+m-1) \tag{6.4}$$

makes $Cact^1$ *into a quasi–operad which is a quasi–suboperad of* $Cacti^1$.

Furthermore the quasi–operad of normalized spineless cacti is homotopy associative. It is homotopic as a quasi–operad to the operad of spineless cacti and is quasi–isomorphic to the operad of spineless cacti. I.e. the homology operads of normalized cacti and cacti are isomorphic.

6.7. *Scaling of a Cactus and Projective Cacti*

Cacti and spineless cacti both come with a universal scaling operation of $\mathbb{R}_{>0}$ which simultaneously scales all radii by the same factor $\lambda \in \mathbb{R}_{>0}$. This action is a free action and the glueing descends to the quotient by this action. We sometimes call these operads projective cacti or spineless projective cacti.

6.8. *Left, Right and Symmetric Cacti Operads*

For the glueing above one has three basic possibilities to scale in order to make the size of the outer loop of the cactus that is to be inserted match the size of the lobe into which the insertion should be made.

(1) Scale down the cactus which is to be inserted. This is the original version – we call it the right scaling version.
(2) Scale up the cactus into which will be inserted. We call it the left scaling version.
(3) Scale both cacti. The one which is to be inserted by the size of the lobe into which it will be inserted and the cactus into which the insertion is going to be taking place by the size of the outer loop of the cactus which will be inserted. We call this it the symmetric scaling version.

All of these versions are of course homotopy equivalent and in the quotient operad of *Cacti* by overall scalings, the projective cacti $Cacti/\mathbb{R}_{>0}$ they all descend to the same glueing.

The advantages of the different versions are that version (1) is the original one and inspired by the rescaling of loops, i.e. the size of the outer loop of the first cactus is constant. Version (2) has the advantage that cacti whose lobes have integer sizes are a suboperad. We will use this later on. And version (3), the symmetric version, is the one we also used in $\mathcal{DA}rc$ and as shown below, in this version there is an embedding of the cacti operad into the cyclic operad $\mathcal{DA}rc$.

6.9. *Cacti as a Suboperad of* $\mathcal{DA}rc$

In the following sections we will show how to realize our species of cacti naturally as suboperads of $\mathcal{DA}rc$. This has the advantage of making their topology transparent.

6.10. *Framing of a Cactus*

We will give a map of cacti into $\mathcal{A}rc$ called a framing. First notice that a cactus can be decomposed by the initial point and the intersection points and the local zeros into a sequence of arcs following the natural orientation given by the data. These arcs are labelled by their lengths as parts of $l_i(S^1_{r_i})$. To frame a given cactus, draw a pointed circle around it and run an arc from each arc of the cactus to the outside circle respecting the linear order given by the outside loop, i.e. starting with the initial arc of the cactus as the first arc emanating from the outside circle in its orientation. Label each such arc by the parameter associated to the arc of the cactus.

We can think of attaching wide bands to the arc of the cactus. The widths of the bands are just the lengths of the arcs to which they are attached. Using these bands we identify the outside circle with the circumference of the cactus. Notice that this "outside" circle appears in the glueing formalism for cacti.

The marked points on the inside boundaries correspond to the local zeros of the inside circles viz. lobes of the cactus.

Two examples of this procedure are provided in figure 24.

Remark 6.10.1. If one frames a spineless cactus, then the image is in the linear trees.

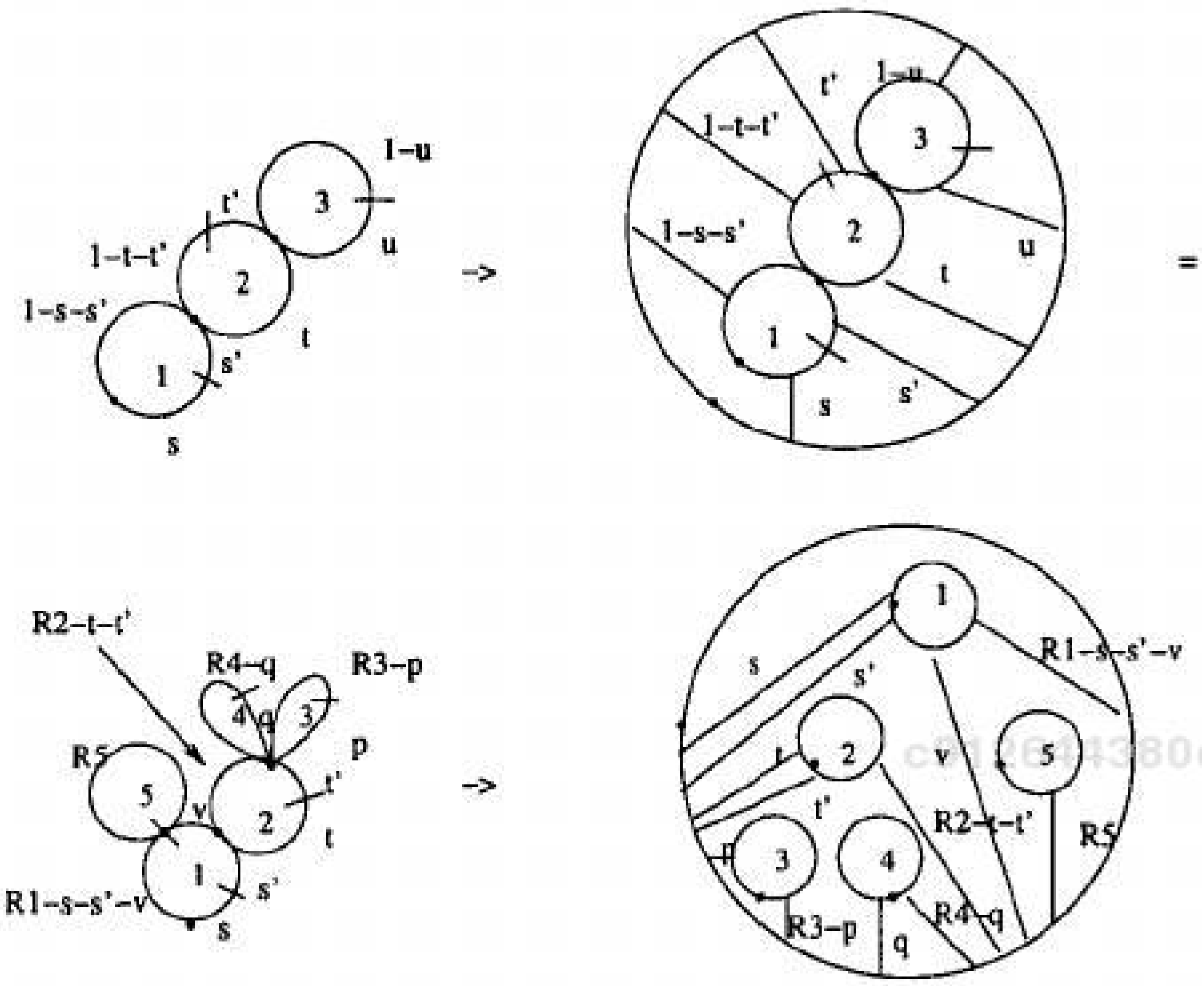


Figure 24. Framings of cacti

6.11. *The Loop of an Arc Family*

Given a surface with arcs we can forget some of the structure and in this way either produce a collection of loops or one loop which is given by using the arcs as an equivalence relation.

6.12. *The Boundary Circles*

Given an exhaustive weighted arc family (α) in the surface F, we can consider the measure-preserving maps

$$\tilde{c}_i^{(\alpha)} : \partial_i(\alpha) \to S^1_{m_i} \tag{6.5}$$

where S^1_r is a circle of radius r and $m_i = \mu^i(\partial_i(\alpha))$ is the total weight of the arc family at the i–th boundary. Combining these maps, we obtain

$$\tilde{c} : \partial(\alpha) \to \coprod_i S^1_{m_i}. \tag{6.6}$$

Choosing a measure on ∂F as in §1 to identify $\partial(\alpha)$ with ∂F, we finally obtain a map

$$circ : \partial F \to \coprod_i S^1_{m_i} \tag{6.7}$$

Notice that the image of the initial points of the bands give well-defined base–points $0 \in S^1_{m_i}$ for each i.

6.13. *The Equivalence Relations Induced by Arcs*

On the set $\partial(\alpha)$ there is a natural reflexive and symmetric relation given by $p \sim_{fol} q$ if p and q are on the same leaf of the partial measured foliation.

Definition 6.13.1.

Let $\sim$ be the equivalence relation on $\coprod_i S^1_{m_i}$ generated by $\sim_{fol}$. In other words $p \sim q$ if there are leaves l_j, for $j = 1, \dots m$, so that $p \in \tilde{c}(\partial(l_1)), q \in \tilde{c}(\partial(l_n))$ and $\tilde{c}(\partial(l_j)) \cap \tilde{c}(\partial(l_{j+1})) \neq \emptyset$.

Remark 6.13.2. It is clear that neither the image of *circ* —which will denote by *circ*$((\alpha))$— as a collection of parameterized circles nor the relation $\sim$ depends upon the choice of measure on ∂F.

Definition 6.13.3. Given a deprojectivized arc family $(\alpha) \in \mathcal{D}\mathcal{A}rc$, we define $Loop((\alpha)) = circ((\alpha))/\sim$ and denote the projection map $\pi : circ((\alpha)) \to Loop((\alpha))$.

Furthermore, we define two maps taking values in the monoidal category of pointed spaces:

$$int((\alpha)) = \coprod_{i=1}^{n} (\pi(\tilde{c}_i^{(\alpha)}(\partial_i(\alpha)), \pi(*_i)) \tag{6.8}$$

$$ext((\alpha)) = (\pi(\tilde{c}_0^{(\alpha)}(\partial_0(\alpha)), \pi(*_0)) \tag{6.9}$$

and call them the internal and external loops of (α) in $Loop((\alpha))$. We denote the space with induced topology given by the collection of images $Loop((\alpha))$ of all $(\alpha) \in \mathcal{D}\mathcal{A}rc(n)$ by $Loop(n)$.

Notice that there are $n+1$ marked points on $Loop((\alpha))$ for $(\alpha) \in \mathcal{D}\mathcal{A}rc(n)$.

Examples of loops of an arc family are depicted in figures 25–27. In figure 27 I the image of the boundary 1 runs along the outside circle and then around the inside circle. The same holds for the boundary 3 in figure 27 II. In both 27 I and II, the outside circle and its base–point are in bold.

Remark 6.13.4. There are two types of intersection points for pairs of loops. The first are those coming from the interiors of the bands; these points are double points and occur along entire intervals. The second type of multiple point arises from the boundaries of the bands via the transitive closure; they can have any multiplicity, but are isolated.

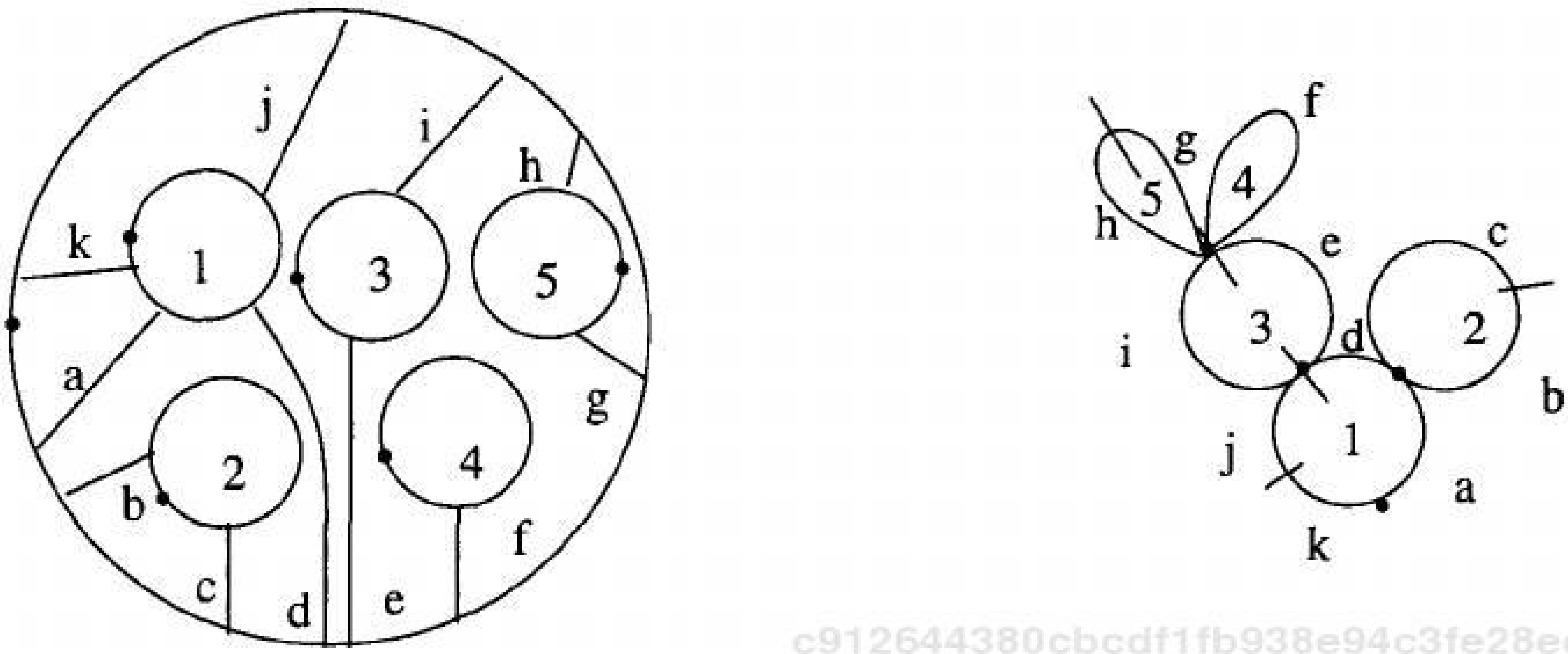


Figure 25. An arc family whose loop is a cactus

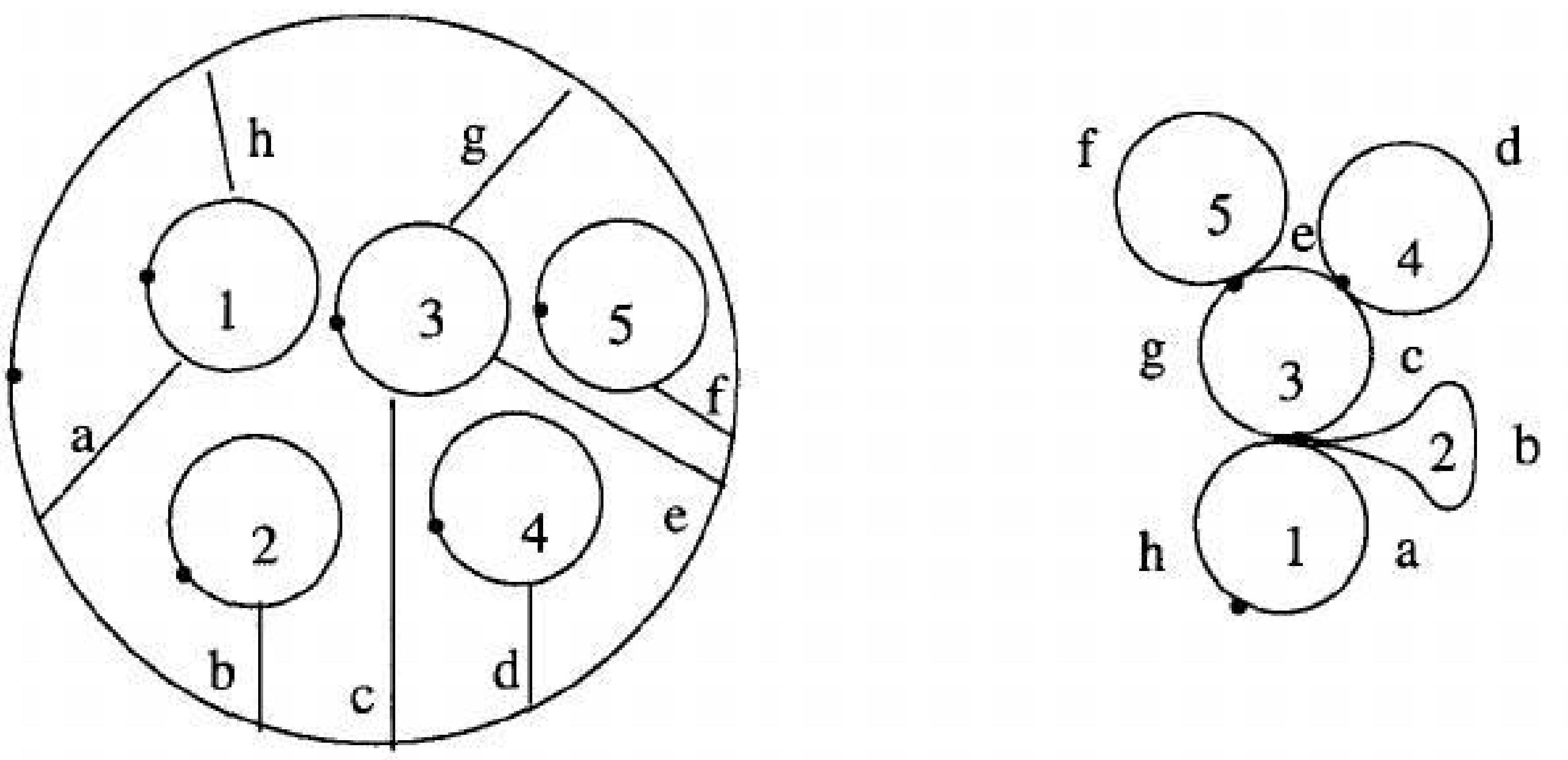


Figure 26. An arc family whose loop is a cactus without spines

6.14. *From Loops to Arcs*

If the underlying surface of an arc family satisfies $g = s = 0$, then its Loop together with the parameterizations uniquely determines the arc family. In other words, the map $frame$ is a section of *Loop*.

Definition 6.14.1. A *configuration of circles* is the image of a surjection $p : \coprod_i S^1_{m_i} \to L$ of metric spaces such that each point of L lies in the image of at least two components and the intersections of the images of more than two components are isolated. Let $Config(n)$ be the space of all such configurations of $n+1$ circles with the natural topology. We call a configuration of circles *planar*, if L can be embedded in the plane with the

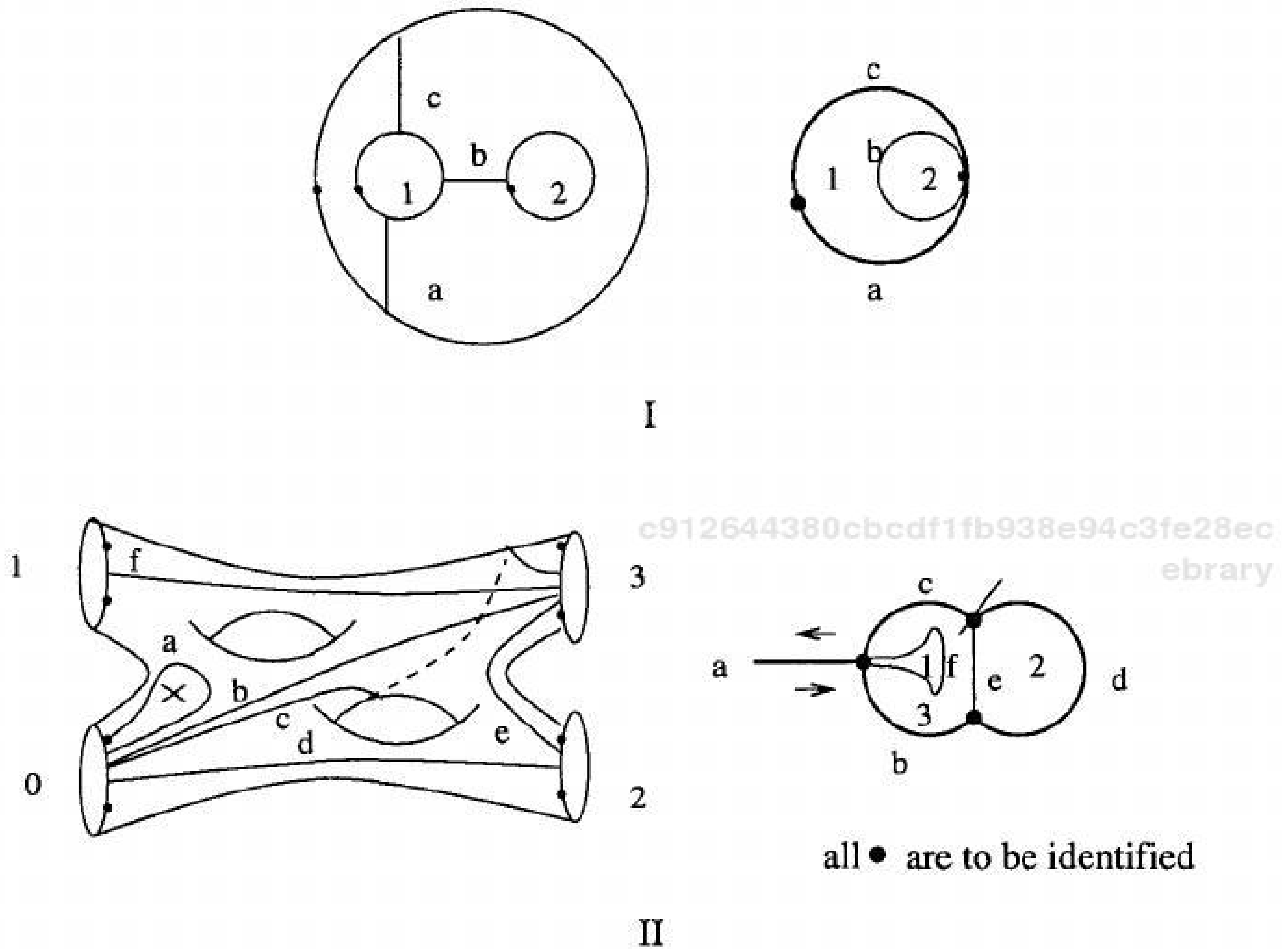


Figure 27. Loops of arc families not yielding cacti: I. genus 0 case; II. genus 2 with one puncture.

natural orientation for all images $S^1_i : i \neq 0$ coinciding with the induced orientation and the opposite orientation for S^1_0. We call the space of planar configurations of $n+1$ circles $Config_p(n)$.

Proposition 6.14.2. *The map* $Loop : \mathcal{D}\mathcal{A}rc(n) \to Config_p(n)$ *is surjective.*

Definition-Proposition 6.14.3. The deprojectivized arc families such that $\pi|_{\bar{c}(\partial_0)}(\alpha)) = \mathcal{L}oop((\alpha))$ constitute a suboperad of $\mathcal{A}rc$. We call this suboperad $\mathcal{L}oop$.

Proposition 6.14.4. *If* $(\alpha) \in \mathcal{L}oop$ *then* $\mathcal{L}oop((\alpha))$ *is a cactus. Furthermore, the operad* $\mathcal{L}oop$ *is identical to the operad of Chinese trees.*

Using the symmetric scaling version of cacti (see 6.8), we obtain:

Theorem 6.14.5. *The framing of a cactus is a section of Loop and is thus an embedding. This embedding identifies (normalized and/or spine-*

less) cacti as (normalized and/or linear) trees.

$$
\begin{array}{ccc}
 & \xleftarrow{Loop} & \\
\textit{(spineless) cacti} & \xrightarrow[frame]{} & \mathcal{D}\mathcal{A}rc \\
\pi \downarrow & & \downarrow \pi \\
\textit{(spineless) cacti}/\mathbb{R}_{>0} & \xrightarrow{\pi frame} & \mathcal{A}rc \\
 & \xleftarrow[\pi Loop]{} &
\end{array}
$$

where $\pi Loop$ *is defined by choosing any lift.*

By inspection of the diagrams for the Gerstenhaber relations we see that up to homotopy all are defined in the image of $Cact$ and the BV structure can be defined in the image of $Cacti$.

Thus we have:

Corollary 6.14.6. [29]. *The chains of Cact carry the structure of a Gerstenhaber algebra up to homotopy. The chains of Cacti carry the structure of a GBV algebra up to homotopy.*

Remark 6.14.7. Actually, up to homotopy all the structures can be obtained using normalized cacti, see below §7.

6.15. *Configurations, Loops and Ribbon Graphs*

Although we did not use the notation of ribbon graphs, it is easy to see that our configurations are essentially ribbon graphs with marked points on some cycles. For this discussion it is easier to restrict to $\mathcal{A}rc_{\#}$ and $s = 0$ (no punctures). In this case the graph we obtain from loop is the dual graph on the surface, which is a ribbon graph. Also on each cycle there is a marked point and a parametrization. In this description it is also easy to see that for $s = 0$ $\mathcal{A}rc_{\#}$ is homotopy equivalent to the decorated moduli space by using Strebel differentials.

6.16. *Comments on an Action on Loop Spaces*

Given a manifold M we can consider its loop space LM. Using the configuration we have maps

$$Arc(n)\times LM^n \overset{Loop\times id}{\longleftarrow} Config(n)\times LM^n \overset{i}{\longleftarrow} L^{Config(n))}M \overset{e}{\rightarrow} LM \quad (6.10)$$

where $L^{Config(n))}M$ are continuous maps of the images L of the configurations into M, i.e., such a map takes a configuration $p : \coprod_i S^1_{m_i} \to L$ and produces a continuous $f : L \to M$; the maps i, e are given by $i(f) = (p : \coprod_i S^1_{m_i} \to L, f(p(S^1_{m_1}), \dots, f(p(S^1_{m_n})))$ and $e(f) = f(p(S^1_{m_0}))$.

One would like apply a Pontrjagin–Thom construction[10,49] so that the maps i and e would in turn induce maps on the level of homology

$$\text{“}H_*(\mathcal{A}rc(n)) \otimes H_*(LM^n) \simeq H_*(\mathcal{D}\mathcal{A}rc(n)) \otimes H_*(LM^n) \overset{Loop_*}{\to}$$
$$H_*(Config(n)) \otimes H_*(LM^n) \overset{i^!}{\to} H_*(L^{Config(n)}M) \overset{e_*}{\to} H_*(LM)\text{”} \quad (6.11)$$

where $i^!$ is the “Umkehr” map.

If we restrict ourselves to this subspace for which the validity of the argument above has been established[8,49,10,6] we obtain

Proposition 6.16.1. *The homology of the loop space of a compact manifold is an algebra over the suboperad of quasi-filling Chinese trees.*

6.17. *Remarks*

(1) This also holds for the appropriate PROP or di-operad setting, in which the arcs only run from distinguished inputs to outputs.

(2) It is clear that one desideratum is the extension of this result to all of $\mathcal{A}rc_\#$.

(3) The first example of an operation of composing loops which are not cacti would be given by the *Loop* of the pair of pants with three arcs as depicted in Figure 28. This kind of composition first appeared in the considerations of closed string field theory.

(4) If the image of Loop is not connected, then the information is partially lost. This can be refined however by using a prop version of our operad.

(5) Factoring the operation of $\mathcal{A}rc$ through *Loop* has the effect that the internal topological structure is forgotten; thus, the torus with two boundary components has the same effect as the cylinder, for instance. This amounts to a certain stabilization.

7. Little Discs, Spineless Cacti and the Cellular Chains of Normalized Spineless Cacti

One motivation for studying spineless cacti is that they give a well adapted chain model for the little discs operad. In fact, we will show below how the

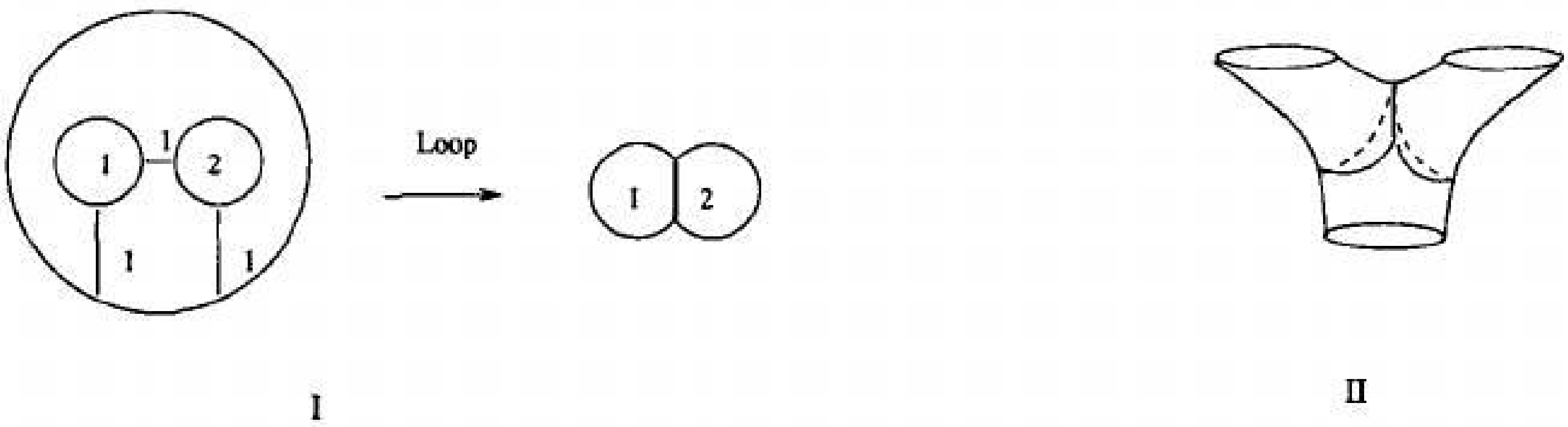


Figure 28. I. The Loop of a symmetric pair of pants; II. A closed string field theory picture of a pair of pants

chains of spineless cacti operate naturally on the Hochschild complex of an associative algebra using the chain decomposition of this section.

7.1. *Cacti as Semi–Direct Products of Normalized Cacti*

In this paragraph we will make the relationship between cacti and normalized cacti explicit using the notion of semi–direct product of quasi–operads[29]. Since we will be dealing with one explicit example of this structure only, where all the relevant operations are explicitly defined[29], we will not review the general constructions here. We include this section to show that the normalized cacti are indeed associative up to homotopy. We will also use this description of normalized cacti to obtain the operadic structure on its cell decomposition in §7

7.2. *The Scaling Operad*

We define the scaling operad $\mathcal{R}_{>0}$ to be given by the spaces $\mathcal{R}_{>0}(n) := \mathbb{R}^n_{>0}$ with the permutation action by $\mathbb{S}_n$ and the following products

$$(r_1,\dots,r_n)\circ_i(r'_1,\dots,r'_m) = (r_1,\dots r_{i-1},\frac{r_i}{R}r'_1,\dots,\frac{r_i}{R}r'_m,r_{i+1},\dots r_n)$$

where $R=\sum_{k=1}^m r'_k$. It is straightforward to check that this indeed defines an operad.

7.3. *The Perturbed Compositions*

We define the perturbed compositions

$$\circ_i^{\mathcal{R}_{>0}} : Cacti^1(n)\times\mathcal{R}_{>0}(m)\times Cacti^1(m)\to Cacti^1(n+m-1) \quad (7.1)$$

via the following procedure: Given $(c,\vec{r}',c')$ we first scale c' according to $\vec{r}'$, i.e. scale the j–th lobe of c' by the j–th entry r_i of $\vec{r}$ for all lobes. Then we scale the i–th lobe of the cactus c by $R = \sum_j r_j$ and glue in the scaled cactus. Finally we scale all the lobes of the composed cactus back to one.

We also use the analogous perturbed compositions for $Cact^1$.

7.4. *The Perturbed Multiplications in Terms of an Action*

We can also describe, slightly more technically, the above compositions in the following form. Fix an element $\vec{r} := (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$ and set $R = \sum_i r_i$ and a normalized cactus c with n lobes. Denote by $\vec{r}(c)$ the cactus where each lobe has been scaled according to $\vec{r}$, i.e. the j–th lobe by the j–th entry of $\vec{r}$. Now consider the chord diagram of the cactus $\vec{r}(c)$. It defines an action on S^1 via

$$\rho : S^1 \xrightarrow{rep^1_R} S^1_R \xrightarrow{cont_{\vec{r}(c)}} S^1_n \xrightarrow{rep^n_1} S^1 \tag{7.2}$$

Where $cont_{\vec{r}}$ acts on S^1_R in the following way. Identify the pointed S^1_R with the pointed outside circle of the chord diagram of $\vec{r}(c)$. Now contract the arcs belonging to the i–th lobe homogeneously with a scaling factor $\frac{1}{r_i}$.

Using this map on the i–lobe of a normalized (spineless) cactus which we think of as an S^1 with base point given by the local zero together with marked points, where the marked points are the intersection points, we obtain maps

$$\begin{aligned} \rho_i &: Cact^1(n) \times \mathcal{R}_{>0}(m) \times Cact^1(m) \to Cact^1(n) \\ \rho_i &: Cacti^1(n) \times \mathcal{R}_{>0}(m) \times Cacti^1(m) \to Cacti^1(n) \end{aligned} \tag{7.3}$$

What this action effectively does is move the lobes and if applicable the root of the cactus c which are attached to the i–th lobe according to the cactus $\vec{r}(c')$ in a manner that depends continuously on $\vec{r}$ and c'.

With this action we can write the perturbed multiplication as

$$\begin{aligned} \circ_i^{\mathcal{R}_{>0}} &: Cacti^1(n) \times \mathcal{R}_{>0}(m) \times Cacti^1(m) \\ &\xrightarrow{id \times id \times \Delta} Cacti^1(n) \times \mathcal{R}_{>0}(m) \times Cacti^1(m) \times Cacti^1(m) \\ &\xrightarrow{\rho_i \times id} Cacti^1(n) \times Cacti^1(m) \xrightarrow{\circ_i} Cacti^1(n+m-1) \end{aligned} \tag{7.4}$$

Theorem 7.4.1. *The operad of spineless cacti is isomorphic to the operad given by the semi–direct product of their normalized version with the scaling operad. The latter is homotopic through quasi–operad maps to the direct product as quasi–operads. The same statements hold true for cacti.*

$$Cact \cong \mathcal{R}_{>0} \ltimes Cact^1 \simeq Cact^1 \times \mathcal{R}_{>0}$$
$$Cacti \cong \mathcal{R}_{>0} \ltimes Cacti^1 \simeq Cacti^1 \times \mathcal{R}_{>0} \tag{7.5}$$

as operads where the operadic compositions are given by

$$(\vec{r}, c) \circ_i (\vec{r}', c') = (\vec{r} \circ_i \vec{r'}, C \circ_i^{\vec{r'}} c') \tag{7.6}$$

From this description we obtain several useful corollaries.

Corollary 7.4.2. *The quasi–operads of normalized cacti and normalized spineless cacti are homotopy associative and thus their homology quasi–operads are actually operads.*

Corollary 7.4.3. *The quasi–operads of normalized cacti and normalized spineless cacti are weakly homotopy equivalent to cacti respectively spineless cacti.*

Furthermore the quasi–operads of normalized cacti and normalized spineless cacti are homotopy equivalent to cacti respectively spineless cacti as quasi–operads.

And lastly:

Corollary 7.4.4. *Normalized cacti and normalized spineless cacti are quasi–isomorphic to cacti respectively spineless cacti. I.e. there homology operads are isomorphic.*

7.5. *Cact(i) and the (Framed) Little Discs Operad*

We would like to collect the following facts.

Theorem 7.5.1. [29] *The operad Cact is (weakly) homotopy equivalent to the little discs operad.*

We proved this fact[29] by using the recognition principle of Fiedorowicz[15]. The A_∞ structure is given by the so–called spineless corolla cacti, whose defining property is that all base points coincide. They correspond to arc families of genus zero with no punctures and exactly one arc from boundary i to boundary 0. The braid structure is shown to hold by using the diagram for the associator. Finally the contractibility of the universal cover follows from the fact that by contracting the $n+1$–st lobe of a cactus with $n+1$ lobes $Cact(n+1)$ is homotopy equivalent to the universal fibration over spineless cacti with n lobes whose fiber over a cactus is the image of that cactus.

Theorem 7.5.2. [49,29] *The operad Cacti is (weakly) homotopy equivalent to the framed little discs operad.*

We also introduced the notion of semi–direct and bi–crossed products for quasi–operads[29] which are the suitable generalization of the same notions for groups. With this notion the relationship between cacti and spineless cacti can be formulated precisely.

Theorem 7.5.3. [29] *The operad of cacti is the bi–crossed product of the operad of spineless cacti with the operad $\mathcal{S}^1$ based on S^1 and furthermore this bi–crossed product is homotopic to the semi–direct product of the operad of cacti without spines with the circle group S^1 which is homotopy equivalent as quasi–operads to the semi–direct product.*

$$\mathcal{C}acti \cong \mathcal{C}act \bowtie \mathcal{S}^1 \simeq \mathcal{C}act \rtimes \mathcal{S}^1 \tag{7.7}$$

This fact that should be compared the fact

Theorem 7.5.4. [47] *The framed little discs operad is the semi–direct product of the little discs with S^1.*

For any monoid, there is a notion of an associated operad[47,29]. In this case the semi–direct product of quasi–operads[29] actually yields and operad[47,29].

Remark 7.5.5. This theorem[47] together with Theorem 7.5.3 and Theorem 7.5.1 imply Voronov's Theorem.

7.6. *A Cell Decomposition for Spineless Cacti*

Recall that the spaces $\mathcal{C}act^1(n)$ are the subspaces of $\mathcal{C}act(n)$ with the restriction that all the radii of the lobes are fixed to one.

This space inherits the obvious action by $\mathbb{S}_n$ of permuting the labels.

Definition 7.6.1. The topological type of a spineless normalized cactus in $\mathcal{C}act^1(n)$ is defined to be the tree $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$ which is its b/w graph together with the labelling induced from the labels of the cactus and the linear order induced on the edges, by the embedding into the plane and the position of the root.

Definition 7.6.2. We define $\mathcal{T}_{bp}^{pp,nt}(n)^k$ to be the elements of $\mathcal{T}_{bp}^{pp,nt}(n)$ with $|E_w| = k$.

Let Δ^n denote the standard n–simplex.

Definition 7.6.3. For $\tau \in \mathcal{T}_{bp}^{pp,nt}$ we define

$$\Delta(\tau) := \times_{v \in V_w(\tau)} \Delta^{|v|} \tag{7.8}$$

Notice that $\dim(|\Delta(\tau)|) = |E_w(\tau)|$.

Theorem 7.6.4. *The space $\mathcal{C}act^1(n)$ is a CW complex whose k-cells are indexed by $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)^k$ with the cell $C(\tau) \sim |\Delta(\tau)|$. Moreover the map $\tau \mapsto C(\tau)$ is a map of differential operads and it identifies $\mathcal{T}_{bp}^{pp,nt}(n)^k$ with $CC_k(\mathcal{C}act^1(n))$, where CC_k are the dimension k cellular chains.*

Proof. Given an element in $\mathcal{C}act^1$, we can view it as given by its topological type and a labelling of the arcs of its underlying arc family with the condition that the sum of all labels at each boundary is one. The number of incident arcs at each boundary i is $|v_i|$ and the condition of the weights summing to one translates to the weights being in $|\Delta^{|v_i|}|$. Vice versa given an element on the right hand side, the summand determines the topological type and it is obvious that any tree in $\mathcal{T}_{bp}^{pp,nt}$ can be realized. Then the barycentric coordinates in the standard orientation define weights to the arcs in their fixed orientation of incidence. We orient the cells $|\Delta(\tau)|$ in the natural orientation induced from the linear order on the white edges. This makes the glueing well defined which can, for instance, be seen from the definition of the arc complex. □

Keeping track of the homotopies which are explicitly given in §7.1, it is evident that $CC_*(\mathcal{C}act^1(n))$ is indeed a chain operad. From our previous analysis about the structure of $\mathcal{C}act$ as a semi–direct product see §7.1 we thus obtain:

Theorem 7.6.5. *The glueings induced from the glueings of spineless normalized cacti make the collection $CC_*(\mathcal{C}act^1(n))$ into a chain operad. And since $\mathcal{C}act, \mathcal{C}act^1$ and D_2 are all homotopy equivalent $CC_*(cact^1)$, is a model for the chains of the little discs operad.*

Remark 7.6.6. If one would like a topological operad in the background, one can choose any chain model $Chain(\mathcal{R}_{>0})$ for the scaling operad, then use the mixed chains for $\mathcal{C}act$ i.e. $CC_*(\mathcal{C}act^1) \otimes Chain(\mathcal{R}_{>0})$. It follows that the inclusion of the cellular chains of $\mathcal{C}act^1$ into the mixed chains is an inclusion of operads up to homotopy.

Finally, given an operation of the $CC_*(\mathcal{C}act^1)$ we can let the mixed chains of $\mathcal{C}act$ act by letting the mixed chains of bidegree $(n, 0)$ act as the component of $\mathcal{C}act^1$ and sending all the others to zero.

7.7. *Orientations of Chains*

To fix the generators and thereby the signs for the chain operad we have several choices, each of which is natural and has appeared in the literature.

To fix a generator $g(\tau)$ of $CC_*(\mathcal{C}act^1)$ corresponding to the cell indexed by $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$ we need to specify an orientation for it, i.e. a parameterizations or equivalently an order of the white edges of the tree the arc family it represents, i.e. a parameterizations or .

The first orientation which we call Nat is the orientation given by the natural orientation of the arc family or equivalently the natural orientation for a planar planted tree. I.e. fixing the order of the white edges to be the one given by the embedding in the plane.

We will also consider the orientation Op which is the enumeration of the white edges which is obtained by starting with the incoming edges of the white vertex labelled one, in the natural orientation of that vertex, then continuing with the incoming white edges into the vertex two, etc. until the last label is reached.

Lastly, for top–dimensional cells, we will consider the orientation of the edges induced by the labels, which we call Lab. It is obtained from Nat as follows: for $\tau \in \mathcal{T}_{b/w}^{pp,nt,fl}$ let $\sigma \in \mathbb{S}_n$ be the permutation which permutes the vertices $v_1, \ldots, v_n$ to their natural order induced by the order $\prec^{(\tau)}$. Then let the enumeration of E_w be $\sigma(Nat)$, where the action of σ on E_w is given by the correspondence out and the correspondence between black and white edges via $(v, N(v)) \mapsto (N(v), N^2(v))$ for top dimensional cells.

To compare with the literature it is also useful to introduce the orientations $\overline{Nat}$, $\overline{Lab}$, and $\overline{Op}$ which are the reversed orientation of Nat, $\overline{Lab}$ and Op, i.e. reading them from right to left.

7.8. *The Differential on* $\mathcal{T}_{bp}^{pp,nt}$

There is a natural differential on $\mathcal{T}_{bp}^{pp,nt}$ which it inherits from its interpretation as $CC_*(\mathcal{C}act)$ see below.

Recall that for a planted planar tree there is a linear order on all edges and therefore a linear order on all subsets of edges.

Definition 7.8.1. Let $\tau \in \mathcal{T}_{bp}^{pp,nt}$. We set $E_{angle} = E(\tau) \setminus (E_{leaf}(\tau) \cup \{e_{root}\})$ and we denote by $num_E : E_{angle} \to \{1, \dots, N\}$ the bijection which is induced by the linear order $\prec^{(\tau,p)}$.

Definition 7.8.2. Let $\tau \in \mathcal{T}_{bp}^{pp,nt}$, $e \in E_{angle}$, $e = \{w, b\}$, with $w \in V_w$ and $b \in V_b$. Let $e- = \{w, b-\}$ be edge preceding e in the cyclic order $\prec^{\tau,w}$ at w. Then $\partial_e(\tau)$ is defined to be the tree with the vertex b and edge of e deleted and the other edges adjacent to b transplanted to the vertex of the next edge keeping their order of w.r.t $\prec^{\tau}$ intact. As a planar picture, one can think of collapsing the angle between the edge e and its predecessor in the cyclic order of w.

Definition 7.8.3. We define the operator ∂ on the space $\mathcal{T}_{bp}^{pp,nt}$ to be given by the following formulas

$$\partial(\tau) := \sum_{e \in E_{angle}} (-1)^{num_E(e)-1} \partial_e(\tau) \tag{7.9}$$

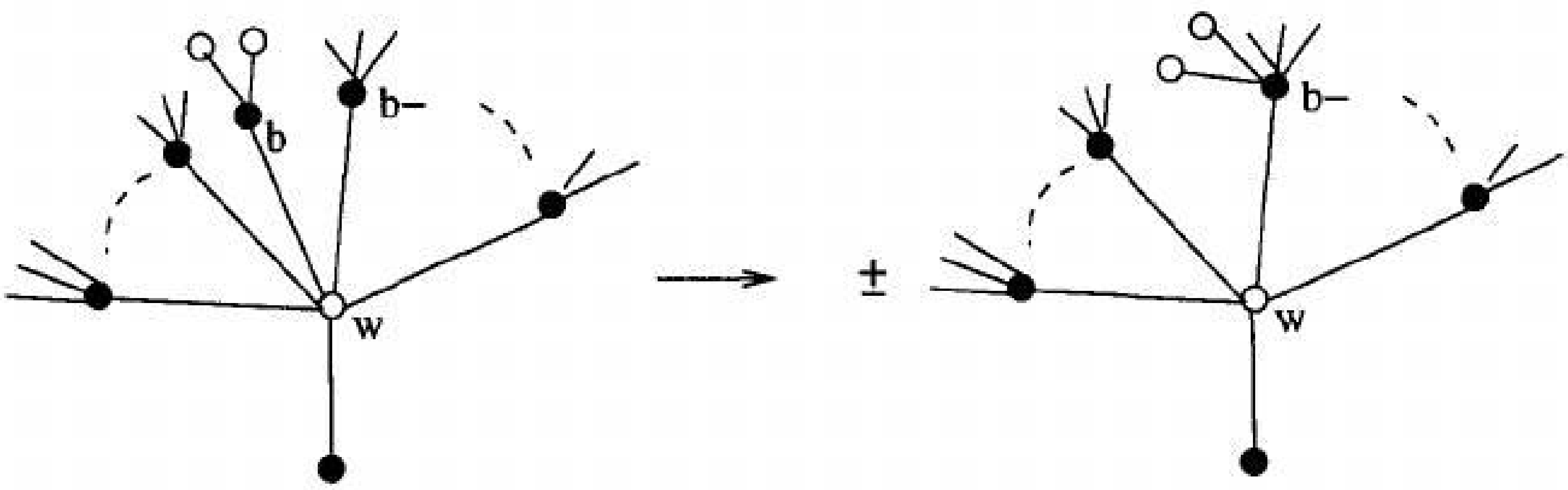


Figure 29. The collapsing of an angle.

Denote by $\mathcal{T}_{bp}^{pp,nt}(n)^k$ the elements of $\mathcal{T}_{bp}^{pp,nt}(n)$ with k white edges.

Proposition 7.8.4. *The map $\partial : \mathcal{T}_{bp}^{pp,nt}(n)^k \to \mathcal{T}_{bp}^{pp,nt}(n)^{k-1}$ is a differential for $\mathcal{T}_{bp}^{pp,nt}$ and turns $\mathcal{T}_{bp}^{pp,nt}$ into a differential operad.*

Proof. The fact that ∂ reduces the number of white edges by one is clear. The fact that $\partial^2 = 0$ follows from a straightforward calculation. Collapsing two angels in one order contributes negatively with respect to the other order. The compatibility of the multiplications $\circ_i$ is also straightforward. All these properties will also follow from the chain interpretation of the trees in §7. □

Theorem 7.8.5. *For the choice of orientation Nat and the induced operad structure* $\circ_i$ *the map* $\tau \mapsto g(\tau)$ *where* $g(\tau)$ *is the generator corresponding to* $C(\tau)$ *fixed in 7.7 is a map of differential operads and it identifies* $\mathcal{T}_{bp}^{pp,nt}(n)^k$ *with* $CC_k(\mathcal{C}act^1(n))$*, where* CC_k *are the dimension* k *cellular chains.*

The analogous statement holds true when passing to operads indexed by sets for both $\mathcal{C}act$ *and* $\mathcal{T}_{bp}^{pp,nt}$.

7.9. *The Operadic Action of* $\mathcal{T}_{bp}^{pp,nt}$

A natural way to let $\mathcal{T}_{bp}^{pp,nt}$ act on a complex $(\mathcal{O}, \delta)$ is given by building a mixed complex by identifying the white edges of a tree with elements from $\mathcal{C}act^1$ and the vertices with elements from $\mathcal{O}$.

First we notice that if we are dealing with planted planar trees, we have the total linear order $\prec$ on the set of vertices and edges. For an action of the operad $\mathcal{T}_{bp}^{pp,nt}$ on a graded space $\mathcal{O} = \sum O(n)$, we will consider maps

$$\rho : \mathcal{T}_{bp}^{pp,nt}(k) \otimes O(n_1) \otimes \cdots \otimes O(n_k) \to O(m) \tag{7.10}$$

$$\tau \otimes f_1 \otimes \cdots \otimes f_k \mapsto \tau(f_1 \otimes \cdots \otimes f_k) \tag{7.11}$$

Actually, $\tau(f_1 \otimes \cdots \otimes f_k)$ will be zero unless $|v_i| = n_i$ and $m = \sum n_i - k$.

In the graded case, we have to fix the order of the tensor product on the l.h.s. of the expression (7.11). We do this by using $\prec^\tau$ to give tensor product the natural operadic order (i.e. $O(i)$ inserted into the vertex v_i. Let $N := |\{V(\tau) \amalg E_w(\tau)\}|$ and let $num : |\{V(\tau) \amalg E(\tau)\}| \to \{1, \dots, N\}$ be the bijection which is induced by $\prec^\tau$. We fix L_1 to be a "Shifted" line i.e. a free generated by an element of degree one. Now set

$$W_i := \begin{cases} O(n_j) & \text{if } num^{-1}(i) = v_j \\ L_1 & \text{if } num^{-1}(i) \text{ is a white edge} \end{cases} \tag{7.12}$$

We then define the order on tensor product on the l.h.s. of the expression (7.11) to be given by

$$W := W_1 \otimes \cdots \otimes W_N$$

Another way would be to include the sign which is necessary to permute the l.h.s. of 7.11 into W into the operation ρ.

7.10. *The Action of the Symmetric Group*

The action of the symmetric group is induced by permuting the labels and permuting the elements of $\mathcal{O}$ respectively. This induces a sign by permutation on W.

Remark 7.10.1. This treatment of the signs is essential if one is dealing with operads and wishes to obtain equivariance with respect to the symmetric group actions. In general the symmetric group action on the endomorphism operads will not produce the right signs needed in the description of the iterations of the universal concatenation $\circ$ of §8. In particular this is the case for Gerstenhaber's product on the Hochschild cochains. The above modification however leads to an agreement of sings for the action of the symmetric group for the subcomplex of the Hochschild complex generated by products and the brace operations, see below §7.9 and §9. Another approach is given by viewing the operations not as endomorphisms of the Hochschild cochains but rather maps of the Hochschild cochains twisted by tensoring with copies of the line L_1[36] and §9. If one is not concerned with the action of the symmetric group, then one can forgo this step.

7.11. *The Action of Chain($\mathcal{A}rc$) on Itself and String Topology*

A good example of the type of action described above is the action of the chains of the $\mathcal{A}rc$ operad on themselves[34]. For the homotopy Gerstenhaber structure we need an action of $CC_*(Cact^1)$ on any choice of chain model for $\mathcal{A}rc$ or any of the suboperads which are stable under the linear trees suboperad. The action ρ is just given by the glueing in $\mathcal{A}rc$.

We get agreement with the signs of the operation on $\mathcal{A}rc$ which agree with those of string topology[8], if we denote the action of τ_1 as $*^{op}$ and τ_2^b as $\cdot$, see §5.13 for the operations and figure 31 for the definitions of the trees.

For the homotopy GBV structure we should consider the chains $CC_*(Cacti^1)$ and again any choice of chain model for $\mathcal{A}rc$ or any of the suboperads which are stable under the action of the trees suboperad.

8. Structures on Operads and Meta–Operads

Before going into the statement and proof of Deligne's conjecture, we would like to digress once more on operads. This helps to explain some choices of signs and explains the naturality of the construction of insertion operads which gives a special role to spineless cacti as their topological incarnation as well as to $\mathcal{A}rc$ as a natural generalization.

This analysis also enables us to relate spineless cacti to the renormalization Hopf algebra of Connes and Kreimer[11]. In particular for a given linear operad or operad which affords a direct sum, we defined a Hopf algebra[30]. The symmetric group coinvariants of the Hopf algebra of the suboperad of symmetric top–dimensional cells of the normalized spineless cacti are exactly the Hopf algebra of Connes and Kreimer.

Given any operad there are certain universal operations, i.e. maps of the operad to itself. We will first ignore possible signs and comment on them later on.

8.1. *The Universal Concatenations*

Given any operad, we have the structure maps

$$\circ_i : O(m) \otimes O(n) \to O(m+n-1)$$

and the concatenations of these, which can be described by their flow charts. These are given by $\tau \in \mathcal{T}^{pp,fl}_{b/w}$. More precisely given k elements $op_k \in O(n_k)$, we can concatenate them with the $\circ_i$ to produce a tree flow chart where the inputs are the leaves and tails and the inner vertices are labelled by the operations op_k, where the a vertex v_k labelled by op_k necessarily has valence n_k. The number of leaves and the degrees $n-i$ of the op_i satisfy the condition

$$wt(\tau) = \sum_v |v| = \sum n_i = \#\text{leaves} + \#\text{inner vertices} = \#\text{leaves} + k \quad (8.1)$$

Notice, we might have white leaves, which allows one to consider operads also with a 0 component such as CH^* see below.

So let $\tau \in \mathcal{T}^{pp,fl}_{b/w}(k)$ and let $n_i : i \in 1, \dots k := |v_i|$ then there is an operation

$$\circ(\tau)(O(n_1) \otimes \cdots \otimes O(n_k)) \to O(m) \quad (8.2)$$

by labelling the vertex v_i by $op_{n_i} \in \mathcal{O}(n_i)$.

Notice that, we used the linear order on a planted planar tree in order to associate the functions to the non–leaf vertices.

In general lifting the restriction on the n_i, we define the operations $\circ(\tau)$ to be zero of $|v_i| \neq n_i$.

The above considerations give rise to a partial non–Σ operad operation of $\mathcal{T}^{pp,fl}_{b/w}$ which can be made into an operation of the operad $\mathcal{T}^{pp,fl}_{b/w}$ by using $\mathbb{S}_n$ equivariance. The partial concatenations $\circ_i$ insert a tree with k–tails into the vertex v_i if $|v_i| = k$, by connecting the incoming edges of v_i to the tail vertices in the linear order at v_i and contracting the tail edges.

Definition 8.1.1. We will fix that for $\mathcal{O}$ in $\mathcal{S}et$ the direct sum which we again denote by $\mathcal{O}$ is given by the free Abelian group generated by $\mathcal{O}$ which we consider to be graded by the arity of the operations $op \in \mathcal{O}$ minus one. If the operad $\mathcal{O}$ is in $\mathcal{C}hain$ we can take the direct sum of the components as $\mathbb{Z}$-modules. In the case of an operad $\mathcal{O}$ in the category $\mathcal{V}ect_k$ we consider its direct sum to be the direct sum over k of its components. In all these cases, we call $\mathcal{O}$ the direct sum and say $\mathcal{O}$ affords a direct sum and write $\mathcal{O} = \bigoplus_{n\in\mathbb{N}} Op(n)$. In all these cases we can consider $\mathcal{O}$ to be graded by $\mathbb{N}$ with the degree of $Op(n)$ being $n-1$.

Remark 8.1.2. The above definition allows one to can make sense, formal linear combinations of operad elements with coefficients ± 1. We could extend the use of the expression to afford a direct sum to mean that, the category which the operad is defined allows one to construct direct sums which are $\mathbb{Z}$ modules.

If we consider an operad which affords a direct sum and let $\mathcal{O}$ be its direct sum then we obtain an operadic map.

$$\mathcal{T}^{pp,fl}_{b/w} \to Hom(\mathcal{O},\mathcal{O})$$

In this sense one can say that $\mathcal{T}^{pp,fl}_{b/w}$ is the universal concatenation partial–operad.

8.2. *The pre–Lie Structure of an Operad*

In an operad which affords direct sums, one can define the analog of the $\circ$ product and the iterated brace operation (cf. [19,25]), see above.

Definition 8.2.1. Given any operad $\mathcal{O}$ in $\mathcal{S}et$, $\mathcal{C}hain$ or $\mathcal{V}ect_k$, we define the following map

$$O(m) \otimes O(n) \to O(m+n-1) \tag{8.3}$$

$$op_m \otimes op_n \mapsto \sum_{i=1}^{m} (-1)^{(i-1)(n+1)} op_m \circ_i op_n \tag{8.4}$$

This extends to a map

$$\circ : \mathcal{O} \otimes \mathcal{O} \to \mathcal{O} \tag{8.5}$$

which we call the $\circ$ product.

We call the map which is obtained from in the same fashion as $\circ$, but with the omission of the signs $(-1)^{(i-1)(n+1)}$ the ungraded $\circ$ product.

Following Gerstenhaber's calculation[16,40] (essentially using associativity), we immediately have the following proposition

Proposition 8.2.2. *The product* $\circ$ *defines on* $O := \bigoplus_{i\in\mathbb{N}} O(n)$ *the structure of a graded pre–Lie algebra. Omitting the sign* $(-1)^{(i-1)(n+1)}$ *in the sum yields the structure of a non–graded pre–Lie algebra.*

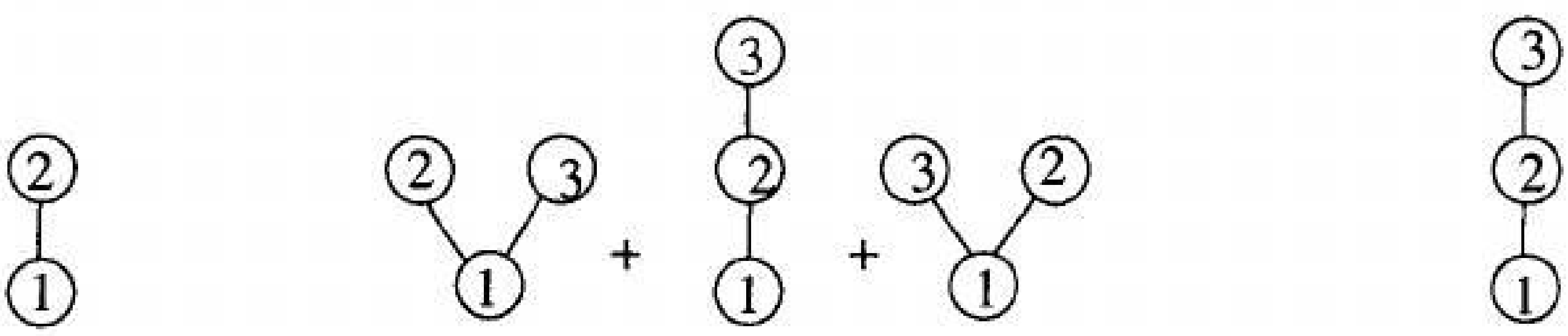
We do not rewrite the proof here, but in graphical notation the proof follows from figure 30 below.

Without signs this notation is related to the one that can be found for rooted trees[7], for the case with signs see §7.7 and §9.8.

8.3. *The Insertion Operad*

The interesting property of the operation $\circ$ is that it effectively removes the dependence on the number of inputs of the factors.

Given an operad in $\mathcal{C}hain$ we can also define other operations similar to $\circ$ which are in natural correspondence with $\mathcal{T}^{pp}$. In fact these operations all appear in the iterations of $\circ$. They are given by inserting the operations into each other according to the scheme of the tree and then distributing tails so that the equation (8.1) is satisfied. Examples of this are given in figure 30. Here the first tree yields the operation $f_1 \circ f_2$, i.e the insertion (at every place) of f_2 into f_1. Iterating this insertion we obtain expression II which shows that inserting f_3 into $f_1 \circ f_2$ gives rise to three topological types: inserting f_3 in front of f_2, into f_2 and behind f_2. In the opposite iteration one just inserts $f_2 \circ f_3$ into f_1 which gives a linear insertion of f_2 into f_1 and f_3 into f_2. From the figure (up to signs) one can read off the symmetry in the entries 2 and 3 of the associator.



I II III

Figure 30. I. $f_1 \circ f_2$ II. $(f_1 \circ f_2) \circ f_3$ and III. $f_1 \circ (f_2 \circ f_3)$

We will not care about signs at the moment, they follow from §7.7 of from 9.8.

8.4. *Notation*

There are some standard trees, which are essential in our study, these are the n–tail tree l_n, the white n–leaf tree τ_n, and the black n–leaf tree τ_n^b, as shown in figure 31

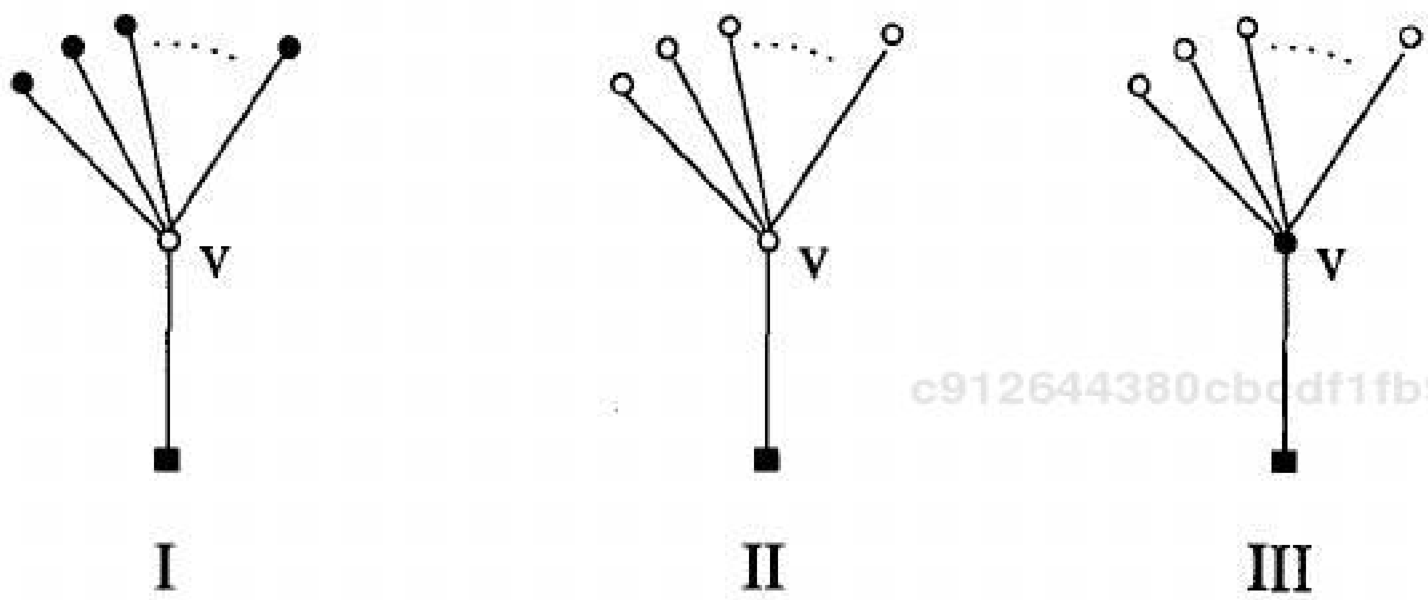


Figure 31. I The n–tail tree l_n, II. The white n–leaf tree τ_n, III. The black n–leaf tree τ_n^b

Essentially, if we would not like to *a priori* specify the number of leaves, i.e. inputs and degrees of the operations, we have to consider trees with all possible decorations by leaves. For this we need foliation operators in the botanical sense. To avoid confusion with the mathematical term "foliation", we choose to abuse the English language and call these operations "foliage" operators.

Recall that there is an operation of $\mathcal{T}_{b/w}^{pp,fl}$ on homogeneous elements of $\mathcal{O}$ of the right degree. We extend this operation to all of $\mathcal{O}$ by extending linearly and setting to zero expressions which do not satisfy degree condition that $op_k \in \mathcal{O}(|v_k|)$

Definition 8.4.1. Let l_n be the tree in $\mathcal{T}_{bp}^{pp}$ with one white vertex labelled by v and n tails as depicted in figure 31. The foliage operator $F : \mathcal{T}_{b/w}^{pp,st,nt} \to \mathcal{T}_{b/w}^{pp,st}$ is defined by the following equation

$$F(\tau) := \sum_{n \in \mathbb{N}} l_n \circ_v \tau$$

Notice that the right hand side is infinite, but since $\mathcal{T}_{b/w}^{pp,st}$ is graded by say the number of leaves, and $F(\tau)$ is finite for a fixed number of black leaves the definition does not pose any problems. Furthermore, one could let F take values in $\mathcal{T}_{b/w}^{pp,st}[[t]]$ where t keeps track of the number of tails which would make the grading explicit.

Also notice that $F : \mathcal{T}_{bp}^{pp,nt} \to \mathcal{T}_{bp}^{pp}$ and $F : \mathcal{T}_{b/w}^{pp,nt,fl} \to \mathcal{T}_{b/w}^{pp,fl}$.

Recall that there is an operation of $\mathcal{T}_{b/w}^{pp,fl}$ on homogeneous elements of $\mathcal{O}$ of the right degree. We extend this operation to all of $\mathcal{O}$ by extending linearly and setting to zero expressions which do not satisfy degree count, i.e. satisfy the equation (8.1).

Given τ in $\mathcal{T}_{b/w}^{pp,nt,fl}(n)$, we can then define the operation

$$\circ_\tau(f_1 \otimes \cdots \otimes f_n) := F(\tau)(f_1 \otimes \cdots \otimes f_n)$$

Notice that, although $F(\tau)$ is an infinite linear combination, for given $f_1, \ldots, f_n$ the expression on the right hand side is finite.

The following is almost automatic.

Theorem 8.4.2. *Any chain operad which affords a direct sum is an algebra over the operad $\mathcal{T}^{pp} = \mathcal{T}_{b/w}^{pp,nt,fl}$ with the insertion product.*

In fact, following Remark 8.4.3, we are forced to look at the insertion product.

Remark 8.4.3. Thinking about F as a formal power–series, e.g. in $\mathcal{T}_{b/w}^{pp,fl}[[t]]$, we can define a product $*$ by the formula

$$F(\tau_1) \circ F(\tau_2) := F(\tau_1 * \tau_2) \tag{8.6}$$

Now, if the $*$ is thought of as operadic, i.e. $\tau_1 * \tau_2 = \gamma(*, \tau_1, \tau_2)$, then by linearity and associativity, we know how to define $\gamma(\tau, t_1, \ldots, t_n)$ for any $\tau \in \mathcal{T}^r \subset \mathcal{T}^{pp}$, fixing the operations of the pre–Lie operad.

8.5. *The Hopf Algebra of an Operad*

We have seen in 8.2 that any operad that affords a direct sum gives rise to a pre–Lie algebra. Now the defining property for a pre–Lie algebra is that the commutator of its product gives a Lie or in the graded case an odd Lie algebra.

Definition 8.5.1. Given an operad $\mathcal{O}$, which affords a direct sum, we define its pre–Lie algebra $PL(\mathcal{O})$ to be the pre–Lie algebra $(\mathcal{O}, \circ)$, its Lie algebra $L(\mathcal{O})$ to be the Lie algebra $(\mathcal{O}, [\ ])$using the Lie bracket $[a, b] := a \circ b = b \circ a$, its Gerstenhaber algebra $G(\mathcal{O})$ to be the Gerstenhaber algebra $(\mathcal{O}, \{\ \})$ where $\{\ \}$ is defined as usual via $\{a\,, b\} := a*b-(-1)^{(|a|+1)(|b|+1)} b*a$. Lastly the Hopf algebra of an operad $H(\mathcal{O})$ is defined to be $U^*(L(\mathcal{O}))$, i.e. the dual of universal enveloping algebra of its Lie algebra.

9. Spineless Cacti as a Natural Solution to Deligne's Conjecture

9.1. *The Hochschild Complex, its Gerstenhaber Structure and Deligne's Conjecture*

Let A be an associative algebra over a field k. We define $CH^*(A, A) := \bigoplus_{q\geq 0} \operatorname{Hom}(A^{\otimes q}, A)$

There are two natural operations

$$\circ_i : CH^m(A, A) \otimes CH^n(A, A) \to CH^{m+n-1}(A, A) \tag{9.1}$$

$$\cup : CH^n(A, A) \otimes CH^m(A, A) \to CH^{m+n}(A, A) \tag{9.2}$$

where the first morphism is as in 4.4 , i.e. for $f \in CH^p(A, A)$ and $g \in CH^q(A, A)$.

$$f \circ_i g(x_1, \ldots, x_{p+q-1}) = f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+q-1}), x_{i+q}, \ldots, x_{p+q-1})$$

and the second is given by the multiplication

$$f(a_1 \ldots, a_m) \cup g(b_1, \ldots, b_n) = f(a_1 \ldots, a_m) g(b_1, \ldots, b_n)$$

9.2. *The Differential on CH^**

The Hochschild complex also has a differential which is also derived from the algebra structure.

Given $f \in CH^n(A, A)$ then

$$\partial(f)(a_1, \ldots, a_{n+1}) := a_1 f(a_2, \ldots, a_{n+1}) - f(a_1 a_2, \ldots, a_{n+1}) + \cdots + (-1)^{n+1} f(a_1, \ldots, a_n a_{n+1}) + (-1)^{n+2} f(a_1, \ldots, a_n) a_{n+1}$$

Definition 9.2.1. The Hochschild complex is the complex (CH^*, ∂), its cohomology is called the Hochschild cohomology and denoted by $HH^*(A, A)$.

9.3. *The Gerstenhaber Structure*

Gerstenhaber[16] introduced the $\circ$ operations: for $f \in CH^p(A, A)$ and $g \in CH^q(A, A)$

$$f \circ g := \sum_{i=1}^{p} (-1)^{(i-1)(q+1)} f \circ_i g$$

and defined the bracket

$$\{f \bullet g\} := f \circ g - (-1)^{(p+1)(q+1)} g \circ f$$

and showed that this is indeed induces what is now called a Gerstenhaber bracket, i.e. odd Poisson for $\cup$, on $HH^*(A, A)$.

9.4. *Deligne's Conjecture*

Since $HH^*(A, A)$ has the structure of a Gerstenhaber algebra one knows from general theory that thereby $HH^*(A, A)$ is an algebra over the homology operad of the little discs operad.

The question of Deligne was: *Can one lift the action of the homology of the little discs operad to the chain respectively cochain level? Or in other words: is there a chain model for the little discs operad that operated on the Hochschild cochains which reduces to the usual action on the homology/cohomology level?*

This question has an affirmative answer in many ways by picking a suitable chain model for the little discs operad[32,48,39,50]. A review of these constructions is also available[40]. We will provide a new and in a sense natural and minimal positive answer to this question, by giving an operation of $CC_*(Cact^1)$ on the Hochschild cochains.

There is a certain minimal set of operations necessary for the proof of such a statement which is given by iterations of the operations $\cup$ and $\circ_i$. These are, as we argue below in bijective correspondence with trees in $\mathcal{T}_{bp}^{pp,nt}$, our model for the chains of the little discs operad $CC_*(\mathcal{C}act^1)$, has chains which are exactly indexed by these trees. Furthermore, the top dimensional cells which control the bracket are the universal concatenation operad. And lastly we will show that the differential of deleting arcs can be seen as a topological version of the Hochschild differential. This makes our new solution natural and minimal.

9.5. *The Operation of $CC_*(\mathcal{C}act^1)$ on $\mathcal{H}om_{CH}$*

For $\mathcal{O} = \mathcal{H}om_{CH}$, we define the map ρ of ed. 7.11, to be given by the operadic extension of the maps which send the tree τ_n to the non-intersecting brace operations i.e. for homogeneous f, g_i of degrees $|f|$ and $|g_i|$, $N = |f| + \sum_i |g_i| - n$

$$f\{g_1,\dots,g_n\}(x_1,\dots,x_N) := \sum_{\substack{1\le i_1\le\cdots\le i_n\le |f|:\\ i_j+|g_j|\le i_{j+1}}} \pm$$

$$f(x_1,\dots,x_{i_1-1},g_1(x_{i_1},\dots,x_{i_1+|g_1|}),\dots,x_{i_n-1},g_n(x_{i_n},\dots,x_{i_n+|g_n|}),\dots,x_N) \tag{9.3}$$

where the sign of the shuffle of the g_j and x_i which is determined by considering the shifted degrees, i.e. the x_i to have degree 1 and the g_j to have degree $|g_j|+1$.

Notice that $f\{g\} = f \circ g$. Brace operations have first been considered by Getzler[19].

In order to make signs match with those of Gerstenhaber[16], we will have to consider the opposite orientation for W i.e. $\overline{W} := W_N \otimes \cdots \otimes W_1$. To implement this change of sign we define $sign^W(\tau)$ to be the sign obtained by passing from W to $\overline{W}$. This basically means that in the orientation of W one would regard the operation $\circ^{op}, \cup^{op}$ on the Hochschild complex, where $f \circ^{op} g = (-1)^{pq+p+1} g \circ f$ and $f \cup^{op} g := (-1)^{pq} g \cup f$.

The action of the tree τ_n is given by:

$$f \otimes L_{e_1} \otimes g_1 \otimes \cdots \otimes L_{e_n} \otimes g_n \mapsto (-1)^{sign^W(\tau)} f\{g_1,\dots,g_n\} \tag{9.4}$$

The action of τ_n^b is given by

$$g_1 \otimes \cdots \otimes g_n \mapsto (-1)^{sign^W(\tau)} g_1 \cup \cdots \cup g_n$$

The operadic extension means that we read the tree as a flow chart at each black vertex $|v|$ the operation $\tau^b_{|v|}$ is performed and at each white vertex the operation $\tau^w_{|v|}$ is performed. The $\mathbb{S}_n$ action is given by permutations and indeed induces the right signs on the Hochschild complex as seen by straightforward calculation.

For the operadic composition in $CC_*(\mathcal{C}act^1)$ we choose $\circ$ with the orientation Nat.

The following is now straightforward:

Proposition 9.5.1. *The above procedure makes $CH^*(A,A)$ into a non-Σ algebra over $CC_*(\mathcal{C}act^1)$.*

9.6. *Signs for the Braces*

It well known[19,33,25] that the set of concatenations of multiplications and brace operations form a suboperad of the endomorphism operad of the Hochschild complex we will call it $\mathcal{B}race$.

The generators of this suboperad are in 1-1 correspondence with elements of $\mathcal{T}_{bp}^{pp,nt}$. Such a tree represents a flow chart. The functions to be acted upon are to be inserted into the white vertices. A black vertex signifies the multiplication of the incoming entities, while a white vertex represents the brace operation of the elements attached to that vertex outside the brace and incoming elements inside the brace.

Notice that in the flow chart of an expression of the type $f\{(g_1),(g_2 \cdot g_3 \cdot g_4\{h_1,h_2\})\}$ the symbols "{" and "," correspond to the white edges.

Proposition 9.6.1. *The association of a flow chart is a non-Σ operadic isomorphism between $\mathcal{B}race$ and $\mathcal{T}_{bp}^{pp,nt}$ of operads with a differential.*

Definition 9.6.2. We define an action of the symmetric group on $\mathcal{B}race$, by considering the symbols "{" and "," to be each of degree one.

The following propositions follow from straightforward computation[30].

Proposition 9.6.3. *With the above action of the symmetric group on $\mathcal{B}race$ the isomorphism of 9.6.1 is an isomorphism of operads.*

Proposition 9.6.4. *The above procedure gives an operation of $CC_*(\mathcal{C}act^1)$ on $CH^*(A,A)$ and an operadic isomorphisms of $\mathcal{B}race$ and $CC_*(\mathcal{C}act^1)$.*

9.7. *The Differential*

If we denote the differential on $CC_*(\mathcal{C}act^1)$ as ∂ and the differential of CH as δ then the action of $CC_*(\mathcal{C}act^1)$ on $\mathcal{H}om_{CH}$ commutes with the differential. On the space W there is a natural differential $\partial_W := \delta + \partial$. The calculations for the chains of the arc operad[34] and the straightforward generalization to action of τ_n yield the following proposition.

Proposition 9.7.1.

$$\rho \circ (\partial_W) = \delta \circ \rho \tag{9.5}$$

9.8. *Another Approach to Signs and Actions*

Another way to fix the signs for the symmetric group actions on the Hochschild complex[36] is achieved by tensoring with one dimensional spaces L_1 and L_2 of degrees -1 and -2 and their duals L_1^* and L_2^*. S Also it is useful to deal with operads indexed by arbitrary sets. For a graded vector space A and an indexing set I one[36] defines

$$C = C^*(A;A) = \oplus_I \underline{Hom}(A^{\otimes I}, A) \otimes (L_2^* \otimes L_1)^{\otimes I}$$

where the sum is taken over all non-empty complectly ordered finite sets, and $\underline{Hom}$ is the internal Hom in the tensor category $Vect_{\mathbb{Z}}$ of $\mathbb{Z}$-graded vector spaces (with Koszul rule of signs). The Gerstenhaber bracket is a map then a map from $C \otimes C \to C \otimes (L_2^* \otimes L_1)^{36}$. Since we will not be in the A_∞ setting, we can omit the reference to the lines L_2.

9.9. *A Second Approach to the Operation of $CC_*(\mathcal{C}act^1)$*

Another way to make $CC_*(\mathcal{C}act^1)$ or $\mathcal{T}_{bp}^{pp,nt}$ act is by using the foliage operator. This approach[36] stresses the fact that a function $f \in CH^q(A, A)$ is naturally depicted by τ_n. Notice for instance the compatibility of the differentials.

9.10. *Natural Operations on CH^* and their Tree Depiction*

Given elements of the Hochschild cochain complex there are two types of natural operations which are defined for them. Suppose f_i is a homogeneous element, then it is given by a function $f : A^{\otimes n} \to A$. So treating the cochains as function, we have the operation of insertion, as in 4.4. The second type of operation comes from the fact that A is an associative algebra; therefore, for each collection $f_1, \ldots, f_n \in CH^*(A, A)$ we have the n! ways of multiplying them together.

We will encode the concatenation of these operations into a black and white bipartite tree as follows: Suppose that we would like to build a cochain by using insertion and multiplication on the homogeneous cochains $f_1, \ldots, f_n$. First we represent each function f_i as a white vertex with $|f_i|$ inputs and one output with the cyclic order according to the inputs $1, \ldots, |f_i|$ of the function. For each insertion of a function into a function we put a black vertex of valence having as input edge the output of the function to be inserted and as an output edge the input of the function into which the insertion is being made. For a multiplication of $k \geq 2$ functions we put a black vertex whose inputs are the functions which are to be multiplied in the order of their multiplication. Finally we add tails to the tree by putting a black vertex at each input edge which has not yet been given a black vertex, and we decorate the tails by variables $a_1, \ldots a_s$ according to their order in the total order of the vertices of the rooted planted planar tree. It is clear that this determines a black and white bipartite tree.

A rooted planted planar bipartite black and white tree whose tails are all black and decorated by variables $a_1, \ldots a_s$ and whose white vertices are labelled by homogeneous elements $f_v \in CH^{|v|}(A, A)$ determines an element in $CH^s(A,A)$ by using the tree as a "flow chart", i.e. inserting for each black vertex of valence one and multiplying for each black vertex of higher valence. Notice that, since the algebra is associative, given an ordered set of elements there is a unique multiplication.

Remark 9.10.1. The possible ways to compose k homogeneous elements of $CH^*(A,A)$ using insertion and cup product are bijectively enumerated by black and white bipartite planar rooted planted trees with tails and k white vertices labelled by k functions whose degree is equal to the valence of the vertex.

We will fix A and use the short hand notation $CH := CH^*(A,A)$. For an element $f \in CH$, we write $f^{(d)}$ for its homogeneous component of degree d.

If we would like to consider non–homogeneous elements, then given a tree we can only use the homogeneous components of the elements of CH with the right degree. This leads to:

Definition 9.10.2. For $\tau \in \mathcal{T}_{bp}^{pp}(n)$ and $f_1,\dots,f_n \in CH$ we let $\tau(f_1,\dots,f_n)$ be the operation obtained in the above fashion by decorating the vertex v_i with label i with the homogeneous component of $f_i^{(|v_i|)}$. Notice that the result is zero if any of the homogeneous components $f_i^{(|v_i|)}$ vanish.

Remark 9.10.3. Up to the signs which are discussed below this gives an operation of $CC_*(Cact)$ on the Hochschild complex.

9.11. *The Operation of $\mathcal{T}_{bp}^{pp,nt}$*

Definition 9.11.1. For a tree $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$ with n white vertices we define a map $op(\tau) \in \mathrm{Hom}(CH^{\otimes n}, CH) = \mathcal{H}om_{CH}(n)$ by

$$op(\tau)(f_1,\dots,f_n) := \pm op(ins(F(\tau),(f_1,\dots,f_n))$$

here ins inserts the function f_i into the label i and the signs are discussed in §7.7 and §9.8.

Proposition 9.11.2. *The Hochschild cochains are an algebra over $\mathcal{T}_{bp}^{pp,nt}$.*

9.12. *The Differential*

Again the differentials are compatible. This can be checked by a straightforward calculation, see §10.6 below. It is also implicit in the work of Kontsevich and Soibelman[36].

Notice that in the tree formalism a black edge is inserted upon differentiating, while in $\mathcal{A}rc$ an arc is erased which in $\mathcal{C}act$ corresponds to contracting an arc. Both these operations reduce the number of parameters by one. Of course omission of a parameter and insertion of a degree one space amounts to the same signs.

Remark 9.12.1. The considerations of this section naturally lead to brace operations in far more general setting. This is explained in detail in §10.5

9.13. *A Solution Of Deligne's Conjecture from Spineless Cacti*

In this section, we sum up the rather technical results of the previous ones.

9.14. *The Action*

Using the cell decomposition 7.6.4 and the interpretation of $CC_*(\mathcal{C}act^1)$ as $\mathcal{T}_{bp}^{pp,nt}$, $CC_*(\mathcal{C}act^1)$ acts naturally on the Hochschild complex, with the signs being fixed by one of the schemes above.

9.15. *Deligne's Conjecture*

Notice that we have proven that $\mathcal{C}act$ is homotopy equivalent to the little discs operad[29] as well as to $\mathcal{C}act^1$ (see Corollary 7.5.1) and hence the cellular chains of $\mathcal{C}act^1$ give us a model of $D(2)$.

From our previous analysis:

Theorem 9.15.1. *Deligne's conjecture is true for the chain model of the little discs operad provided by $CC_*(\mathcal{C}act^1)$ and moreover $CH^*(A, A)$ is even a dg–algebra over $CC_*(\mathcal{C}act^1)$.*

Remark 9.15.2. This operad of spineless cacti and its cellular chains thus give a simple minimal topological description of the Gerstenhaber structure of the Hochschild complex.

Remark 9.15.3. As mentioned in §7.6.6 choosing a chain model $Chain(\mathcal{C}act)$ of $\mathcal{C}act$ by fixing a chain model for the scaling operad, we can let $Chain(\mathcal{C}act)$ by sending all cells of $Chain(\mathcal{C}act)$ which are not the product of a cell of $CC_*(\mathcal{C}act^1)$ and a zero dimensional cell of $\mathbb{R}^n_{>0}$ to zero and letting the cells of $CC_*(\mathcal{C}act^1)$ times a zero dimensional cell of $\mathbb{R}^n_{>0}$ i.e. cells of the type $\Delta(\tau) \times pt$ act via τ.

Remark 9.15.4. As noticed before, the fact that the chain operads of $\mathcal{C}act$ and $\mathcal{C}act^1$ possess the structure of a Gerstenhaber algebra up to homotopy, means that this structure exists also on $\mathcal{H}om_{CH^*(A,A)}$ up to homotopy and on $\mathcal{H}om_A$ on the nose – the latter being Gerstenhaber's original theorem[16]. It is interesting to note that our homotopies can be seen as a natural geometric depiction of the homotopies Gerstenhaber used.

10. The Relation of $\mathcal{C}act$ to Connes–Kreimer's Hopf Algebra and Generalizations

10.1. *Connes–Kreimer's Hopf Algebra as the Hopf Algebra of an Operad*

Connes and Kreimer[11] defined a Hopf algebra based in order to explain the procedure of renormalization in terms of the antipode of this Hopf algebra. This Hopf algebra was described directly, but also as the dual to the universal enveloping algebra of certain Lie algebra which is the Lie algebra associated to the free pre–Lie algebra in one generato[7]r.

Definition 10.1.1. By the $\mathbb{S}_n$ coinvariants of an operad which affords a direct sum, we mean $\bigoplus_{n\in\mathbb{N}}(\mathcal{O}(n))_{\mathbb{S}_n}$. Here $\bigoplus$ is the shorthand notation explained in §8.1.1.

In our notation we can rephrase the results[11,7] about this Hopf algebra as

Proposition 10.1.2. *The renormalization Hopf algebra of Connes and Kreimer H_{CK} is the Hopf algebra of $\mathbb{S}_n$ coinvariants of $H(\mathcal{T}^{r,fl})$ which agrees with the $\mathbb{S}_n$ coinvariants of $H(\mathcal{P}l)$.*

For the reader unfamiliar with this particular Hopf algebra this can also be a definition.

10.2. *The Top Dimensional Cells of Spineless Cacti and the Pre–Lie Operad*

We denote the top–dimensional cells of $CC_n(\mathcal{C}act(n))$ by $CC_n^{top}(n)$. These cells again form an operad and they are indexed by trees with black vertices of valence one (recall that means one input). Furthermore, the symmetric combinations of these cells which are the image of $\mathcal{T}^{r,fl}$ under the embedding *cppin* form an sub–operad.

From our previous description, one obtains[30]

Lemma 10.2.1. *In the orientation $\overline{Lab}$ for the top–dimensional cells for $\tau \in \mathcal{T}^{r,fl}(n), \tau'\mathcal{T}^{r,fl}(m)$*

$$cppin(\tau) \circ_i cppin(\tau') = \pm cppin(\tau \circ_i \tau')$$

Definition 10.2.2. Let $\mathcal{G}Pl$ be the quadratic operad in the category $Vect_{\mathbb{Z}}$ obtained as the quotient of free operad $\mathcal{F}$ generated by the regular representation of $\mathbb{S}_2$ by the quadratic relations defining a graded pre–Lie algebra, i.e. the quotient of F by the ideal R generated by the graded $\mathbb{S}_3$ submodule generated by the relation $r = (x_1 * x_2) * x_3 - x_1 * (x_2 * x_3) - (-1)^{|x_2||x_1|}((x_1 * x_3) * x_2 - x_1 * (x_3 * x_2))$. Where $\mathcal{F}$ and R are considered to be graded by given the degree $n-1$ to $\mathcal{F}(n)$.

Theorem 10.2.3. *The operad $CC_n^{top}(n)^{\mathbb{S}} \otimes k$ is isomorphic to the operad $\mathcal{G}Pl$ for graded pre–Lie algebras. Furthermore the shifted operad $(CC_n^{top} \otimes L^{\otimes E_w})^{\mathbb{S}}(n) \otimes k$ is isomorphic to the operad $\mathcal{P}l$ defining pre–Lie algebras.*

Here we have used the notation of tensor products indexed by arbitrary sets[14] and used the "shifted line" L which a the free object generated by an element

Proof. In view of Lemma 10.2.1 and the definition of the map *cppin* the second statement follows from the operadic isomorphism of $\mathcal{T}^{r,fl}$ and the pre–Lie operad $\mathcal{P}l$[7]. This also proves the first statement up to signs. The matching of the signs is guaranteed by the shift. The fact that the relation r holds and generates the respective ideal is explicitly verified in the presentation of Gerstenhaber structure on the chains of the arc operad[34] □

Remark 10.2.4. These statements also hold over $\mathbb{Z}$. Thus, from now on we will omit the explicit tensoring with k.

As an immediate consequence, we obtain:

Corollary 10.2.5. *The direct sum of an operad which affords a direct is an algebra over the symmetric top dimensional chains of the little disc operad of the chain model provided $CC_*(\mathcal{C}act^1)$ as well as over the shifted chains $(CC_n^{top})^{\mathbb{S}} \otimes L^{\otimes E_w}$.*

of degree -1. Recall that E_w is the set of white edges.

Corollary 10.2.6. *The pre–Lie algebra of $\mathbb{S}_n$ coinvariants $((CC_n^{top} \otimes L^{\otimes E_w})^{\mathbb{S}}(n))_{\mathbb{S}_n}$ is isomorphic to the free pre–Lie algebra in one generator.*

Likewise the graded pre–Lie algebra of $\mathbb{S}_n$ coinvariants $(CC_n^{top})^{\mathbb{S}}(n))_{\mathbb{S}_n}$ is isomorphic to the graded free pre–Lie algebra in one generator

Proof. The first statement follows from the references[7,11] and thus so does the second up to signs. These are guaranteed to agree by the shifting procedure and Theorem 10.2.3. □

10.3. *A Cell Interpretation of H_{CK}*

As shown in Theorem 10.2.3 there is cell and thus a topological interpretation of the pre–Lie operad and the graded inside $\mathcal{C}act^1$ and thus inside the $\mathcal{A}rc$ operad. In this interpretation H_{CK} is also the Hopf algebra of the coinvariants of the shifted chain operad $CC_*^{top}(\mathcal{C}act)^{\mathbb{S}} \otimes L^{\otimes E_w}$.

Corollary 10.3.1. *H_{CK} is equal the Hopf algebra of $\mathbb{S}_n$ coinvariants of the sub–operad of top–dimensional symmetric combinations of shifted cells $CC_*^{top}(\mathcal{C}act^1)^{\mathbb{S}} \otimes L^{\otimes E_w}$ of the shifted cellular chain operad of normalized spineless cacti $CC_*(\mathcal{C}act^1) \otimes L^{\otimes E_w}$.*

It is interesting to note that also the G and BV structures[34] are inside the symmetric (graded symmetric) combinations.

10.4. *Comments on Operads and H_{CK}*

We have shown that any operad is an algebra over the operad $\mathcal{T}^{r,fl}$ in a natural way and thus the Hopf algebra H_{CK} naturally appears in any context involving operads, such as Deligne's conjecture. We have furthermore shown that there is a topological incarnation of the insertion product, which is based on surfaces. In this setting, we have constructed a chain representation of the algebra H_{CK}. This links the algebra H_{CK} and its underlying bracket for instance to string topology.

Remark 10.4.1. We expect to obtain other interesting examples of such Hopf algebras by considering other tree operads.

10.5. *Operad Algebras and a Generalized Deligne Conjecture*

Definition 10.5.1. We define an operad algebra to be an operad $\mathcal{O}$ which affords a direct sum together with an element $\cup \in O(2)$ which is associative, i.e. define $a \cup b$ to be $(-1)^{|a|}(\cup \circ_1 a) \circ_{|a+1|} b$ then $(a \cup b) \cup c = a \cup (b \cup c)$, recall that $|a| = n-1$ if $a \in \mathcal{O}(n)$.

Definition 10.5.2. We formulate the generalized Deligne conjecture as the statement that the direct sum of any operad algebra which affords a direct sum is an algebra over the chains of the little discs operad in the sense that there is a map of differential operads of $CC_*(\mathcal{C}act^1)$ action as specified in §7.9.

Definition 10.5.3. For $f \in \mathcal{O}(m), g_i \in \mathcal{O}(n_i)$, we define the generalized brace operations

$$f\{g_1,\dots,g_n\} := \sum_{\substack{1 \le i_1 \le \cdots \le i_n \le m: \\ i_j + |g_j+1| \le i_{j+1}}} \pm(\cdots((f \circ_{i_1} g_1) \circ_{i_2} g_2) \circ_{i_3} \cdots) \circ_{i_n} g_n \tag{10.1}$$

where the sign is defined to be the same one as in equation (9.3)

Lemma 10.5.4. *There is an operadic action of $\mathcal{T}_{bp}^{pp,nt}$ of any operad algebra.*

Proof. We can view the bipartite tree as a flow chart. For the white vertices, we use the brace operations above and for black vertex with n incoming edges, we use the operation of applying $\cup$ $n-1$ times. Notice that the order in which we perform these operations does not matter, since we took $\cup$ to be associative. □

Definition-Proposition 10.5.5. Generalizing Gerstenhaber's[16] definition to an operad algebra, we define a differential on the direct sum by $\partial f = f \circ \cup - (-1)^{|f|} \cup \circ f$.

Proof. The fact that this is a differential follows from the calculations of Gerstenhaber[16]. □

10.6. *Differential on Trees with Tails*

Definition 10.6.1. For a tree τ with tails in $\mathcal{T}_{bp}^{pp}$ and vertex $v \in V_b \setminus \{v_{root}\}$ we define τ_v^+ to be the b/w tree obtained by adding a black vertex $b+$ and an edge $e^+ := \{b+, v\}$, if $|v| \neq 0$ and if $|v| = 0$, the tree obtained by adding two vertices $b+$ and b_{st} and two edges $e+ = \{b+, v\}$ and $e_{st} = \{b_{st}, v\}$ to τ.

We call a linear order $\prec'$ on τ_v^+ compatible with the order $\prec$ on τ if a) $e+ \prec' e_{st}$ if applicable and b) the order induced on τ by $\prec'$ by contracting e^+ and e_{st} (if applicable) coincides with $\prec$. We define E_{w-int} to be the white internal edges, i.e. white edges which are not leaves and set $E_{b-angle}$: $E(\tau) \setminus E_{w-int}$. For a linear order $\prec'$ on τ_v^+

$$sign(\prec') := (-1)^{|\{e|e \in E_{b-angle}, e \prec' v\}|}$$

and set

$$\partial_v(\tau) := \sum_{\text{compatible} \prec'} sign(\prec')(\tau_v^+, \prec')$$

we recall, that tail edges are considered to be black. Finally we define

$$\partial(\tau) := \sum_{v \in V_b \setminus \{v_{root}\}} \partial_v(\tau) \tag{10.2}$$

Remark 10.6.2. There is again a tree depiction for the operations of insertion an cup product. This is analogous to tree picture explained in 9.10 where we now replace functions by elements of the operad. The tree differential then describes the insertion of the new edges at all angles corresponding to the black vertices which amounts to inserting a $\cup$ product. Using this interpretation and the tree notation for the known calculations [16,25] it is straightforward to check that the tree differential (10.2) defined above agrees with the differential induced by the differential on the operad, which we defined in 10.5.5. Our differential also agrees with differential induced by the differential of Kontsevich and Soibelman[36] via st_∞.

Theorem 10.6.3. *The generalized Deligne conjecture holds.*

Proof. By the preceding Lemma 10.5.4, we have an operadic action of $\mathcal{T}_{bp}^{pp,nt}$ and thus an action of the chains $CC_*(\mathcal{C}act^1)$ which is a chain model for the little discs operad. The compatibility of the differentials follows directly from their definitions by a straightforward calculation as remarked above. □

10.7. *A Cyclic Version of Deligne's Conjecture*

We have shown that spineless cacti naturally act on the Hochschild cohomology of an associative algebra thereby providing a solution to Deligne's conjecture. Recently[31] we have generalize this fact to an action of a cell model of cacti on the Hochschild cohomology of an associative algebra which is isomorphic as a bi–algebra to its dual. This means that there is a BV structure on the Hochschild cochains up to homotopy and a BV structure inducing the Gerstenhaber structure on the Hochschild cohomology of such an algebra.

Our treatment is again of a more general nature. It is easy to abstract from it to a general setup of cyclic operad algebras.

11. Outlook and Speculations

11.1. *Operation on the Hochschild Complex of an A_∞ algebra*

There is a natural action of $\mathcal{T}^{pp,st,nt}_{b/w}$ on $CH^*(A,A)$[36] was constructed which allows to solve Deligne's conjecture. Here the black vertices stand for the higher multiplications μ_n of the A_∞ structure. The solution was then established by constructing a quasi–isomorphism of the free operad of $\mathcal{M}$ onto the operad of the Fulton–MacPherson compactification of the configurations of $\mathbb{R}^2$. This beautiful construction is however indirect, as one has to invert the quasi–isomorphism and furthermore the map to the Fulton–MacPherson compactification is rather involved and has considerably many choices. We would hope to find a direct interpretation of $\mathcal{T}^{pp,st,nt}_{b/w}$.

11.2. *A Putative Cell Decomposition*

In our situation an A_∞ version of Deligne's could be established, if we had a cellular decomposition of a suitable version of $Cact$ (e.g. $Cact^{\leq}$ where this is the space of cacti whose lobes have radius not greater than $2^{ht(v)}$) such that the cells are indexed by $\mathcal{T}^{pp,st,nt}_{b/w}$ and are given by

$$cell(\tau) := \prod_{v \in V_W} C_{|v|} \times \prod_{v \in V_B} K_{|v|} \tag{11.1}$$

where $C_{|v|}$ is the $|v|$–dimensional cyclohedron and $K_{|v|}$ is the $|v|$–dimensional Stasheff polytope or associahedron[40].

Let $Cell(\mathcal{T}^{pp,st,nt}_{b/w})$ be the CW complex glued from the cells $cell(\tau)$ using the natural differential δ for the cyclohedra and associahedra and let ∂ be the tree differential[36]. It is then straightforward to show that:

Proposition 11.2.1. *The differential ∂ and δ agree on $CC_*(Cell(\mathcal{T}^{pp,st,nt}_{b/w}))$*

This leads us to the conjecture

Conjecture 11.2.2. *There is a suitable suboperad of Cact which is quasi-isomorphic to Cact and whose cell decomposition is given by eq. (11.1).*

11.3. *Truncation of Simplices and Stasheff Polytopes*

A good candidate for the space for which the cell decomposition should be possible is the space $Cact^{\leq}$ of restricted cacti.

It is straightforward to see that this space is indeed a suboperad.

To find the cell decomposition, we should have a special realization of cyclohedra and associahedra. In particular, we need a realization of an n associahedron in the hyperplane $\sum_i x_i = 2^{n-1} \subset \mathbb{R}^{n-1}$ and the n cyclohedron in $\mathbb{R}^{2n+1}$.

The corners for the Associahedron K_n should be given by n tuples indexed by a binary tree and should be $2^{n-ht(v)}$ which are the weights obtained by using the homotopy associative operation $\cdot$ on $\mathcal{DA}rc$ using the bracketing given by a binary tree and reading off the total weights on the boundaries $\partial_i(F)$.

For $n = 2$ this is the point $(1,1)$ for $n = 3$ one can take the interval on the line through $(1,1,2)$ and $(2,1,1)$. For $n = 4$ the corners of the pentagon should be $(1,1,2,4)$, $(2,1,1,4)$, $(2,2,2,2)$, $(4,1,1,2)$ and $(4,2,1,1)$.

The cyclohedra should come from blowing up the n dimensional simplex Δ^n which corresponds to the τ_n, so that the tree boundary for the cyclohedra coincides with the geometric boundary of the operad.

Conjecture 11.3.1. *We conjecture that there is a truncation of simplices with the above corners that yields the above realizations of associahedra and cyclohedra.*

This has been checked for low dimensions, but we currently lack a general scheme. One step in this direction would be the construction of an explicit map of the compactification of the configuration space of n points on S^1 with one point fixed at the origin to the n-th cyclohedron.

11.4. *Relations to the Fulton–MacPherson Compactification*

Finally, we expect a relation of spineless Cacti or the $\mathcal{A}rc$ operad to the Fulton–MacPherson compactification of the configuration space of points in $\mathbb{R}^2$. The idea is to use a variant of polar coordinates and then to keep track of the collision speeds in the width of the bands. A thorough analysis of this fact should result in a positive answer to a conjecture made by Kontsevich and Soibelman[36]. This will be elaborated on elsewhere.

11.5. *Actions of $\mathcal{A}rc$*

Denote by $\mathcal{A}rc_{\#}$ the suboperad of $\mathcal{A}rc$ of surfaces without punctures whose image under *Loop* is a graph whose genus and number of cycles coincides with the genus of the surface and the number of its boundary components.

We also expect that $\mathcal{A}rc_{\#}$ operates on $CH^*(A, A)$, for an algebra A together with a non-degenerate invariant as a cyclic operad. This has two steps, first the non-trees and second higher genus. The operation of *Cacti* has recently been established[31].

Conjecture 11.5.1. *We conjecture that a suitable chain model of $\mathcal{A}rc_{\#}$ acts on CH^*.*

For the action of loop spaces there are several conjectures.

Conjecture 11.5.2. *We conjecture that of $\mathcal{A}rc_{\#}$ acts on LM, the loop space of a compact manifold M.*

If we consider all of $\mathcal{A}rc$ then we can obtain the surfaces whose loop has the wrong genus or number of cycles as images of the stabilization with respect to the genus operator[34]. The map *Loop* is not sensitive to this stabilization and thus if there is an action of $\mathcal{A}rc$ which factors through the map *Loop*, then one is essentially dealing with the stabilized moduli space. We consider the suboperad $\mathcal{A}rc^{C}$ of $\mathcal{A}rc$ given by surfaces without punctures whose $\mathcal{L}oop$ is connected.

The above considerations and the fact that the sequences[39] have an interpretation in terms of loops of arc families on higher genus surfaces leads us to conjecture:

Conjecture 11.5.3. *We conjecture that if the $\mathcal{A}rc^{C}$ operad acts on M by a map that factors through Loop then M has the homotopy type of an infinite loop space. In particular the stabilization of $\mathcal{A}rc^{C}$ has the homotopy type of an infinite loop space.*

In this conjecture, we can probably replace $\mathcal{A}rc^{C}$ by the sub-operad of Chinese trees[34].

Lastly,

Conjecture 11.5.4. *We conjecture that the Arc operad will act on LM by a combinatorial Gromov–Witten type setup.*

This we understand as follows. If we wish to concatenate loops in the free loop space LM that do not intersect, we have to be able to move the loops into the a position in which they do. Regarding the image of a surfaces with arcs in M whose boundaries $1, \dots, n$ are exactly the loops to be multiplied will give us a way to move and multiply them into the loop 0. The versions of the discussed Props yield straightforward generalizations.

With a suitable version of mapping spaces and (virtual) fundamental classes one could hope to construct combinatorial invariants, pulling back families of loops and integrating over the moduli space.

11.6. *Rankin–Cohen Brackets*

Recently[12] it was discovered that the Rankin–Cohen brackets can be realized inside the foliation Hopf Algebra introduced by Connes and Moscovici. Since these brackets have a conjectured form in terms of naturally grown trees and due to the relationship of $\mathcal{A}rc$ to moduli spaces we formulate:

Conjecture 11.6.1. *We conjecture that Rankin Cohen brackets are also realizable on Cact.*

11.7. *Open Ends and Questions*

It still remains to find out the exact relationship of cacti to chord diagrams, the dihedral algebra[23] and polylogarithms and higher zeta values. For the latter the clarification of the relationship to configuration spaces and cacti should be key as well as the relationship to moduli spaces.

We wish to conclude by remarking that the new feature of the Chinese tree operad is that the chord diagrams[29] are no longer planar. Cutting at zero, we obtain not only rainbow diagrams as for genus zero, but (for) high enough genus any trivalent diagram of the types depicted in figure 32 which are known from high energy physics and knot theory[2] and are making their appearance in biology in the form of folding problems for RNA.

References

1. M. Atiyah, Topological quantum field theories, *Inst. Hautes Études Sci. Publ. Math.* **68** (1988), 175–186.
2. D. Bar-Natan. *On the Vassiliev knot invariants.* Topology 34 (1995), 423–472.

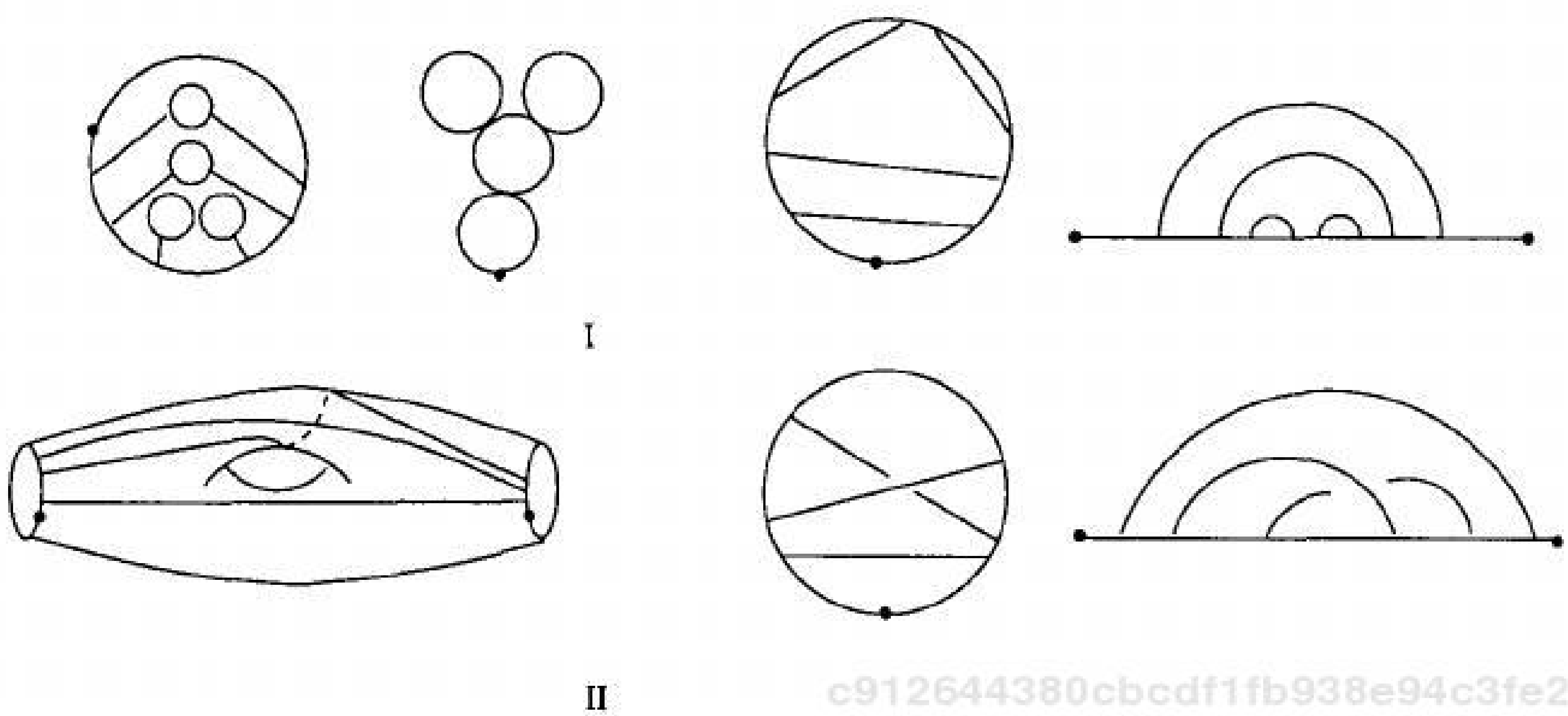


Figure 32. I. A genus one arc family, its cactus, cord diagram and rainbow diagram, II A genus one diagram giving a braid move

3. F. R. Cohen. The homology of C_{n+1} -spaces, $n \geq 0$. In The homology of iterated loop spaces., volume 533 of Lecture Notes in Mathematics. Springer, 1976.
4. F. R. Cohen. Artin's braid groups, classical homotopy theory, and sundry other curiosities. In Braids (Santa Cruz, CA, 1986), volume 78 of Contemp. Math., pages 167–206. Amer. Math. Soc., Providence, RI, 1988.
5. A. Cattaneo and G. Felder. *A path integral approach to the Kontsevich quantization formula.* Comm. Math. Phys. 212 (2000), no. 3, 591–611.
6. R. L. Cohen and V. Godin. *A polarized view of string topology.* Preprint-math.AT/0303003
7. F. Chapoton, M. Livernet. *Pre-Lie algebras and the rooted trees operad.* Internat. Math. Res. Notices 2001, no. 8, 395–408.
8. M. Chas and D. Sullivan. *String Topology.* Preprint math.GT/9911159
9. M. Chas and D. Sullivan. *Closed string operators in topology leading to Lie bialgebras and higher string algebra.* Preprint math.GT/0212358 .
10. R. L. Cohen and J.D.S. Jones *A homotopy theoretic realization of string topology* Math. Ann. 324 (2002), no. 4, 773–798.
11. A. Connes and D. Kreimer *Hopf Algebras, Renormalization and Noncommutative Geometry.* Commun. Math. Phys. 199 (1998) 203-242.
12. A. Connes and H. Moscovici. *Rankin-Cohen Brackets and the Hopf Algebra of Transverse Geometry.* Preprint math.QA/0304316 .
13. W. Chen and Y. Ruan *Orbifold Gromov-Witten Theory* Preprint math.AG/0103156.
14. P. Deligne.*Catégories tannakiennes.* The Grothendieck Festschrift, Vol. II, 111–195,
15. Z. Fiedorowicz. *The symmetric Bar Construction.* Preprint (available at http://www.math.ohio-state.edu/~fiedorow/) and
Z. Fiedorowicz. *Constructions of E_n operads.* Preprint math.AT/9808089,

1999.
16. M. Gerstenhaber. *The cohomology structure of an associative ring*, Ann. of Math. 78 (1963), 267-288.
17. E. Getzler. *Two-dimensional topological gravity and equivariant cohomology.* Comm. Math. Phys. 163 (1994), 473–489.
18. E. Getzler. *Operads and moduli spaces of genus 0 Riemann surfaces.* The moduli space of curves (Texel Island, 1994), 199–230, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.
19. E. Getzler. *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology.* Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), 65–78, Israel Math. Conf. Proc., 7, Bar-Ilan Univ., Ramat Gan, 1993.
20. V. Ginzburg and M. Kapranov. *Koszul duality for operads.* Duke Math. J. 76 (1994), 203–272.
21. E. Getzler. *Operads and moduli spaces of genus 0 Riemann surfaces.* The moduli space of curves (Texel Island, 1994), 199–230, Progr. Math., 129, Birkhuser Boston, Boston, MA, 1995.
22. E. Getzler and M.M. Kapranov. *Modular operads.* Compositio Math. 110 (1998), no. 1, 65–126.
23. A.B. Goncharov *Multiple zeta-values, Galois groups, and geometry of modular varieties.* Talk at the European Congress of Math., 2000. Preprint math.AG/0005069
24. Yu. I. Manin and A. B. Goncharov. *Multiple zeta-motives and moduli spaces* $M_{0,n}$ Preprint math.AG/0204102.
25. M. Gerstenhaber and A. A. Voronov. *Higher-order operations on the Hochschild complex.* Funktsional. Anal. i Prilozhen. 29 (1995), no. 1, 1–6, 96; translation in Funct. Anal. Appl. 29 (1995), no. 1, 1–5.
26. T. Jarvis, R. Kaufmann and T. Kimura, Pointed Admissible G-Covers and G-equivariant Cohomological Field Theories, Preprint math.AG/0302316.
27. R. Kaufmann. *Orbifolding Frobenius algebras.* math.AG/0107163, Internat. J. of Math. 14 (2003), 573-619
28. R. M. Kaufmann. *Second quantized Frobenius algebras.* To apperar in Comm. Math. Phys.
29. R. M. Kaufmann. *On several varieties of cacti and their relations.* Preprint MPI-2002-113, math.QA/0209131.
30. R. M. Kaufmann. *On Spineless Cacti, Deligne's Conjecture and Connes-Kreimer's Hopf Algebra..* Preprint. math.QA/0308005.
31. Kaufmann, Ralph M. *A proof of a cyclic version of Deligne's conjecture via Cacti.* Preprint, math.QA/0403340.
32. M. Kontsevich. *Operads and Motives in Deformation Quantization.* Lett.Math.Phys. 48 (1999) 35-72.
33. Kadeishvili, T. The structure of the A(∞)-algebra, and the Hochschild and Harrison cohomologies. Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 91 (1988).
34. R. M. Kaufmann, M. Livernet and R. B. Penner. *Arc Operads and Arc Algebras.* Geometry and Topology 7 (2003), 511-568.

35. R. M. Kaufmann, M. Livernet and R. B. Penner. *Homological arc operads.* In preparation.
36. M. Kontsevich and Y. Soibelman. *Deformations of algebras over operads and Deligne's conjecture.* Conférence Moshé Flato 1999, Vol. I (Dijon), 255–307, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000.
37. Yu. Manin. *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces.* AMS 1999.
38. Yu. Manin. *Gauge Field Theory and Complex Geometry.* Springer, Berlin–Heidelberg–New York, 1988.
39. J. E. McClure and J. H. Smith, Jeffrey H. *A solution of Deligne's Hochschild cohomology conjecture.* Recent progress in homotopy theory (Baltimore, MD, 2000), 153–193, Contemp. Math., 293, Amer. Math. Soc., Providence, RI, 2002.
40. M. Markl, S. Shnider and J. Stasheff. *Operads in algebra, topology and physics.* Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002. x+349 pp.
41. R. C. Penner. *The simplicial compactification of Riemann's moduli space.* Proceedings of the 37th Taniguchi Symposium, World Scientific (1996), 237-252.
42. R. C. Penner. *Decorated Teichmüller theory of bordered surfaces.* Preprint math.GT/0210326.
43. R. C. Penner. *Cell decomposition and compactification of Riemann's moduli space in decorated Teichmüller theory..* This volume.
44. R. C. Penner with J. L. Harer. *Combinatorics of Train Tracks*, Annals of Mathematical Studies 125, Princeton Univ. Press (1992); second printing (2001).
45. D. Sullivan. *Open and Closed String field theory interpreted in classical Algebraic Topology.* Preprint math.QA/0302332 .
46. S. Stolz and P. Teichner. *What is an elliptic object?.* Preprint.
47. P. Salvatore and N. Wahl. *Framed discs operads and the equivariant recognition principle.* Preprint math.AT/0106242
48. Tamarkin, D., Another proof of M. Kontsevich formality theorem, math/9803025. and Formality of Chain Operad of Small Squares, math/9809164.
49. A. A. Voronov *Notes on universal algebra.* Preprint math.QA/0111009
50. A. A. Voronov *Homotopy Gerstenhaber algebras.* Conf erence Moshé Flato 1999, Vol. II (Dijon), 307–331, Math. Phys. Stud., 22, Kluwer Acad. Publ., Dordrecht, 2000
51. B. Zwiebach. *Closed string field theory: quantum action and the Batalin-Vilkovisky master equation.* Nuclear Phys. B 390 (1993), no. 1, 33–152.