Lett Math Phys (2010) 94:165–195
DOI 10.1007/s11005-010-0427-z

Global Stringy Orbifold Cohomology, K-Theory and de Rham Theory

RALPH M. KAUFMANN
Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA.
e-mail: rkaufman@math.purdue.edu

Received: 28 July 2010 / Accepted: 3 September 2010
Published online: 22 September 2010 –

Abstract. There are two approaches to constructing stringy multiplications for global quotients. The first one is given by first pulling back and then pushing forward. The second one is given by first pushing forward and then pulling back. The first approach has been used to define a global stringy extension of the functors K_0 and K^{top} by Jarvis–Kaufmann–Kimura, A^* by Abramovich–Graber–Vistoli, and H^* by Chen–Ruan and Fantechi–Göttsche. The second approach has been applied by the author in the case of cyclic twisted sector and in particular for singularities with symmetries and for symmetric products. The second type of construction has also been discussed in the de Rham setting for Abelian quotients by Chen–Hu. We give a rigorous formulation of de Rham theory for any global quotient from both points of view. We also show that the pull–push formalism has a solution by the push–pull equations in the setting case of cyclic twisted sectors. In the general, not necessarily cyclic case, we introduce ring extensions and treat all the stringy extension of the functors mentioned above also from the second point of view. A first extension provides formal sections and a second extension fractional Euler classes. The formal sections allow us to give a pull–push solution while fractional Euler classes give a trivialization of the co-cycles of the pull–push formalism. The main tool is the formula for the obstruction bundle of Jarvis–Kaufmann–Kimura. This trivialization can be interpreted as defining the physics notion of twist fields. We end with an outlook on applications to singularities with symmetries aka. orbifold Landau–Ginzburg models.

Mathematics Subject Classification (2000). 14N35, 53D45, 81T45, 14F40, 14C35, 19E08.

Keywords. orbifolds, stringy geometry, K-theory, twist fields, orbifold field theory, de Rham theory, singularity theory, Landau–Ginzburg theory.

0. Introduction

For global quotients by finite group actions, there is by now a standard approach to constructing stringy products via first pulling back and then pushing forward [3,5,7,12]. We will call this construction the push–pull, which stands for push-forward after pulling back.

However, going back to [13,14], there is another mechanism that first pushes forward and then pulls back. We will call this the pull–push approach–read pull after pushing. This approach has been very successful for singularities [14,18] and for special cases of the group, for instance $G = \mathbb{S}_n$, see [16]. The advantage of this approach is that one is left with solving an algebraic co-cycle equation. In many cases this cocycle is unique up to normalized discrete torsion [14–18].

In fact, as we proved in [14,16] the solutions of the co-cycle equations are equivalent to the possible stringy multiplications if the twisted sectors are cyclic modules over the untwisted sector. In the Abelian case an adaption of this technique was discussed in [6]. The authors studied de Rham chains and presented arguments involving the idea of fractional Thom forms. Unfortunately, making strict sense of these arguments would involve dividing by nilpotent elements and the ideas are limited to the Abelian case. We give a rigorous setup for the de Rham case for any global quotient.

We are also able to give the mathematical definition of the notion of twist fields that is prevalent in the physics literature on orbifold conformal field theory.

For the reader's convenience, we review the general setup in Section 1. In Section 2 we treat the case of cyclic twisted sectors. Here, both approaches exist for all the geometric functors considered in [12]. We prove that the push–pull formula of [12] gives a solution for the pull–push formalism. The explicit co-cycle is given by a push-forward of the obstruction bundle. Of course, if we have one solution it can be twisted by discrete torsion [17]. The key in this situation is the existence of sections of the pull–back maps which allow us to prove the relevant theorems using only the projection formula. We go on to show that by adjoining fractional Euler classes the multiplication co-cycles become trivial. The Euler-classes are defined by adjoining roots much like the formal roots in the splitting principle.

In the general setting, see Section 3, we trivialize the multiplication by making a ring extension in two sets of variables. The first set again consists of fractional Euler classes. The second set consists of formal symbols of Euler classes of the negative normal bundles of the fixed point sets. The relations we impose turn these symbols into formally defined sections of the pull–back maps. The trivialization is in terms of the fractional Euler classes of the rational K-theory classes S_m appearing in the definition of the obstruction bundle [11]. These fractional Euler classes can hence be identified as the twist fields.

In Section 4, we give a rigorous treatment in the de Rham setting. Here, we work on the chain level and the push-forward is given by Thom pushforwards. All the formulas of the previous study hold at least up to homotopy, that is up to exact forms. Again we trivialize the co-cycles by adjoining fractional Thom-classes.

Finally, in Section 5, we axiomatize the setting of our calculations in terms of admissible functors and close with a discussion about possible applications to orbifold Landau–Ginzburg theories that is singularities with symmetries.

Conventions

We will use at least coefficients in $\mathbb{Q}$ if nothing else is stated. For some applications such as de Rham forms we will use $\mathbb{R}$ coefficients. All statements remain valid when passing to $\mathbb{C}$.

1. General Setup

We will work in the same setup as in the global part of [12]. That is we simultaneously treat two flavors of geometry, algebraic and differential. For the latter, we consider a stably almost complex manifold X with the action of a finite group G such that the stably almost complex bundle is G equivariant, while for the former X is taken to be a smooth projective variety.

In both situations for $m \in G$ we denote the fixed point set of m by X^m and let

$$I(X) = \amalg_{m\in G} X^m \tag{1.1}$$

be the inertia variety.

We let $\mathcal{F}$ be any of the functors $H^*, K_0, A^*, K^{\text{top}}$, that is cohomology, Grothendieck K_0, Chow ring or topological K-theory with $\mathbb{Q}$ coefficients, and define

$$\mathcal{F}_{\text{stringy}}(X, G) := \mathcal{F}(I(X)) = \bigoplus_{m\in G} \mathcal{F}(X^m) \tag{1.2}$$

additively.

If E is a bundle we set

$$\mathrm{Eu}_{\mathcal{F}}(E) = \begin{cases} c_{\text{top}}(E) & \text{if } \mathcal{F} = H^* \text{ or } A^* \\ \lambda_{-1}(E^*) & \text{if } \mathcal{F} = K_0 \text{ or } K^{\text{top}} \end{cases} \tag{1.3}$$

Notice that on bundles Eu is multiplicative. For general K-theory elements we set

$$\mathrm{Eu}_{\mathcal{F},t}(E) = \begin{cases} c_t(E) & \text{if } \mathcal{F} = H^* \text{ or } A^* \\ \lambda_t(E^*) & \text{if } \mathcal{F} = K_0 \text{ or } K^{\text{top}} \end{cases} \tag{1.4}$$

Remark 1.1. Notice $\mathrm{Eu}_{\mathcal{F},t}$ is *always* multiplicative and it is a power series that starts with 1 and hence is invertible in $\mathcal{F}(X)[[t]]$.

DEFINITION 1.2. For a positive element E—i.e., E can be represented by a bundle—with rank $r = \mathrm{r}k(E)$ we have that $\mathrm{Eu}_{\mathcal{F}}(E) = \mathrm{Eu}_{\mathcal{F},t}(E)|_{t=-1}$ for $\mathcal{F}$ either K_0 or K^{top} and $\mathrm{Eu}_{\mathcal{F}}(E) = \text{Coeff of } t^r \text{ in } [\mathrm{Eu}_{\mathcal{F},t}(E)]$ if $\mathcal{F}$ is A^* or H^*. To be able to deal with both situations, for E, r as above, we define

$$\mathrm{eval}_{\mathcal{F}|r}(\mathrm{Eu}_{\mathcal{F},t}(E)) = \begin{cases} \mathrm{Eu}_{\mathcal{F},t}(E)|_{t=-1} & \text{if } \mathcal{F} \text{ is } K_0 \text{ or } K^{\text{top}} \\ \text{Coeff of } t^r \text{ in } [\mathrm{Eu}_{\mathcal{F},t}(E)] & \text{if } \mathcal{F} \text{ is } A^* \text{ or } H^* \end{cases} \tag{1.5}$$

we then have $\mathrm{eval}_{\mathcal{F}|r}(\mathrm{Eu}_{\mathcal{F},t}(E)) = \mathrm{Eu}_{\mathcal{F}}(E)$.

Remark 1.3. Notice that for $\mathcal{F}$ as above and each subgroup $H \subset G$, $\mathcal{F}(X^H)$ is an algebra. We will call the internal product $\mathcal{F}(X^H) \otimes \mathcal{F}(X^H) \to \mathcal{F}(X^H)$ the *naïve product*. There is, however, a "stringy" product which preserves the G-grading. To define it, we recall some definitions from [12].

Notation 1.4. If $\mathcal{F}$ is fixed, we will just write Eu_t for $\mathrm{Eu}_{\mathcal{F},t}$ and Eu for $\mathrm{Eu}_{\mathcal{F}}$.

1.1. THE STRINGY PRODUCT VIA PUSH–PULL

For $m \in G$, we let X^m be the fixed point set of m, and for a triple $\mathbf{m} = (m_1, m_2, m_3)$ such that $\prod m_i = \mathbf{1}$ (where $\mathbf{1}$ is the identity of G) we let $X^{\mathbf{m}}$ be the common fixed point set, that is, the set fixed under the subgroup generated by them.

In this situation, recall the following definitions. Fix $m \in G$ and let $r = ord(m)$ be its order. Furthermore, let $W_{m,k}$ be the sub-bundle of $TX|_{X^m}$ on which m acts with character $\exp(2\pi i \frac{k}{r})$; then

$$S_m = \bigoplus_k \frac{k}{r} W_{m,k} \tag{1.6}$$

Notice this formula is invariant under stabilization.

We also wish to point out that using the identification $X^m = X^{m^{-1}}$

$$S_m \oplus (S_{m^{-1}}) = N_{X^m/X} \tag{1.7}$$

where for an embedding $X \to Y$ we will use the notation $N_{X/Y}$ for the normal bundle.

Recall from [12] that in such a situation there is a product on $\mathcal{F}(X, G)$ which is given by

$$v_{m_1} * v_{m_2} := \check{e}_{m_3 *}(e_1^*(v_{m_1}) e_2^*(v_{m_2}) \mathrm{Eu}(\mathcal{R}(\mathbf{m}))) \tag{1.8}$$

where the obstruction bundle $\mathcal{R}(\mathbf{m})$ can be defined by

$$\mathcal{R}(\mathbf{m}) = S_{m_1}|_{X^{\mathbf{m}}} \oplus S_{m_2}|_{X^{\mathbf{m}}} \oplus S_{m_3}|_{X^{\mathbf{m}}} \ominus N_{X^{\mathbf{m}}/X} \tag{1.9}$$

and the $e_i : X^{\mathbf{m}} \to X^{m_i}$ and $\check{e}_3 : X^{\mathbf{m}} \to X^{m_3^{-1}}$ are the inclusions. Notice, that as it is written $\mathcal{R}(\mathbf{m})$ only has to be an element of K-theory with rational coefficients, but is actually indeed represented by a bundle [12].

Remark 1.5. The first appearance of a push–pull formula was given in [5] in terms of a moduli space of maps. The product was for the G invariants, that is, for the H^* of the inertia orbifold (in the differential category of orbifolds) and is known as Chen–Ruan cohomology. In [7] the obstruction bundle was given using Galois covers establishing a product for H^* on the inertia variety level (this is the variety defined in Eq. (1.1)). This yields a G-Frobenius algebra as defined in [13,14], which is commonly referred to as the Fantechi–Göttsche ring. The invariants under the G actions reproduce the Chen–Ruan multiplication. In [11], we put this global structure back into a moduli space setting and proved the trace axiom. The multiplication on the Chow ring A^* for the inertia stack was defined in [3]. The representation of the obstruction bundle in terms of the S_m and hence the passing to the differentiable setting as well as the two flavors of K-theory stem from [12].

The following is the key diagram:

$$\begin{array}{ccc} & X & \\ i_1 \nearrow & \uparrow i_2 & \nwarrow \check{i}_3 \\ X^{m_1} & X^{m_2} & X^{m_3^{-1}} \\ e_1 \nwarrow & \uparrow e_2 & \nearrow \check{e}_3 \\ & X^{\mathbf{m}} & \end{array} \tag{1.10}$$

Here, we used the notation of [12], where $e_3: X^{\mathbf{m}} \to X^{m_3}$ and $i_3: X^{m_3} \to X$ are the inclusion, $\vee: I(X) \to I(X)$ is the involution which sends the component X^m to $X^{m^{-1}}$ using the identity map and $\check{i}_3 = i_3 \circ \vee$, $\check{e}_3 = \vee \circ e_3$. This is short-hand notation for the general notation of the inclusion maps $i_m: X^m \to X$, $\check{i}_m := i_m \circ \vee = i_{m^{-1}}$.

Notation 1.6. For $a \in \mathcal{F}(X^m), b \in \mathcal{F}(X^{m^{-1}}): \langle a, b\rangle = \int_{X^m} a \cdot \check{b}$ where $\int_{X^m}$ is the push-forward to a point and $\check{b} = \vee^* b$.

LEMMA 1.7. *Let* $\check{\mathbf{m}}$ *be the triple* $(m_2^{-1}, m_1^{-1}, m_3^{-1})$

$$N_{X^{\mathbf{m}}/X} = \mathcal{R}(\mathbf{m}) \oplus \mathcal{R}(\check{\mathbf{m}}) \oplus N_{X^{\mathbf{m}}/X^{m_1}} \oplus N_{X^{\mathbf{m}}/X^{m_2}} \oplus N_{X^{\mathbf{m}}/X^{m_3}} \tag{1.11}$$

Moreover,

$$\Gamma(\mathbf{m}) := \mathcal{R}(\mathbf{m}) \oplus N_{X^{\mathbf{m}}/X^{m_3}} = S_{m_1} \oplus S_{m_2} \ominus S_{m_3^{-1}} \tag{1.12}$$

Proof. This follows directly from (1.7)

$$\begin{aligned} &\mathcal{R}(\mathbf{m}) \oplus \mathcal{R}(\check{\mathbf{m}}) \oplus N_{X^{\mathbf{m}}/X^{m_1}} \oplus N_{X^{\mathbf{m}}/X^{m_2}} \oplus N_{X^{\mathbf{m}}/X^{m_3}} \\ &\quad = S_{m_1} \oplus S_{m_1^{-1}} \oplus S_{m_2} \oplus S_{m_2^{-1}} \oplus S_{m_3} \oplus S_{m_3^{-1}} \\ &\qquad \oplus N_{X^{\mathbf{m}}/X^{m_1}} \oplus N_{X^{\mathbf{m}}/X^{m_2}} \oplus N_{X^{\mathbf{m}}/X^{m_3}} \ominus 2N_{X^{\mathbf{m}}/X} \\ &\quad = N_{X^{m_1}}|_{X^{\mathbf{m}}} \oplus N_{X^{m_2}}|_{X^{\mathbf{m}}} \oplus N_{X^{m_3}}|_{X^{\mathbf{m}}} \oplus N_{X^{\mathbf{m}}/X^{m_1}} \oplus N_{X^{\mathbf{m}}/X^{m_2}} \oplus N_{X^{\mathbf{m}}/X^{m_3}} \\ &\qquad \ominus 2N_{X^{\mathbf{m}}} \\ &\quad = 3N_{X^{\mathbf{m}}/X} \ominus 2N_{X^{\mathbf{m}}/X} = N_{X^{\mathbf{m}}/X} \end{aligned}$$

□

We also define the bundle

$$\mathcal{S}(\mathbf{m}) := \mathcal{R}(\mathbf{m}) \oplus N_{X^{\mathbf{m}}/X} = e_1^*(S_{m_1}) \oplus e_2^*(S_{m_2}) \oplus e_3^*(S_{m_3}) \tag{1.13}$$

1.2. THE $\mathcal{F}(X)$ MODULE STRUCTURE AND AN ALTERNATIVE FORMULATION FOR THE PRODUCT

Notice that each $\mathcal{F}(X^m)$ is an $\mathcal{F}(X)$ module in two ways which coincide. First via the naïve product and pull back, i.e., $a \cdot v_m := i_m^*(a)v_m$ and secondly via the stringy multiplication $(a, v_m) \mapsto a * v_m$. Now using (1.7) it is straightforward to check that

$$a \cdot v_m = i_m^*(a)v_m = a * v_m \tag{1.14}$$

In the next two sections, we will give an alternate formulation of the product using the maps i_k in lieu of the maps e_k and $\mathcal{F}(X)$-module structure on each of the $\mathcal{F}(X^m)$. This construction first "pushes forward" by using sections of the pull back maps i_1^* and i_2^* and then pulls back along i_3.

First, in Section 2, we will construct such a product on $\mathcal{F}(I(X))$ for the functors $\mathcal{F} \in \{H^*, K_0, A^*, K^{\text{top}}\}$, in case that such sections exist. In Section 4 we construct the sections and the product in the de Rham setting without any additional assumptions, but for this we will need to pass to the chain level.

2. Pull–Push: The Cyclic Case

In this section, we will assume that sections exist. This implies that each $\mathcal{F}(X^m)$ is a cyclic $\mathcal{F}(X)$-module as we prove. In the cyclic case the multiplication $*$ corresponds to a relative cocycle $\gamma : G \times G \to \mathcal{F}(X)$ (in the sense of [14]) which we compute. Cyclic examples are, for instance, given by symmetric products $(X^{\times n}, \mathbb{S}_n)$, see [14,15] or by manifolds whose fixed loci are empty or points. In particular, Theorem 2.11 applied to symmetric products gives a new way to show the existence of the unique co-cycles in this situation first constructed in [16]. In light of [12] this gives the direct relation between the calculations of [7] and [16].

2.1. SECTIONS

DEFINITION 2.1. We say that $\mathcal{F}$ *admits sections* for (X, G) if for every $m \in G$ the inclusion map $i_m : X^m \to X$ the induced pull–back map $i_m^* : \mathcal{F}(X) \to \mathcal{F}(X^m)$ has a section $i_{ms} : \mathcal{F}(X^m) \to \mathcal{F}(X)$, that is, $i_m^* \circ i_{ms} = id : \mathcal{F}(X^m) \to \mathcal{F}(X^m)$.

We say $\mathcal{F}$ *admits sections for* (X, G) *to order two* if furthermore the maps e_i^* have sections. Sections of order two are called Γ *normalized* if $\check{e}_{3*}(\text{Eu}(\mathcal{R}(\mathbf{m})) = \check{e}_{3s}(\text{Eu}(\Gamma(\mathbf{m}))$. Sections of order two are called *normalized* if in addition $i_{3*}e_{3*}(\text{Eu}(\mathcal{R}(\mathbf{m}))) = i_{3s}e_{3s}(\text{Eu}(\mathcal{S}(\mathbf{m})))$. Here, e_j and i_j are the usual shorthand notation for e_{m_j} and i_{m_j}.

LEMMA 2.2. *If $\mathcal{F}$ admits sections for (X, G), then for all m and all $a \in \mathcal{F}(X^m)$ the element $i_{m*}(a)$ is divisible by $i_{m*}(1)$. This determines $i_{ms}(a)$ modulo the annihilator of $i_{m*}(1)$.*

Proof. Since i_{ms} is indeed a section

$$i_{m*}(ab) = i_{m*}(i_m^*(i_{ms}(a))b) = i_{ms}(a)i_{m*}(b) \tag{2.1}$$

and hence

$$i_{m*}(a) = i_{ms}(a)i_{m*}(1) \tag{2.2}$$

□

Remark 2.3. Notice that as sections are unique up to the kernel of i_j^*, respectively, e_j^*, and $e_i^*(e_{i*})(a) = a\mathrm{Eu}(N_{X^{m_i}/X^{\mathbf{m}}})$, Γ-normalization is always possible and similarly second-order normalization can always be achieved.

LEMMA 2.4. *If $\mathcal{F}$ admits sections for (X, G), then $\mathcal{F}(X^m)$ is a cyclic $\mathcal{F}(X)$ module, where the module structure for $v_m \in \mathcal{F}(X^m)$ is given by $a \cdot v_m := i^*(a)v_m$. Moreover, a cyclic generator for the $\mathcal{F}(X)$ module $\mathcal{F}(X^m)$ is the identity element 1_m for the naïve product on $\mathcal{F}(X^m)$.*

Proof. Using Eq. (1.14)

$$v_m = i_m^*(i_{ms}(v_m)) = i_{ms}(v_m) \cdot 1_m \tag{2.3}$$

□

LEMMA 2.5. *In the situation above, we have for all m and $a, b \in \mathcal{F}(X^m)$:*

$$i_{m*}(ab) = i_{ms}(a)i_{m*}(b) = i_{ms}(ab)i_{m*}(1_m) \tag{2.4}$$

and

$$i_m^*(i_{ms}(a)i_{ms}(b)) = i_m^*(i_{ms}(a))i_m^*(i_{ms}(b)) = ab = i_m^*(i_{ms}(ab)) \tag{2.5}$$

Proof. The first equation follows from the projection formula

$$i_{m*}(ab) = i_{m*}(i_m^*(i_{ms}(a))b) = i_{ms}(a)i_*(b)$$

the rest are straightforward. □

LEMMA 2.6. *Let* $\mathbf{m} = (m_1, m_2, m_3)$, *s.t.* $\prod_i m_i = \mathbf{1}$. *Using the notation of Section 1.1 let* $I_j = ker(i_j^*)$ *then for all* $a \in X^{\mathbf{m}}$

$$(I_1 + I_2)\check{\imath}_{3s}\check{e}_{3*}(a) \subset I_3 \tag{2.6}$$

Proof.

$$\check{\imath}_3^*((I_1 + I_2)\check{\imath}_{3s}\check{e}_{3*}(a)) = \check{\imath}_3^*(I_1 + I_2)\check{e}_{3*}(a)$$
$$= \check{e}_{3*}(\check{e}_3^*\check{\imath}_3^*(I_1 + I_2)a) = \check{e}_{3*}(e_1^* i_1^*(I_1) + e_2^* i_2^*(I_2)a) = 0$$

□

2.2. STRINGY MULTIPLICATION COCYCLES IN THE CYCLIC CASE

We fix the generators 1_m above. If $\mathcal{F}$ admits sections for (X, G) set

$$\gamma_{m_1,m_2} := i_{m_1 m_2 s}(1_{m_1} * 1_{m_2}) \in \mathcal{F}(X) \tag{2.7}$$

The product $*$ is determined by a collection of elements γ_{m_1,m_2} in the following sense:

LEMMA 2.7. *For all* $v_{m_1} \in F(X^{m_1})$, $w_{m_2} \in F(X^{m_2})$

$$v_{m_1} * w_{m_2} = i_{m_1 s}(v_{m_1}) i_{m_2 s}(w_{m_2}) \gamma_{m_1,m_2} 1_{m_1 m_2} \tag{2.8}$$

Proof. By Eqs. (1.14), (2.3), the fact that 1_m is an identity for the naïve multiplication, commutativity, and associativity, we obtain

$$v_{m_1} * w_{m_2} = (i_{m_1 s}(v_{m_1}) \cdot 1_{m_1}) * (i_{m_2 s}(w_{m_2}) \cdot 1_{m_2}) = i_{m_1 s}(v_{m_1}) i_{m_2 s}(w_{m_2}) \gamma_{m_1,m_2} 1_{m_1 m_2}$$

□

2.3. THE RECONSTRUCTION PROGRAM

The above considerations are a special case of what is called the reconstruction program in [13,14]. This program aims to classify all possible G-Frobenius algebra structures on a given G-graded vector space which has some other given additional data (such as a G action) and assumptions, e.g., the graded pieces are Frobenius algebras; see *loc. cit.* for full details. In the cyclic case it is proved in [13,14] that such a multiplication is given by a relative co-cycle $\gamma_{m_1,m_2}, m_1, m_2 \in G$ taking values in the Frobenius algebra corresponding to the piece graded by the identity element of G. What this means in particular is that vice versa given a collection γ_{m_1,m_2} satisfying certain properties (given in [13,14]), the multiplication *defined* by (2.8) will be associative and braided commutative, i.e., yield a G-Frobenius algebra. One requirement for the multiplication to be well defined is that

$$(I_1+I_2)\gamma_{m_1m_2}\subset I_3 \tag{2.9}$$

Using the push–pull version of the product, we can find a particular solution to the reconstruction program by translating it to a pull–push formula. Mathematically this defines a stringy multiplication and physically this corresponds to fixing the three-point functions of the twist fields. For more on twist field, see Section 3 below. *A priori* it could happen that there is no solution at all. *A posteriori* in the current situation, this will not be the case as the push–pull formalism produces a solution. Here the G graded space is simply $\mathcal{F}(I(X))$.

Another interesting point that the reconstruction program addresses is that there could possibly be more solutions. First, given a solution, there is always the possibility to twist by discrete torsion [17]. But in principle it is possible that there are solutions that are not related to each other by a twist. In some cases one can prove that this is, however, not the case. This happens, for instance, in the case of symmetric products [16].

THEOREM 2.8. *Assume $\mathcal{F}$ admits sections for (X,G). Let $*$ be the product defined in (1.8), and*

$$\gamma_{m_1,m_2}=\check{i}_{3s}\check{e}_{3*}(\mathrm{Eu}(\mathcal{R}(\mathbf{m}))) \tag{2.10}$$

then,

$$v_{m_1}*v_{m_2}=\check{i}_3^*[i_{1s}(v_{m_1})i_{2s}(v_{m_2})\gamma_{m_1,m_2}] \tag{2.11}$$

Proof. Using the projection formula, the defining equation for the sections $i_j^*\circ i_{js}=id$, and the fact that if $\iota: X^{\mathbf{m}}\to X$ is the inclusion, then $e_k^*\circ i_k^*=\iota^*=\check{e}_3^*\circ\check{i}_3^*$

$$\begin{aligned}\check{e}_{3*}[e_1^*(v_1)e_2^*(v_2)\mathrm{Eu}(\mathcal{R}(\mathbf{m}))]&=\check{e}_{3*}[e_1^*i_1^*i_{1s}(v_1)e_2^*i_2^*i_{2s}(v_2)\mathrm{Eu}(\mathcal{R}(\mathbf{m}))]\\&=\check{e}_{3*}[\check{e}_3^*\check{i}_3^*i_{1s}(v_1)\check{e}_3^*\check{i}_3^*i_{2s}(v_2)\mathrm{Eu}(\mathcal{R}(\mathbf{m}))]\\&=\check{i}_3^*[i_{1s}(v_{m_1})i_{2s}(v_{m_2})]\check{e}_{3*}(\mathrm{Eu}(\mathcal{R}(\mathbf{m})))]\\&=\check{i}_3^*[i_{1s}(v_{m_1})i_{2s}(v_{m_2})\check{i}_{3s}\check{e}_{3*}(\mathrm{Eu}(\mathcal{R}(\mathbf{m})))]\end{aligned}$$

□

Remark 2.9. Note that in view of Lemma 2.6, indeed (2.9) holds. We can further decompose the cocycles γ by passing to solutions in $\mathcal{F}(X,G)[[t]]$. This form suggests that the cocycles are trivial when passing to a ring extension.

PROPOSITION 2.10. *Assume $\mathcal{F}$ admits sections for (X, G). Let $*$ be the product defined in (1.8), set $r = \mathrm{r}k(\mathcal{R}(\mathbf{m}))$ and*

$$\begin{aligned}\gamma_{m_1,m_2}(t) &= i_{1s}(\mathrm{Eu}_t(S_{m_1}))i_{2s}(\mathrm{Eu}_t(S_{m_2}))i_{3s}(\mathrm{Eu}_t(S_{m_3})e_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})))\\ &= i_{1s}(\mathrm{Eu}_t(S_{m_1}))i_{2s}(\mathrm{Eu}_t(S_{m_2}))\check{i}_{3s}(\mathrm{Eu}_t(\ominus S_{m_3^{-1}})\check{e}_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X^{m_3}})))\end{aligned} \tag{2.12}$$

then,

$$\begin{aligned}&v_{m_1} * v_{m_2}\\ &\quad= eval_{\mathcal{F}|r}\left\{\check{i}_3^*[i_{1s}(v_{m_1})i_{2s}(v_{m_2})\gamma_{m_1,m_2}(t)]\right\}\\ &\quad= eval_{\mathcal{F}|r}\left\{\check{i}_3^*[i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))\right.\\ &\qquad\left.\times i_{3s}(\mathrm{Eu}_t(S_{m_3})e_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})))]\right\}\end{aligned} \tag{2.13}$$

Proof. Again using the projection formula, the defining equation for the sections $i_j^* \circ i_{js} = id$, and the fact that $e_k^* \circ i_k^* = \check{e}_3^* \circ \check{i}_3^*$

$$\begin{aligned}&\check{e}_{3*}[e_1^*(v_{m_1})e_2^*(v_{m_2})\mathrm{Eu}_t(S_{m_1}|_{X^{\mathbf{m}}} \oplus S_{m_2}|_{X^{\mathbf{m}}} \oplus S_{m_3}|_{X^{\mathbf{m}}} \ominus N_{X^{\mathbf{m}}/X})]\\ &\quad= \check{e}_{3*}[e_1^*(i_1^*(i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))))e_2^*(i_2^*(i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))))\\ &\qquad\times e_3^*(i_3^*(i_{3s}(\mathrm{Eu}_t(S_{m_3}))))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})]\\ &\quad= \check{e}_{3*}[\check{e}_3^*(\check{i}_3^*(i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))))\\ &\qquad\times \check{e}_3^*(\check{i}_3^*(i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))))\check{e}_3^*(\check{i}_3^*(i_{3s}(\mathrm{Eu}_t(S_{m_3}))))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})]\\ &\quad= \check{i}_3^*[i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))i_{3s}(\mathrm{Eu}_t(S_{m_3}))\check{i}_{3s}(\check{e}_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})))]\\ &\quad= \check{i}_3^*[i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))i_{3s}((\mathrm{Eu}_t(S_{m_3}))e_{3*}(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}}/X)))]\end{aligned} \tag{2.14}$$

So that taking the coefficient of t^r with $r = \mathrm{r}k(\mathcal{R}(\mathbf{m}))$ we obtain the second claimed equality. For the first equality we can use the fact (2.5)

$$\begin{aligned}&\check{e}_{3*}[e_1^*(v_{m_1})e_2^*(v_{m_2})\mathrm{Eu}_t(S_{m_1}|_{X^{\mathbf{m}}} \oplus S_{m_2}|_{X^{\mathbf{m}}} \oplus S_{m_3}|_{X^{\mathbf{m}}} \ominus N_{X^{\mathbf{m}}/X})]\\ &\quad= \check{e}_{3*}[e_1^*(i_1^*(i_{1s}(v_{m_1}\mathrm{Eu}_t(S_{m_1}))))e_2^*(i_2^*(i_{2s}(v_{m_2}\mathrm{Eu}_t(S_{m_2}))))\\ &\qquad\times e_3^*(i_3^*(i_{3s}(\mathrm{Eu}_t(S_{m_3}))))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})]\\ &\quad= \check{e}_{3*}[e_1^*(i_1^*(i_{1s}(v_{m_1})i_{1s}(\mathrm{Eu}_t(S_{m_1}))))e_2^*(i_2^*(i_{2s}(v_{m_2})i_{2s}(\mathrm{Eu}_t(S_{m_2}))))\\ &\qquad\times e_3^*(i_3^*(i_{3s}(\mathrm{Eu}_t(S_{m_3}))))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})]\end{aligned} \tag{2.15}$$

and proceed as above. Finally, for (2.12), we notice that $N_{X^{\mathbf{m}}/X} = N_{X^{\mathbf{m}}/X^{m_3}} \oplus N_{X^{m_3}/X}|_{X^{\mathbf{m}}}$ and use (1.7). □

Using these calculations we can show that the $*$-product from the push–pull formalism, indeed, gives rise to a cocycle γ that allows the product to be written in a pull–push formula and moreover, gives the particular form of these cocycles.

THEOREM 2.11. *Let* $\mathcal{F} \in \{A^*, H^*, K_0, K^*_{\text{top}}\}$ *and assume that* $\mathcal{F}$ *admits sections for* (X, G)*; then the Eq.* (2.13) *solves the re-construction program of* [14] *with the co-cycles*

$$\gamma_{m_1,m_2} = \check{\imath}_{3s}\check{e}_{3*}\text{Eu}(\mathcal{R}(\mathbf{m})) = eval_{\mathcal{F}|r}\gamma_{m_1,m_2}(t)$$

Furthermore, if $\mathcal{F}$ *admits sections for* (X, G) *to order two, we have the following alternative representation:*

$$\gamma_{m_1m_2} = \check{\imath}_{3s}(\check{e}_{3s}[\text{Eu}(e_1^*(S_{m_1}) \oplus e_2^*(S_{m_2}) \ominus \check{e}_3^*(S_{m_1m_2}) \ominus N_{X^{\mathbf{m}}/X^{m_3}})]e_{3*}(1)) \tag{2.16}$$

Finally, if $\mathcal{F}$ *admits sections for* (X, G) *to order two and these sections are* Γ*-normalized, we have*

$$\gamma_{m_1m_2} = \check{\imath}_{3s}\check{e}_{3s}[\text{Eu}(e_1^*(S_{m_1}) \oplus e_2^*(S_{m_2}) \ominus \check{e}_3^*(S_{m_1m_2}))] = \check{\imath}_{3s}\check{e}_{3s}(\text{Eu}(\Gamma(\mathbf{m}))) \tag{2.17}$$

Proof. This follows from the two propositions above and a direct calculation. *A forteriori*, since the product $*$ is well defined and associative, the formulas are independent of the choice of lift and the $\gamma_{m_1,m_2} := \text{eval}_{\mathcal{F}|r}\gamma_{m_1,m_2}(t)$ are, indeed, co-cycles and section independent co-cycles in the sense of [14]. This can independently be checked by a direct computation using Lemma 2.6.

We use the relations $N_{X^{\mathbf{m}}/X^{m_3}} = N_{X^{m_3}/X} \oplus N_{X^{\mathbf{m}}/X^{m_3}}$ and Eq. (1.7) for the third equality and for the last statement, we used the definition of Γ normalized sections. □

3. Twist Fields: Trivializing the Cocycles

In this section, we will construct a ring extension, in which the cocycles γ can be trivialized. The ring contains the so-called fractional Euler classes. In particular, this allows us to identify the fractional Euler classes of the K-theory classes S_m of [12] as the twist fields that are used in physics.

In the de Rham case, we can represent theses classes by fractional Thom classes, see Section 4.

3.1. MOTIVATION

In this section we discuss the motivation and heuristics of our constructions which are carried out rigorously in the following paragraphs. In the physics literature, correlation functions for orbifold models are described using the so-called twist fields. There is one twist field σ_m for each twist by a group element m. In the mathematical formalism one representation of these fields would be given by some elements σ_m which lie in a ring extension $\mathcal{F}(X^m)$, such that the three-point functions in this extended ring satisfy

$$\langle v_{m_1}\sigma_{m_1}, v_{m_2}\sigma_{m_2}, v_{m_3}\sigma_{m_3}\rangle = \langle v_{m_1} * v_{m_2}, v_{m_3}\rangle \tag{3.1}$$

where the left-hand side should be suitably interpreted. One such interpretation is given in Definition 3.4, see also Remark 3.5. We will construct twist fields in the presence of sections i_{ms} and realize the σ_m as elements of a ring extension of $\mathcal{F}(X^m)$ roots of Euler classes. In Section 4, we will realize the twist fields as fractional Thom forms. If there are no sections, one can formally add them by using ring extensions. This is another way of interpreting the twist fields in the general case.

If one wishes to look at the multiplication directly, instead of just the three-point functions, one has to "divide" or "strip off" the twist field σ_{m_3}. One way to do this is to introduce a new unknown term in the pull–push formula which is an "inverse twist field" $\tilde{\sigma}_m$.

$$\check{e}_{3*}(e_1^*(v_{m_1})e_2^*(v_{m_2})\mathrm{Eu}(\mathcal{R}(\mathbf{m}))) =: \check{i}_3^*(i_{1*}(v_{m_1}\sigma_{m_1})i_{2*}(v_{m_2}\sigma_{m_2})\check{i}_{3*}(\tilde{\sigma}_{m_3})) \tag{3.2}$$

For a rigorous interpretation using power series, see Sections 3.3 and 3.4.

If $1_m = i_m^*(1)$ is the unit of $\mathcal{F}(X^m)$, then this formula applied to $1_m * 1_{m^{-1}} = e_{3*}(1_m 1_m) = i_{m*}(1)$ implies

$$i_{m*}(\sigma_m)\check{i}_{m*}(\sigma_{m^{-1}}) = i_{m*}(1) \tag{3.3}$$

while applying it to $1 * 1_m = 1_m$ we obtain

$$i_m^*(i_{m*}(\sigma_m)\check{i}_{m*}(\tilde{\sigma}_m)) = 1_m = i_m^*(1) \tag{3.4}$$

which shows the need for the "inverse twists" $\tilde{\sigma}$.

If we interpret the l.h.s. of (3.1), that is the three-point functions, as an integral over push forwards and express the right-hand side of (3.1) using (3.2), the equation transforms to

$$\int i_{1*}(v_{m_1}\sigma_{m_1})i_{2*}(v_{m_2}\sigma_{m_2})i_{3*}(v_{m_3}\sigma_{m_3}) = \int i_{1*}(v_{m_1}\sigma_{m_1})i_{2*}(v_{m_2}\sigma_{m_2})\check{i}_{3*}(\tilde{\sigma}_{m_3})i_{3*}(v_{m_3}) \tag{3.5}$$

where for the r.h.s. we used that i_{3*} and i_3^* are adjoint. Supposing that these morphisms are still adjoint when extended to twist fields, one obtains:

$$\int i_3^*[i_{1*}(v_{m_1}\sigma_{m_1})i_{2*}(v_{m_2}\sigma_{m_2})]v_{m_3}\sigma_{m_3} = \int i_3^*[i_{1*}(v_{m_1}\sigma_{m_1})i_{2*}(v_{m_2}\sigma_{m_2})]v_{m_3}i_3^*(\check{i}_{3*}(\tilde{\sigma}_{m_3})) \tag{3.6}$$

A stronger version implying the above equation is

$$\sigma_m = i_m^*(\check{i}_{m*}(\tilde{\sigma}_m)) = i_m^* i_{m*}(\vee^*(\tilde{\sigma}_m)) = \vee^*(\tilde{\sigma}_m)\mathrm{Eu}(N_{X^m/X}) \tag{3.7}$$

The Eq. (3.7) is, indeed, stronger, since the postulated equality only needs to hold inside the three-point functions (3.5). Again, there is a solution in a formal power series, see Eq. (3.30).

3.2. TRIVIALIZING BY ADJOINING FRACTIONAL EULER CLASSES

In this subsection, we suppose that $\mathcal{F}$ admits sections for (X, G).

3.2.1. *Positive fractional Euler-Classes*

We construct a ring extension of $\mathcal{F}(X, G)$ which contains fractional Euler classes. This will be a construction in several steps.

STEP 1: ADJOINING FRACTIONAL EULER CLASSES.

For each of the rings $R_m := \mathcal{F}(X^m)$ we will adjoin fractional Euler classes corresponding to the S_m. This can be done by a general procedure. By the splitting principle [8,10], given a set of bundles, we can pass to a splitting cover, where these bundles split. This also yields a ring extension of R_m which contains all the Chern classes of the virtual line bundles, in particular the Euler classes.

In our case this set of bundles on X^m is given by the bundles $N_{X^m/X}$ or equivalently by the isotypical components $W_{m,k}$. We will call the resulting ring extensions R^s_m.

In general, we adjoin rth roots as follows: Let $\mathscr{L}$ be a line bundle, e.g., one obtained from the splitting principle, $u = \mathrm{Eu}(\mathscr{L})$ and R be a ring that contains u. We adjoin rth roots to R by passing to $R' = R[w]/(w^r - u)$.

When extending R^s_m the line bundles $\mathscr{L}$ for which we adjoin roots are enumerated as $\mathscr{L}_{m,k,l}$, where for fixed m and k the $\mathscr{L}_{m,k,l}$ are the line bundles that split the bundles $W_{m,k}$. We start with R^s_m and successively adjoin all $|m|$th roots of the various $\mathrm{Eu}(\mathscr{L}_{m,k,l})$ and at each step denote by $w_{m,k,l}$ a generator of that extension. Let the resulting ring be called R^{sr}_m where s, r stand for split and roots.

The original $\mathcal{F}(X^m)$ is a subring of R^{sr}_m and hence, we can read off formulae on this subring analogously to the procedure used in the splitting principle. For this one just uses the Galois group of both of the extensions, splitting and roots. Note that at the end of the day, we are working over $\mathbb{Q}$ and hence by Artin's theorem, we actually only have to check the invariance under the cyclic subgroups of the cyclic extensions and the symmetric group invariance from the splitting.

We furthermore extend Eu to $\mathfrak{Eu}$ which is defined on the monoid of isomorphism classes of vector bundles on X^m adjoined elements $\frac{1}{|m|}\mathscr{L}_{m,k,l}$ which satisfy $m\left[\frac{1}{|m|}\mathscr{L}_{m,k,l}\right] = \mathscr{L}_{m,k,l}$ by setting $\mathrm{Eu}\left(\frac{1}{|m|}\mathscr{L}_{m,k,l}\right) := \mathfrak{Eu}\left(\frac{1}{|m|}\mathscr{L}_{m,k,l}\right) := w_{m,k,l}$ and extending the property of Eu as a map of monoids, between the additive structure on vector bundles and the multiplicative structure in the recipient ring of Eu.

By definition of $\mathfrak{Eu}$:

$$\mathfrak{Eu}(x \oplus y) = \mathfrak{Eu}(x)\mathfrak{Eu}(y) \tag{3.8}$$

and *if* $x + y = E$ with E a bundle

$$\mathfrak{Eu}(x)\mathfrak{Eu}(y) = \mathrm{Eu}(E) \tag{3.9}$$

which is guaranteed by the choice extensions.

Notice for the equation $x \oplus y = E$ to hold both sides of the equation have to be invariant under the Galois group. We will use this equation for $\mathcal{R}(\mathbf{m}),\Gamma(\mathbf{m})$ and $\mathcal{S}(\mathbf{m})$ where we know this to be true, see Section 1 and [12].

In this notation we get

$$\mathfrak{Eu}(S_m) := \prod_{(k,l):k\neq 0} w_{m,k,l}^k \tag{3.10}$$

We also extend the maps $\vee^*$ in the obvious fashion. Recall that $\vee$ in components is just the identity map $\vee : X^m \to X^{m^{-1}} = X^m$.

STEP 2: EXTENDING THE PULL BACKS e_i^*.

Let $e_i : X^{\mathbf{m}} \to X^{m_i}$ be any of the inclusions. Parallel to step 1, we first go to a splitting cover which splits all bundles $e_i^*(W_{m_i,k})$. Let the resulting ring extension of $R_{\mathbf{m}} := \mathcal{F}(X^{\mathbf{m}})$ be called $R^s_{\mathbf{m}}$. Now the relevant set of virtual line bundles is the set of the $e_i^*(\mathscr{L}_{m_i,k,l})$. We then, again as in step 1, adjoin the roots of the Euler classes bundles $e_i^*(\mathscr{L}_{m_i,k,l})$ to the rings $R^s_{\mathbf{m}}$ to obtain rings $R^{sr}_{\mathbf{m}}$. Again, we extend the monoid of bundles on $X^{\mathbf{m}}$ by elements $\frac{1}{|m_i|}e_i^*(\mathscr{L}_{m_i,k,l})$. Furthermore, we extend the maps e_i^* by defining

$$e_i^*\left(\frac{1}{|m_i|}\mathscr{L}_{m_i,k,l}\right) = \frac{1}{|m_i|}e_i^*(\mathscr{L}_{m_i,k,l}) \tag{3.11}$$

on the extended monoid of bundles and set

$$e_i^*\left(\mathfrak{Eu}\left(\frac{1}{|m_i|}\mathscr{L}_{m_i,k,l}\right)\right) = \mathfrak{Eu}\left(e_i^*\left(\frac{1}{|m_i|}\mathscr{L}_{m_i,k,l}\right)\right) \tag{3.12}$$

as a map $e_i^* : R^{sr}_{m_i} \to R^{sr}_{\mathbf{m}}$. This also guarantees the compatibility of e_i^* with $\mathfrak{Eu}$.

Again the maps for $\check{e}_i^*$ follow automatically.

STEP 3: EXTENDING THE SECTIONS e_{is}, THE PULL- BACKS e_i^* AND THE PUSH- FORWARDS e_{i*}.

In order to extend the section e_{is}, we have to enlarge the rings R^{sr}_m to R^{sar}_m (split all roots) by adjoining $|m_j|$th roots of the elements $e_{is}(\mathrm{Eu}(e_j^*(\mathscr{L}_{m_j,k,l})))$, for $i \neq j$ and fix a generator $e_{is}(\mathrm{Eu}(e_j^*(\mathscr{L}_{m_j,k,l})))^{\frac{1}{m_j}}$. After each such an extension, if it is non-trivial, we recursively extend e_i^* as a ring homomorphism by setting

$$e_i^*\left(e_{is}(\mathrm{Eu}(e_j^*(\mathscr{L}_{m_j,k,l})))^{\frac{1}{|m_i|}}\right) := \mathfrak{Eu}\left(\frac{1}{|m_j|}e_j^*(\mathscr{L}_{m_j,k,l})\right) \tag{3.13}$$

We now extend the map e_{is} as follows; We fix a sequence of extensions of Step 2 and define the maps step by step. Let $R^s_{\mathbf{m}}$ be the ring of the splitting principle in which all the line bundles $e_j^*(\mathscr{L}_{m_j,k,l})$ split and fix an order of non-trivial extensions $R^s_{\mathbf{m}} \subset R^1_{\mathbf{m}} \subset \cdots \subset R^p_{\mathbf{m}} \subset R^{\mathrm{sar}}_{\mathbf{m}}$ such that each extension is of the form $R^{q+1}_{\mathbf{m}} \simeq R^q_{\mathbf{m}}[u]/(u^{|m_j|} - e_j^*(\mathscr{L}_{m_j,k,l}))$.

In the chosen order of extensions of $R_{\mathbf{m}}$, if the extension of $R^q_{\mathbf{m}}$ is by m_ith roots of $\mathrm{Eu}(e_i^*(\mathscr{L}_{m_i,k,l}))$, we fix a $\mathbb{Q}$-basis $a_\alpha\left(\mathfrak{Eu}\left(\frac{n_\alpha}{m_i}e_i^*(\mathscr{L}_{m_i,k,l})\right)\right)$ of $R^{q+1}_{\mathbf{m}}$ with the $a_\alpha \in R^q_{\mathbf{m}}$

$$e_{is}\left(a_\alpha \mathfrak{Eu}\left(\frac{n_\alpha}{|m_i|}e_i^*(\mathscr{L}_{m_i,k,l})\right) := e_{is}(a_\alpha)\mathfrak{Eu}\left(\frac{n_\alpha}{m_i}\mathscr{L}_{m_i,k,l}\right)\right). \tag{3.14}$$

And for $i \neq j$ if, in the given order of extensions, the extension is by $\mathfrak{Eu}\left(\frac{1}{|m_j|}e_j^*(\mathscr{L}_{m_j,k,l})\right)$ we again fix a $\mathbb{Q}$-basis $a_\alpha(\mathrm{Eu}(e_j^*(\mathscr{L}_{m_j,k,l})))^{\frac{n_\alpha}{m_j}}$ and set

$$e_{is}(a_\alpha \mathfrak{Eu}\left(\frac{n_\alpha}{|m_j|}e_j^*(\mathscr{L}_{m_j,k,l})\right) := e_{is}(a_\alpha)e_{is}(\mathrm{Eu}(e_j^*(\mathscr{L}_{m_j,k,l})))^{\frac{n_\alpha}{m_j}} \tag{3.15}$$

We extend the push-forwards e_{i*} by

$$e_{i*}(x) := e_{is}(x)e_{i*}(1) \tag{3.16}$$

and again extend the constructions to $\check{e}_i$ in the obvious way.

STEP 4: EXTENDING THE SECTION i_{js}, THE PUSH- FORWARDS i_{j*} AND THE PULL- BACKS i_j^*.

Let $i_j : X^{m_j} \to X$ be the inclusions. We now enlarge the ring $R = \mathcal{F}(X)$ to $R^{\mathrm{sar}} = \mathcal{F}(X)$ by adjoining $|m_j|$th roots of $i_{js}(\mathrm{Eu}(\mathscr{L}_{m_j,k,l}))$. Again choose primitive roots $i_{js}(\mathrm{Eu}(\mathscr{L}_{m_j,k,l}))^{\frac{1}{|m_j|}}$.

For non-trivial extensions, we define

$$\begin{aligned} i_j^*\left(i_{js}(\mathrm{Eu}(\mathscr{L}_{m_j,k,l}))^{\frac{1}{|m_j|}}\right) &= \mathfrak{Eu}\left(\frac{1}{|m_j|}\mathscr{L}_{m_j,k,i}\right) \\ i_j^*\left(i_{j's}(\mathrm{Eu}(\mathscr{L}_{m_{j'},k,l}))^{\frac{1}{|m_{j'}|}}\right) &= e_{js}\left(\mathfrak{Eu}\left(\frac{1}{|m_{j'}|}e_{j'}^*(\mathscr{L}_{m_{j'},k,i})\right)\right) \quad j \neq j' \end{aligned} \tag{3.17}$$

For the i_{js}, we proceed exactly as in Step 3, we fix an order of ring extensions of each of the R_{m_j} made in Steps 1 and 3. We now extend i_{js} recursively via choosing a basis as in Step 3 and setting

$$\begin{aligned} i_{js}(a_\alpha \mathfrak{Eu}\left(\frac{n_\alpha}{|m_j|}\mathscr{L}_{m_j,k,i}\right) &:= i_{js}(\mathrm{Eu}(\mathscr{L}_{m_j,k,l}))^{\frac{n_\alpha}{|m_j|}} i_{js}(a_\alpha) \\ i_{js}\left(a_\alpha e_{js}\left(\mathfrak{Eu}\left(\frac{n_\alpha}{|m_{j'}|}e_{j'}^*(\mathscr{L}_{m_{j'},k,l})\right)\right)\right) &:= i_{j's}(\mathrm{Eu}(\mathscr{L}_{m_j,k,l}))^{\frac{n_\alpha}{|m_{j'}|}} i_{js}(a_\alpha) \end{aligned} \tag{3.18}$$

respectively.

We finally set

$$i_{m*}(x) := i_{ms}(x)i_*(1) \tag{3.19}$$

LEMMA 3.1. *There are ring injections* $R \hookrightarrow R^{\mathrm{sar}}$, $R_m \hookrightarrow R_m^{\mathrm{sar}}$ *and* $R_{\mathbf{m}} \hookrightarrow R_{\mathbf{m}}^{\mathrm{sar}}$. *For the above ring extensions, the morphisms* e_j^*, i_j^* *and their* $\vee$*-checked analogues are ring homomorphisms. The following formulas and their* $\vee$*-checked analogs hold*:

$$e_j^*(e_{js}(x)) = x, \quad i_j^*(i_{js}(x)) = x, \quad e_j^* i_j^* = e_l^* i_l^*, \quad e_j^*(e_{j*}(x))) = x\mathrm{Eu}(N_{X^{\mathbf{m}}/X^{m_j}}) \tag{3.20}$$

Furthermore, the projection equation holds on elements of the original rings.

Proof. The injections are clear by construction. All the properties except for the last one follow from the definitions. Now

$$e_j^*(e_{j*}(x)) = e_j^*(e_{js}(x)) e_j^*(e_{j*}(1)) = x\mathrm{Eu}(N_{X^{\mathbf{m}}/X^{m_j}})$$

□

The fractional Euler classes are the twist fields in the following sense:

THEOREM 3.2. *If we have sections of order two*

$$e_1^*(v_{m_1}\mathfrak{Eu}(S_{m_1})) e_2^*(v_{m_2}\mathfrak{Eu}(S_{m_2})) = \check{e}_3^*[(v_1 * v_2)\mathfrak{Eu}(S_{m_3})] \tag{3.21}$$

$$\check{e}_{3s}\check{e}_3^*[\check{i}_3^*[i_{1s}[v_{m_1}\mathfrak{Eu}(S_{m_1})] i_{2s}[v_{m_2}\mathfrak{Eu}(S_{m_2})]] = (v_1 * v_2)\mathfrak{Eu}(S_{m_3}) \tag{3.22}$$

Proof.

$$\begin{aligned} & e_1^*(v_{m_1}\mathfrak{Eu}(S_{m_1})) e_2^*(v_{m_2}\mathfrak{Eu}(S_{m_2})) \\ & \quad = e_1^*(v_{m_1}) e_2^*(v_{m_2}) \mathrm{Eu}(e_1^*(S_{m_1}) \oplus e_2^*(S_{m_2}) \oplus e_3^*(S_{m_3}) \ominus N_{X^{\mathbf{m}}/X}) \\ & \qquad \times \mathfrak{Eu}(\check{e}_3^*(S_{m_3^{-1}})) \mathrm{Eu}(N_{X^{\mathbf{m}}/X^{m_3^{-1}}}) \\ & \quad = \check{e}_3^* \check{e}_{3*}[e_1^*(v_{m_1}) e_2^*(v_{m_2}) \mathrm{Eu}(\mathcal{R}(\mathbf{m}))] \check{e}_3^*(\mathfrak{Eu}(S_{m_3^{-1}}))] \\ & \quad = \check{e}_3^*([(v_1 * v_2)\mathfrak{Eu}(S_{m_3})] \end{aligned}$$

where for the first equality we used Lemma 3.1 and the fact that $e_1^*(S_{m_1}) \oplus e_2^*(S_{m_2}) = \Gamma(\mathbf{m}) \oplus S_{m_3^{-1}} = \mathcal{R}(\mathbf{m}) \oplus S_{m_3} \oplus N_{X^{\mathbf{m}}/X^{m_3^{-1}}}$ in the extended monoid of bundles.

The second equation follows from the definition of $*$ and the self-intersection formula of Lemma 3.1.

Using Lemma 3.1 we can proceed analogously to the proof of Theorem 2.8 to obtain:

$$\check{e}_{3s}\check{e}_3^*[\check{i}_3^*(i_{1s}(v_{m_1}\mathfrak{Eu}(S_{m_1})) i_{2s}(v_{m_2}\mathfrak{Eu}(S_{m_2}))] = \check{e}_{3s}[e_1^*(v_{m_1}\mathfrak{Eu}(S_{m_1})) e_2^*(v_{m_2}\mathfrak{Eu}(S_{m_2}))]$$

□

Remark 3.3. We notice that there is a projection term $\check{e}_{3s}\check{e}_3^*$ which we cannot *a priori* exclude. In terms of twist fields these projection will be built into the definition of the three-point function. Up to this projection term, we have trivialized the co-cycles. Notice that we do not divide by the fractional Euler class $\mathfrak{Eu}(S_{m_3^{-1}})$. This operation is not well defined unless we localize, but as these elements are nilpotent localization would render the zero ring.

DEFINITION 3.4. We define the space of fields as $\mathcal{H} := \bigoplus_m \mathcal{F}(X^m)\mathfrak{Eu}(S_m)$ and for second-order normalized sections, we define the 3-point functions as

$$\langle u_m \mathfrak{Eu}(S_{m_1}), v_{m_2}\mathfrak{Eu}(S_{m_2}), w_{m_3}\mathfrak{Eu}(S_{m_3})\rangle \tag{3.23}$$

$$:= \delta_{m_1 m_2 m_3, \mathbf{1}} \int_X i_{3s}e_{3s}e_3^* i_3^*[i_{m_1 s}(u_{m_1}) i_{m_2 s}(v_{m_2}) i_{m_3 s}(w_{m_3})]$$

$$\times\, i_{3s}e_{3s}e_3^* i_3^*[i_{m_1 s}\mathfrak{Eu}(S_{m_1}) i_{m_2 s}\mathfrak{Eu}(S_{m_2}) i_{m_3 s}\mathfrak{Eu}(S_{m_3})] \tag{3.24}$$

Remark 3.5. Notice that by definition if $\prod m_i = \mathbf{1}$

$$\langle u_m \mathfrak{Eu}(S_{m_1}), v_{m_2}\mathfrak{Eu}(S_{m_2}), w_{m_3}\mathfrak{Eu}(S_{m_3})\rangle$$

$$= \int_X i_{3s}e_{3s}[e_1^*(u_{m_1})e_2^*(v_{m_2})e_3^*(w_{m_3})] i_{3s}e_{3s}(\mathrm{Eu}(\mathcal{S}(\mathbf{m})))$$

$$= \int_X i_{3s}e_{3s}[e_1^*(u_{m_1})e_2^*(v_2)e_3^*(w_{m_3})] i_{3*}e_{3*}(\mathrm{Eu}(\mathcal{R}(\mathbf{m})))$$

$$= \int_X i_{3*}e_{3*}(e_1^*(u_{m_1})e_2^*(v_{m_2})e_3^*(w_{m_3})\mathrm{Eu}(\mathcal{R}(\mathbf{m})))$$

$$= \int_{X^{\mathbf{m}}} e_1^*(u_{m_1})e_2^*(v_{m_2})e_3^*(w_{m_3})\mathrm{Eu}(\mathcal{R}(\mathbf{m}))$$

$$= \langle u_{m_1} * v_{m_2}, w_{m_3}\rangle \tag{3.25}$$

So that the two- and three-point functions agree with the usual ones.

3.3. AN EXCESS INTERSECTION CALCULATION

We now drop the assumption of having sections above. As a motivation for the general case we give a calculation in $\mathcal{F}(X, G)[[t]]$.

To this end we set $r = \mathrm{r}k(\mathcal{R}(\mathbf{m}))$ and rewrite (3.2) as

$$\mathrm{eval}_{\mathcal{F}|r}[\check{i}^{3*}(i_{1*}(v_{m_1}\sigma_{1,t}) i_{2*}(v_{m_2}\sigma_{2,t})\check{i}_{3*}(\tilde{\sigma}_{3,t}))]$$

$$= \mathrm{eval}_{\mathcal{F}|r}[\check{e}_{3*}(e_1^*(v_{m_1})e_2^*(v_{m_2})\mathrm{Eu}_t(\mathcal{R}(\mathbf{m})))] \tag{3.26}$$

where now the $\sigma_{i,t}$ and $\tilde{\sigma}_{i,t}$ are power series.

The main tool will be the excess intersection formula [8,21] on the Cartesian square

$$\begin{array}{ccc} X^{\mathbf{m}} & \stackrel{\check{e}_3}{\longrightarrow} & X^{m_3^{-1}} \\ (e_1,e_2,\check{e}_3)\circ(\Delta,id)\circ\Delta\downarrow & & \downarrow(\check{\imath}_3,\check{\imath}_3,\check{\imath}_3)\circ(\Delta,id)\circ\Delta \\ X^{m_1}\times X^{m_2}\times X^{m_3^{-1}} & \stackrel{(i_1,i_2,\check{\imath}_3)}{\longrightarrow} & X\times X\times X \end{array}$$

which has excess bundle

$$E = N_{X^{m_1}/X}|_{X^{\mathbf{m}}} \oplus N_{X^{m_2}/X}|_{X^{\mathbf{m}}} \oplus N_{X^{m_3^{-1}}/X}|_{X^{\mathbf{m}}} \ominus N_{X^{\mathbf{m}}/X^{m_3^{-1}}} \tag{3.27}$$

Using it we can transform the l.h.s. of Eq. (3.26) as follows:

$$\begin{aligned} \text{l.h.s.}\,(3.26) &= \check{\imath}_3^*[i_{1*}(v_{m_1}\sigma_1)i_{2*}(v_{m_2}\sigma_2)\check{\imath}_{3*}(\tilde{\sigma}_3)] \\ &= e_{3*}[e_1^*(v_{m_1}\sigma_1\mathrm{Eu}(N_{X^{m_1}/X}))e_2^*(v_{m_2}\sigma_2\mathrm{Eu}(N_{X^{m_2}/X})) \\ &\quad \times e_3^*(\tilde{\sigma}_3\mathrm{Eu}(N_{X^{m_3^{-1}}/X}))\mathrm{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_3^{-1}}})] \\ &= \mathrm{eval}_{\mathcal{F}|k}\left\{\check{e}_{3*}[e_1^*(v_{m_1}\sigma_1\mathrm{Eu}_t(N_{X^{m_1}/X}))e_2^*(v_{m_2}\sigma_2\mathrm{Eu}_t(N_{X^{m_2}/X}))\right. \\ &\quad \left.\times\,\check{e}_3^*(\tilde{\sigma}_3\mathrm{Eu}_t(N_{X^{m_3^{-1}}/X}))\mathrm{Eu}_t(\ominus N_{X^{X^{\mathbf{m}}/m_3^{-1}}})]\right\} \end{aligned} \tag{3.28}$$

where $k = \mathrm{r}k(E)$.

While the r.h.s. can be transformed to

$$\begin{aligned} \text{r.h.s.}\ (3.26) &= \mathrm{eval}_{\mathcal{F}|r}\{\check{e}_{3*}[e_1^*(v_{m_1}\mathrm{Eu}_t(S_{m_1}))e_2^*(v_{m_2}\mathrm{Eu}_t(S_{m_2})) \\ &\quad \times e_3^*(v_{m_3}\mathrm{Eu}_t(S_{m_3}))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X})]\} \\ &= \mathrm{eval}_{\mathcal{F}|r}\left\{\check{e}_{3*}[e_1^*(v_{m_1}\mathrm{Eu}_t(S_{m_1})\mathrm{Eu}_t(N_{X^{m_1}/X})\mathrm{Eu}_t(\ominus N_{X^{m_1}/X}))\right. \\ &\quad \times e_2^*(v_{m_2}\mathrm{Eu}_t(S_{m_2})\mathrm{Eu}_t(N_{X^{m_2}/X})\mathrm{Eu}_t(\ominus N_{X^{m_2}/X})) \\ &\quad \left.\times\, e_3^*(\mathrm{Eu}_t(S_{m_3}))e_3^*(\mathrm{Eu}_t(\ominus N_{X^{m_3}/X}))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X^{m_3^{-1}}})]\right\} \\ &= \mathrm{eval}_{\mathcal{F}|r}\left\{\check{e}_{3*}[e_1^*(v_{m_1}\mathrm{Eu}_t(\ominus S_{m_1^{-1}})\mathrm{Eu}_t(N_{X^{m_1}/X}))\right. \\ &\quad \left.\times\, e_2^*(v_{m_2}\mathrm{Eu}_t(\ominus S_{m_2})\mathrm{Eu}_t(N_{X^{m_2}/X}))\check{e}_3^*(\mathrm{Eu}_t(\ominus S_{m_3^{-1}}))\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X^{m_3^{-1}}})]\right\} \end{aligned} \tag{3.29}$$

3.4. A FORMAL SOLUTION

Comparing the two sides, that is, Eqs. (3.28) and (3.29), we set

$$\begin{aligned} \sigma_{1,t} &= \mathrm{Eu}_t(\ominus S_{m_1^{-1}}) = \mathrm{Eu}_t(S_{m_1})\mathrm{Eu}_t(\ominus N_{X^{m_1}/X}) \\ \sigma_{2,t} &= \mathrm{Eu}_t(\ominus S_{m_2^{-1}}) = \mathrm{Eu}_t(S_{m_2})\mathrm{Eu}_t(\ominus N_{X^{m_2}/X}) \\ \tilde{\sigma}_{3,t} &= \mathrm{Eu}_t(\ominus S_{m_3^{-1}} \ominus N_{X^{m_3^{-1}}/X}) = \vee^*(\mathrm{Eu}_t(S_{m_3})\mathrm{Eu}_t(\ominus N_{X^{m_3}/X})^2) \end{aligned} \tag{3.30}$$

as formal twist fields.

One now is tempted to use a kind of evaluation map, that is, to set $\sigma_i = \mathrm{eval}_{\mathcal{F}|\mathrm{v}r(\sigma_i)}(\sigma_{i,t})$ and $\tilde{\sigma}_3 := \mathrm{eval}_{\mathcal{F}|\mathrm{v}r(\tilde{\sigma}_3)}(\sigma_{3,t})$ where vr denotes the virtual rank. This is, however, not possible, since it is not clear that the respective power series converges for -1 nor is it clear what the coefficient at a rational power or a negative virtual rank means. We are faced with two challenges: how to make sense out of evaluating the $\mathrm{Eu}_t(S_m)$ and the $\mathrm{Eu}_t(\ominus N_{X^{m_i}/X})$ at their virtual rank.

For the former, we can simply use the ring extension above and replace the evaluation of $\mathrm{Eu}_t(S_m)$ by $\mathfrak{Eu}(S_m)$. The evaluation of the elements $\mathrm{Eu}_t(\ominus N_{X^{m_i}/X})$ poses more of a problem. These should of course be inverses to $\mathrm{Eu}(N_{X^{m_i}/X})$ which are nilpotent. Localizing would hence yield the zero ring. The answer is that the evaluations should be interpreted as formal sections.

That is, we will basically adjoin two sets of variables $\mathfrak{S}_1 := \{\mathfrak{Eu}(S_m)\}$ and $\mathfrak{S}_2 := \{\mathfrak{Eu}(\ominus N_{X^m/X})\}$ and mod out by appropriate relations. We think of $\mathfrak{S}_1$ as fractional Euler classes and $\mathfrak{S}_2$ as formal sections. The extension for the variables $\mathfrak{S}_1$ is analogous to the one discussed in the previous section. We will now give the details for the second adjunction.

3.4.1. *Motivation*

For a given inclusion $i: Y \to X$, the self intersection formula yields

$$i^*(i_*(a)) = a\mathrm{Eu}(N_{X/Y}) \tag{3.31}$$

this is why we can think

$$\text{“}i_s(a) := i_*(a\mathfrak{Eu}(\ominus N_{X/Y}))\text{”} \tag{3.32}$$

We will put equations like this in quotes for the time being.

Indeed, then using the same logic

$$\text{“}i^*(i_s(a)) := i^*(i_*(a\mathfrak{Eu}(\ominus N_{X/Y}))) = a\mathrm{Eu}(N_{X/Y})\mathfrak{Eu}(\ominus N_{X/Y}) = a\text{”} \tag{3.33}$$

Notice that if i_s is indeed a section

$$i_*(ab) = i_*(i^*(i_s(a))b) = i_s(a)i_*(b) \tag{3.34}$$

and hence

$$i_*(a) = i_s(a)i_*(1) \tag{3.35}$$

So that we see that if there are sections, indeed,

$$\text{“}i_s(a) = i_*(a)/i_*(1)\text{”} \tag{3.36}$$

where this equation should be read as Eq. (3.35) which essentially defines the i_s, see Remark 2.3.

3.4.2. *Adjoining formal sections*

We now define a further ring extension by additionally adjoining formal symbols encoding the properties of $\mathfrak{Eu}(\ominus N_{X^m/X})$ and $\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^m})$. To achieve this, we add formal sections *before* adding the fractional classes. We again proceed in steps.

STEP 1: We extend the rings $R^s_{m_j}$ by symbols $a\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}})$ for all $a \in R^s_{\mathbf{m}}$.

We now define e_{js} as follows:

$$e_{js}(a) := e_{j*}(a\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}})) \quad \text{for } a \in R^s_{\mathbf{m}} \tag{3.37}$$

and extend e_j^* as a ring morphism by setting

$$e_j^*(e_{j*}(a\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}}))) := a \tag{3.38}$$

hence the e_{js} are sections.

We then take the quotient of the above ring by the relations

$$\begin{aligned} &e_{j*}(a\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}}))e_{j*}(b) - e_{j*}(ab) &&\text{for} \quad a, b \in \mathcal{F}(X^{\mathbf{m}}) \\ &e_{j*}(a\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}}))e_{j*}(b\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}})) \\ &\quad - e_{j*}(ab\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}})) &&\text{for} \quad a, b \in \mathcal{F}(X^{\mathbf{m}}) \end{aligned} \tag{3.39}$$

and call this ring $\tilde{R}^s_{m_j}$.

Notice that under e_j^* these relations go to zero and hence e_j^*, e_{j*} and e_{js} pass to maps between $R^s_{\mathbf{m}}$ and $\tilde{R}^s_{m_j}$.

STEP 2. To $\mathcal{F}(X)$ we adjoin elements $i_{j*}(a\mathfrak{Eu}(\ominus N_{X^{m_j}/X}))$ for $a \in \mathcal{F}(X^{m_j})$, $j = 1, 2, 3$ and elements $\iota_*(a\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X}))$ for $a \in \mathcal{F}(X^{\mathbf{m}})$.

We define i_j^* as a ring homomorphism to the non quotiented rings of step 1 via:

$$\begin{aligned} &i_j^* i_{j*}(a\mathfrak{Eu}(\ominus N_{X^{m_j}/X})) := a \\ &i_j^* \iota_*(a\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X})) := e_{j*}(a\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}})) \\ &i_j^* i_{k*}(a\mathfrak{Eu}(\ominus N_{X^{m_j}/X})) := e_{j*}(e_k^*(a)\mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}})) \quad j \neq k \end{aligned} \tag{3.40}$$

Likewise, we define i_{js} as follows: For $a \in \mathcal{F}(X^{m_j})$ and $a_i \in \mathcal{F}(X^{\mathbf{m}})$

$$\begin{aligned} &i_{js}\left(a \prod_{i \in I} e_{j*}(a_i \mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}}))\right) \\ &\quad = i_{j*}(a\mathfrak{Eu}(\ominus N_{X^{m_j}/X})) \prod_{i \in I} \iota_*(a_i \mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X})) \end{aligned} \tag{3.41}$$

We also extend i_{j*} by

$$i_{j*}\left(a \prod_{i \in I} e_{j*}(a_i \mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X^{m_j}}))\right) = i_{j*}(a) \prod_{i \in I} \iota_*(a_i \mathfrak{Eu}(\ominus N_{X^{\mathbf{m}}/X})) \tag{3.42}$$

We form a quotient $\tilde{R}$ of the ring extension of $\mathcal{F}(X)$, by modding out by the relations

$$
\begin{aligned}
& i_{j*}(a\mathfrak{Eu}(\ominus N_{X^{m_j}/X}))i_{j*}(b) - i_{j*}(ab) && \text{for} \quad a,b\in R^s_{m_j}\\
& i_{j*}(a\mathfrak{Eu}(\ominus N_{X^{m_j}/X}))i_{j*}(b\mathfrak{Eu}(\ominus N_{X^{m_j}/X})) && \\
& \quad - i_{j*}(ab\mathfrak{Eu}(\ominus N_{X^{m_j}/X})) && \text{for} \quad a,b\in R^s_{m_j}\\
& \iota_*(a\mathfrak{Eu}(\ominus N_{X^{m_j}/X}))i_{j*}e_{j*}(b) - i_{j*}e_{j*}(ab) && \text{for} \quad a,b\in R^s_{\mathbf{m}}\\
& \iota_*(a\mathfrak{Eu}(\ominus N_{X^{m_j}/X}))\iota_*(b\mathfrak{Eu}(\ominus N_{X^{m_j}/X})) - \iota_*(ab\mathfrak{Eu}(\ominus N_{X^{m_j}/X})) && \text{for} \quad a,b\in R^s_{\mathbf{m}}
\end{aligned}
\tag{3.43}
$$

It is now a straightforward check that the maps i^*_j, i_{j*}, i_{js} induce maps between $\tilde{R}^s_m$ and $\tilde{R}^s_{m_j}$.

STEP 3. Adjoin the fractional Euler classes as in Section 3.2.1.

THEOREM 3.6. *Theorems 2.8, 2.11 and 3.2 hold in the formal setting as well.*

Proof. The only relations were needed in the proofs are guaranteed by the above constructions. □

Remark 3.7. This means that after adding formal sections there is a pull–push stringy multiplication in terms of trivializable co-cycles just as in the cyclic case. This is rather surprising, since *a priori* from an algebraic standpoint, if the twisted sectors $\mathcal{F}(X^m)$ are not cyclic as modules over $\mathcal{F}(X)$ the cocycles describing the stringy multiplication are matrix valued after choosing generators. We now see *a posteriori* that these matrices can be chosen to be "constant", that is, they only depend on the stringy product of the units of $\mathcal{F}(X^m)$, which on top only depends on the group elements m. Of course there might be some dependence on the connected components, but this is handled completely through the geometry of the fixed point sets.

One can artificially create such matrix-valued products, by, for instance, taking two copies of X with the diagonal G action and twisting each copy of the stringy multiplication by different discrete torsions. In a sense this is of course not a very serious perturbation, as we move from constant twists by discrete torsion to locally constant twists. An interesting question is whether one can find examples in the non-cyclic case of more complicated stringy multiplications given by "non-constant" matrix co-cycles.

The calculation of the three-point functions also gives mathematical rigor to the physical notion of twist fields, which exists in the formal setting. The trivialization can be restated in this setting as saying that there is indeed only one twist field per group element.

4. The de Rham Theory for Stringy Cohomology of Global Quotients

In view of Lemma 2.4, there is no section of the functor $\mathcal{F}=H^*$, K^* itself, unless the modules $H^*(X^m)$, $K^*(X^m)$ are cyclic. But although the pull back e_i^* is not surjective on cohomology in general or by the usual Chern isomorphism on K-theory, on the level of de Rham chains the pull back is surjective.

Notice that in the proof of Proposition 2.10, we only used the following three properties: (1) projection formula, (2) the defining equation for the sections, and (3) the fact that the pull-back is an algebra homomorphism. So after establishing these facts for forms, we can proceed analogously to the calculation in the last section.

Notation 4.1. In this section, we fix coefficients to be $\mathbb{R}$ and we denote by $\Omega^n(X)$ the n-forms on X. Likewise, for a bundle $E\to B$ with compact base we denote $\Omega^n_{cv}(E)$ the n forms on E with compact vertical support and let $H^*_{cv}(E)$ be the corresponding cohomology with compact vertical support.

4.1. DE RHAM CHAINS AND THOM PUSH-FORWARDS

In this section, we will use de Rham chains and the Thom construction [4]. The advantage is that every form on every X^m is a "pull-back" from a tubular neighborhood.

We recall the salient features adapted to our situation from [4]. Let $i: X\to Y$ be an embedding; then there is a tubular neighborhood $Tub(N_{X/Y})$ of the zero section of the normal bundle $N_{X/Y}$ which is contained in Y. We let $j: Tub(N_{X/Y})\to Y$ be the inclusion.

Now the Thom isomorphism $\mathcal{T}: H^*(X)\to H^{*+\mathrm{codim}(X/Y})_{cv}(N_{X/Y})$ can be realized on the level of forms via capping with a Thom form Θ: $\mathcal{T}(\omega)=\pi^*(\omega)\wedge\Theta$. The Thom map is inverse to the integration along the fiber π_* and hence $\pi_*(\Theta)=1$. In fact, the class of this form is the unique class whose vertical restriction is a generator and whose integral along the fiber is 1. For any given tubular neighborhood $Tub(N_{X/Y})$ of the zero section of the normal bundle one can find a form representative Θ such that the $supp(\Theta)\subset Tub(N_{X/Y})$.

4.2. PUSH-FORWARD

In this situation the Thom push-forward $i_*: H^*(X)\to H^*(Y)$ is given by $\mathcal{T}$ followed by the extension by zero j_*. These maps are actually defined on the form level. That is, we choose Θ to have support strictly inside the tube, and hence the extension by zero outside the tube is well defined for the forms in the image of the Thom map.

$$i_*(\omega) := j_*(T(\omega)) = j_*(\pi^*(\omega)\wedge\Theta) \tag{4.1}$$

Notice that for two consecutive embeddings $X\xrightarrow{e} Y\xrightarrow{i} Z$, on cohomology we have $e_*\circ i_* = (e\circ i)_*: H^*(X)\to H^*(Z)$. On the level of forms depending on the choice of representatives of the Thom form either the identity holds on the nose, since the Thom classes are multiplicative [4] or the two push-forwards differ by an exact form $e_* \circ i_*(\omega) = (e \circ i)_* + d\tau$.

4.3. THE PROJECTION FORMULA ON THE LEVEL OF FORMS

The following proposition follows from standard facts [4]:

PROPOSITION 4.2 (Projection Formula for Forms). *With $i : X \to Y$ and embedding and i_* defined as above, for any form $\phi \in \Omega^*(X)$ and any closed form $\omega \in \Omega^*(Y)$ there is an exact form $d\tau \in \Omega^*(Y)$ such that*

$$i_*(i^*(\omega) \wedge \phi) = \omega \wedge i_*(\phi) + d\tau \tag{4.2}$$

Proof. Denote the zero section by $z : X \to N_{X/Y}$ and projection map of the normal bundle by $\pi : N_{X/Y} \to X$; then $i = j \circ z$.

$$X \overset{\pi|Tub}{\underset{z}{\leftrightarrows}} Tub(N_{X/Y}) \overset{j}{\to} Y \tag{4.3}$$

Since π is a deformation retraction, π^* and z^* are chain homotopic [4] and hence $\pi^* \circ z^*(\omega) = \omega + d\tau$. We can now calculate

$$\begin{aligned} i_*(i^*(\omega) \wedge \phi) &= j_*(\pi^*(i^*(\omega) \wedge \phi) \wedge \Theta) \\ &= j_*(\pi^*(z^*(j^*(\omega)) \wedge \pi^*(\phi) \wedge \Theta)) \\ &= j_*((j^*(\omega) + d\tau) \wedge \pi^*(\phi) \wedge \Theta) \\ &= \omega \wedge j_*(\pi^*(\phi) \wedge \Theta) + j_*(d\tau \wedge \pi^*(\phi) \wedge \Theta) \\ &= \omega \wedge i_*(\phi) + dj_*(\tau \wedge \pi^*(\phi) \wedge \Theta) \end{aligned} \tag{4.4}$$

where the penultimate question holds true since Θ has support inside $Tub(N_{X/Y})$ and the last equation holds true since d commutes with the extension by zero and pull-back. □

4.4. SECTIONS

To construct a section on the level of forms, we first notice that the Thom class can be represented by using a bump function f so that if X^{m_i} is given locally on U by the equations $x_k = \cdots = x_N = 0$

$$\mathcal{T}(1)|_F = f dx_k \wedge \cdots \wedge dx_N \tag{4.5}$$

where f is a bump function along the fiber F that can be chosen such that $supp(f)$, the support of f, lies strictly inside the tubular neighborhood and moreover $supp(f)$ lies strictly inside this neighborhood. We consider a characteristic

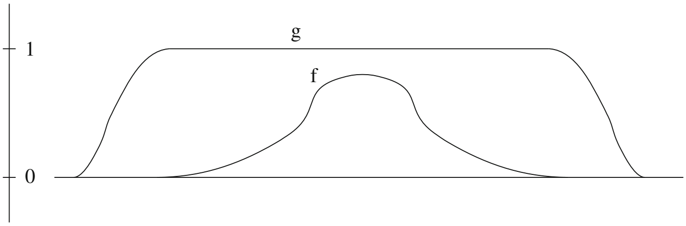

Figure 1. A bump function f of the Thom class representative and a characteristic function g.

function g of an open subset U with $supp(f) \subset U \subset Tub(N)$ inside the tubular neighborhood, see Figure 1. Here, characteristic function means that on U g has value 1, and there is an open V such that $U \subset V \subset \bar{V} \subset Tub(N)$ such that $g = 0$ outside $\bar{V}$ Notice that $fg(x) = f(x)$. We let $\mathbf{g}$ be a 0-form with compact vertical support whose restriction to the fiber is given by g.

For any form $\omega \in \Omega^*(X)$, we define

$$i_{ms}(\omega) := j_*(\mathbf{g}\pi_m^*(\omega)) \tag{4.6}$$

Then

$$i_m^*(j_*(\mathbf{g}\pi_m^*(\omega))) = z_m^*(j^*(j_*(\mathbf{g}\pi_m^*(\omega)))) = z_m^*(\mathbf{g})z_m^*(\pi_m^*(\omega)) = \omega + \mathrm{d}\tau \tag{4.7}$$

Remark 4.3. Actually, $i_*(\omega) := j_*(\mathcal{T}(w))$ is divisible by $j_*(\mathcal{T}(1))$:

$$\begin{aligned} i_{m*}(\omega) &= j_*(\mathcal{T}(w)) = j_*(\pi_m^*(\omega) \wedge \Theta) \\ &= j_*(\mathbf{g}\pi_m^*(\omega) \wedge \Theta) \\ &= i_{ms}(\omega) \wedge \Theta \end{aligned} \tag{4.8}$$

Notation 4.4. For a cohomology class α we let $\Upsilon(\alpha)$ be a choice of form representative: $[\Upsilon(\alpha)] = \alpha$.

COROLLARY 4.5. *In the situation above we can choose a form representative, such that*

$$\Upsilon(\mathrm{Eu}(N_{X/Y})) = i^* i_*(1) = i^*(\mathcal{T}(1)) = i^*(\Theta) \tag{4.9}$$

and up to closed forms:

$$i_s(\Upsilon(\mathrm{Eu}(N_{X/Y}))) = \Theta + \tau \ \textit{with}\ i^*(\tau) = 0 \tag{4.10}$$

Proof. The first equation is a well-known property of the Thom-form [4]. The second equation follows immediately from the fact that i_s is a section of i^* on homology. □

Remark 4.6. The Eq. (1.11) holds in K theory over $\mathbb{Z}$. In particular, this means that the two bundles are stably equivalent. That means there is a trivial bundle τ_n such that when forming the sum with both sides, the bundles become isomorphic.

$$N_{X^{\mathbf{m}}} \oplus \tau_n \simeq \mathcal{R}(\mathbf{m}) \oplus \mathcal{R}(\check{\mathbf{m}}) \oplus N_{X^{\mathbf{m}}/X^{m_1}} \oplus N_{X^{\mathbf{m}}/X^{m_2}} \oplus N_{X^{\mathbf{m}}/X^{m_3}} \oplus \tau_n \tag{4.11}$$

Hence we can treat $\mathcal{R}(\mathbf{m}) \oplus N_{X^{m_3^{-1}}}/X$ as a subbundle of $N_{X^{\mathbf{m}}} \oplus \tau_n$. In order to factor the inclusion through the respective tubular neighborhoods, we factor the inclusion of $X^{\mathbf{m}} \hookrightarrow X$ as

$$X^{\mathbf{m}} \overset{\check{i}_3 \circ \check{e}_3}{\to} X \overset{z}{\to} X' = X \times B^n \overset{\pi_1}{\to} X$$

where B^n is a ball, z is the zero section, and π_1 the first projection. Notice that $H^*(X) \simeq H^*(X')$ and we will identify these cohomologies.

Now $N_{X^{\mathbf{m}}/X\times B^n} = N_{X^{\mathbf{m}}} \oplus \tau_n$ and $\Gamma(\mathbf{m}) = \mathcal{R}(\mathbf{m}) \oplus N_{m_3^{-1}}$ is a subbundle and we can factor

$$X^{\mathbf{m}} \overset{e_\Gamma}{\to} Tub(\Gamma) \overset{\tilde{i}_\Gamma}{\to} Tub(N_{X^{\mathbf{m}}/X\times\mathbb{R}^n}) \overset{j_\Gamma}{\to} X'$$

We let $i_\Gamma = \tilde{i}_\Gamma j_\Gamma$.

PROPOSITION 4.7. *The following equations hold up to choices of form representatives and closed forms*

$$\begin{aligned}
&\text{Coeff of } t^r \text{ in } \{\, \check{i}_3^*[i_{s1}(\omega_{m_1}) i_{s2}(\omega_{m_2}) i_{s1}(\Upsilon(\mathrm{Eu}_t(S_{m_1}))) i_{s2}(\Upsilon(\mathrm{Eu}_t(S_{m_1})))\\
&\qquad \times i_{s3}(\Upsilon(\mathrm{Eu}_t(S_{m_3}))\Upsilon(\mathrm{Eu}_t(\ominus N_{X^{\mathbf{m}}/X}))\Upsilon(\mathrm{Eu}(N_{X^{\mathbf{m}}/X^{m_3}})))]\}\\
&\quad = \check{i}_3^*[i_{1s}(\omega_{m_1}) i_{2s}(\omega_{m_2}) \check{i}_{3s}\check{e}_{3*}(\Upsilon(\mathrm{Eu}(\mathcal{R}(\mathbf{m}))))]\\
&\quad = \check{i}_3^*[i_{1s}(\omega_{m_1}) i_{2s}(\omega_{m_2}) i_{\Gamma s}(\Theta_{\Gamma(\mathbf{m})})]
\end{aligned} \tag{4.12}$$

Here $\Upsilon(v)$ is a closed form representative of the class v and $\Theta_{\Gamma(\mathbf{m})}$ is a Thom form for the vector bundle.

Proof. The first equality is by definition of $\mathcal{R}(\mathbf{m})$. For the second we use the factorization of Remark 4.6 and replace X with X' extending the maps i_j appropriately. Then up to choices for form representatives and closed forms

$$\begin{aligned}
\check{i}_{3s}\check{e}_{3*}(\Upsilon(\mathrm{Eu}(\mathcal{R}(\mathbf{m})))) &= \check{i}_{3s}\check{e}_{3s}[\Upsilon(\mathrm{Eu}(\mathcal{R}(\mathbf{m}))) \wedge \Upsilon(\mathrm{Eu}(N_{X^{m_3^{-1}}/X^{\mathbf{m}}})]\\
&= i_{\Gamma s} e_{\Gamma s}[\Upsilon(\mathrm{Eu}(\mathcal{R}(\mathbf{m}))) \wedge \Upsilon(\mathrm{Eu}(N_{X^{m_3^{-1}}/X^{\mathbf{m}}})]\\
&= i_{\Gamma s}(\Theta_{\mathcal{R}(\mathbf{m})} \wedge \Theta_{N_{X^{m_3^{-1}}/X^{\mathbf{m}}}})\\
&= i_{\Gamma s}(\Theta_{\Gamma(\mathbf{m})})
\end{aligned}$$

□

DEFINITION 4.8. We define the form level product as given by any of the Eq. (4.12) above.

THEOREM 4.9.

$$\omega_{m_1} * \omega_{m_2} = e_{m_3 *}(e_1^*(\omega_{m_1}) e_2^*(\omega_{m_2}) \Upsilon(\mathrm{Eu}(\mathcal{R}(\mathbf{m})))) + \mathrm{d}\tau \tag{4.13}$$

for some exact form $\mathrm{d}\tau$.

Proof. Completely parallel to the proof of Proposition 2.10, since we have established all equalities up to chain homotopy. □

COROLLARY 4.10. *The three-point functions coincide with the ones induced by* (1.8)*. That is if* Υ *denotes the lift of a class to a form and* $\Upsilon(v_{m_i}) = \omega_{m_i}$*, then*

$$\begin{aligned} \langle \omega_{m_1} * \omega_{m_2}, \omega_{m_3} \rangle &:= \int_X \omega_{m_1} * \omega_{m_2} \wedge \omega_{m_3} \\ &= \int_X \Upsilon(v_{m_1} * v_{m_2}) \wedge \omega_{m_3} \\ &= (v_{m_1} * v_{m_2} \cup v_{m_3}) \cap [X] \qquad (4.14) \\ &= \langle v_{m_1} * v_{m_2}, v_{m_3} \rangle \qquad (4.15) \end{aligned}$$

where $[X]$ *is the fundamental class of* X *and hence the three-point functions are independent of the lift.*

Proof. Straightforward by Stokes. □

4.5. TRIVIALIZING THE CO-CYCLES, FRACTIONAL THOM FORMS

Now using the formalism of Section 3.2.1 and passing to a local trivializing neighborhood U, where the line bundles $\mathscr{L}_{m,k}$ have first Chern class represented by the forms $\mathrm{d}x_l, \ldots \mathrm{d}x_N$, we get a Thom-form representative of $\mathfrak{Eu}(S_m)$

$$\Theta_{\mathfrak{Eu}(S_m)}|_U = f^{k/|m|} \prod_{k \neq 0, i} (\mathrm{d}x)^{k/|m|} \tag{4.16}$$

These forms trivialize the co-cycles as explained above. In the Abelian case this type of expression was used in the arguments of [6].

What we have now is the generalization to an arbitrary group as well as a trivialization of the co-cycles in terms of roots, thus completing (re)-construction program of [13,14] in the de Rham setting of global quotients. The surprising answer is that there is always a stringy multiplication arising from a co-cycle that is trivializable in a ring extension obtained by adjoining roots; see also Remark 3.7.

5. Admissible Functors and Outlook

5.1. ADMISSIBLE FUNCTORS

Here we collect the formal properties of the functors $\mathcal{F}$ we used in our calculations.

DEFINITION 5.1. Let $\mathcal{F}$ be a functor together with an Euler-class Eu_t which has the following properties:

(1) $\mathcal{F}$ The Euler class Eu_t is defined for elements of rational K-theory and is multiplicative and takes values in $\mathcal{F}(X)[[t]]$.
(2) $\mathcal{F}$ is contravariant, i.e., it has pullbacks and the Euler-class is natural with respect to these.
(3) $\mathcal{F}$ has push-forwards i_* for closed embeddings $i: X \hookrightarrow Y$.
(4) $\mathcal{F}$ has an excess intersection formula for closed embeddings. That is, we have an evaluation morphism $\mathrm{Eu} := \mathrm{eval}_{\mathcal{F}|r}(\mathrm{Eu}_{\mathcal{F},t}) : \mathcal{F}(X)[[t]] \to \mathcal{F}(X)$ such that for the Cartesian squares

$$\begin{array}{ccc} Z & \stackrel{e_2}{\to} & Y_2 \\ \downarrow e_1 & & \downarrow i_2 \\ Y_1 & \stackrel{i_1}{\to} & X \end{array} \tag{5.1}$$

we have the following formula:

$$i_2^*(i_{1*}(a)) = e_{2*}(e_1^*(a)\epsilon_j) \tag{5.2}$$

where $\epsilon := \mathrm{Eu}(E)$ with E the excess bundle $E := N_{Y_1/X}|_Z \ominus N_{Z/Y_2}$ and r is its rank.

We call such a functor *admissible*.

All the functors $\mathcal{F}$ studied above are admissible and the calculations of this section—formal and non-formal—carry over to admissible functors. Actually, de Rham forms are admissible up to homotopy, see below, so that *mutatis mutandis* we can use the same arguments on the level of forms.

5.1.1. *Forms as an admissible functors*

In this case, which we worked out in the previous paragraph, we have an Euler class and all the properties of an admissible functor are valid on the chain level up to homotopy, that is up to closed forms.

(1) The Thom push-forward on the chain level induces the push-forward in cohomology induced by the Poincaré pairing, since the Thom class and the Poincaré dual can be represented by the same form [4].

(2) The projection formula holds, since the pull-back of the Thom class is the Euler class of the normal bundle [4].
(3) The excess intersection formula holds up to homotopy. Since it holds in cobordism theory and cohomology [21] we know that for closed ω the two forms $i_2^* i_{1*}(\omega)$ and $e_{2*}e_1^*(\omega\Upsilon(\mathrm{Eu}(E)))$ differ by a closed form.
(4) In particular, we can use the Thom pushforward and then the divisibility of the push-forward by the Thom class to give us sections.

5.2. OUTLOOK: APPLICATIONS TO SINGULARITIES WITH SYMMETRIES AKA. ORBIFOLD LANDAU–GINZBURG THEORIES

In conclusion, we wish to make some remarks about singularities with symmetries as regarded in [14,18]. We will restrict to the case of a trivial character $\chi\equiv 1$ for the G-Frobenius algebra. Recall that such a character is part of the data of any G-Frobenius algebra, [13,14]. In this case, the formula (1.8) adapted to this setting produces a solution to the stringy multiplication problem as we outline below. In the general case, some more care has to be taken, but it is also possible to write down a solution; see [19] for full details.

We recall that the relevant data are a pair (f, G) of a singularity $f:\mathbb{C}^n\to\mathbb{C}$ with an isolated critical point at zero and a finite group G with embedding into $Gl(n,\mathbb{C})$ such that $g^*(f)=f$. The character χ is given by $\chi(g):=\det(g)$.

5.2.1. *Euler classes and the solution*

The analog of the fixed point sets X^m are just subsets $Fix(m)\subset\mathbb{C}^n$ with the singularity given by restriction of the function f and likewise for the double intersection.

Pull backs and push forwards are given in this situation. In order to use the presented setup, all we need are Euler classes. We set

$$\mathrm{Eu}(f):=\mathrm{hess}(f)=\det(\mathrm{Hess}(f)) \tag{5.3}$$

Using the basic principles of Chern–Weil theory, we can even define a total Chern class in this situation:

$$\mathrm{Eu}_t(f):=\sum_i tr(\Lambda^i \mathrm{Hess}(f))t^i \tag{5.4}$$

To get an expression for the Euler class and the total Chern class of $\mathcal{R}(\mathbf{m})$, we first notice that the role of the tangent space is played by $\mathbb{C}^n$ together with its G action. Each subgroup $\langle g\rangle$, $g\in G$ then defines a representation, and we can define $S_g\in Rep(G)\otimes\mathbb{Q}$ by the formula (1.6) by noticing that the Eigenbundles in this case are just subrepresentations.

For any subrepresentation V of G we define

$$\mathrm{Eu}_t(V) = \sum_i tr(\Lambda^i \mathrm{Hess}(f|_V))t^i \tag{5.5}$$

In the Abelian case $\mathcal{R}(\mathbf{m})$ is the subrepresentation V which is given as follows: simultaneously diagonalize the action of G. Let $g = diag(\exp(2\pi i \lambda_j(g))$, with $\lambda_j(g) \in [0, 1)$; then, V is spanned by the simultaneous Eigenvectors e_j whose log-Eigenvalues satisfy

$$\lambda_j(g) + \lambda_j(h) = \lambda_j(gh) + 1$$

In the non-Abelian case, we just regard the S_m as elements of $K_G(pt)$ or as virtual representation. Analogous to Remark 4.6, we can stabilize the normal bundle and regard $\mathcal{R}(\mathbf{m})$ as a subbundle. In order to evaluate the Euler class, we also stabilize the singularity by adding squares. These two operations of stabilization are compatible. Indeed, in K-theory stabilization (see, e.g., [1]) means that we add trivial bundles. In the theory of singularities (see, e.g., [2]) stabilization means that instead of $f(\mathbf{z})$ one considers the function $F(\mathbf{z}, \mathbf{w}) = f(z) + w_1^2 + \cdots + w_l^2$ which has the same Milnor ring. Trivially extending the action of G, we obtain the compatibility of the two stabilizations.

Hence, (1.8) defines a multiplication on the orbifold Milnor ring $\bigoplus_{g\in G} M(f|_{Fix(g)})$ (cf. [14,18]) where $M(f|_{Fix(g)})$ denotes the Minor ring of the function $f|_{Fix(g)}$ which again has an isolated singularity. Pull-back is the restriction of functions and push-forward is the adjoint map to pull-back. Here, "adjoint" is taken in the sense of maps between Frobenius algebras. Given two Frobenius algebras A and B with non-degenerate forms $\langle , \rangle_A$, $\langle , \rangle_B$ and a morphism $r: A \to B$ its adjoint $r^\dagger: B \to A$ is defined by

$$\langle a, r^\dagger(b)\rangle_A = \langle r(a), b\rangle_B \tag{5.6}$$

5.2.2. *Sections and fractional classes*

There are even sections i_{ms} which are given by considering a function of fewer variables to be a function of more variables (cf. [14,16,18]). If we furthermore assume that $\mathrm{Hess}(f)$ is diagonal, the expressions become particularly appealing. We can even give the expression for the fractional Euler classes:

$$\mathrm{Eu}_t(S_g) = \prod_j (1 + \partial^2/\partial z_j^2(f)t)^{\lambda_j(g)} \tag{5.7}$$

and

$$\mathfrak{Eu}(S_g) = \prod_j (\partial^2/\partial z_j^2(f))^{\lambda_j(g)} \tag{5.8}$$

5.2.3. *Remarks on mirror symmetry*

It turns out that this multiplication in general does not respect the bi-grading for orbifold singularities given in [18].

In the case $f_n = z_0^n + \cdots + z_{n-1}^n$ this gives a multiplication which is part A-model and part B-model: the untwisted sector behaving like the B-side and the twisted sectors behaving like the A-side. Here, A- and B-side are the usual sides of mirror symmetry. In this particular situation, we can either use the definitions of [18], or the general $N = 2$ framework from physics [9,20] which distinguishes the two sides for instance by their bi-grading.

What geometry this describes is an intriguing question, to which we plan to return in a subsequent paper.

Acknowledgements

The main ideas of this paper was first presented at the "Workshop on Quantum Cohomology of Stacks" at the IHP in Feb. 2007. We wish to thank the organizers for the wonderful conference. We thank Takashi Kimura, Yongbin Ruan, and Maxim Kontsevich for stimulating discussions. We also to thank the Max-Planck-Institute for Mathematics in Bonn for its hospitality, where a significant part of this work was carried out and the IHES where the paper received its final form. The author also gratefully acknowledges support from the NSF under grant DMS-0805881 and support from the Humboldt foundation for part of this project.

References

1. Atiyah, M.F.: K-Theory, v+166+xlix pp. W. A. Benjamin, Inc., New York (1967)
2. Arnold, V.I., Goryunov, V.V., Lyashko, O.V., Vasilev, V.A.: Singularity Theory I, iv+245 pp. Springer-Verlag, Berlin (1998)
3. Abramovich, D., Graber, T., Vistoli, A.: Algebraic orbifold quantum products. Orbifolds in mathematics and physics (Madison, WI, 2001). Contemp. Math. **310**, 1–24. Amer. Math. Soc., Providence, RI (2002) and Gromov–Witten theory of Deligne–Mumford stacks. Preprint math.AG/0603151
4. Bott, R., Tu, L.W.: Differential forms in algebraic topology. In: Graduate Texts in Mathematics, vol. 82. Springer-Verlag, New York (1982)
5. Chen, W., Ruan, Y.: A new cohomology theory for orbifold. Commun. Math. Phys. **248**(1), 1–31 (2004)
6. Chen, B., Hu, S.: A de Rham model for Chen-Ruan cohomology ring of abelian orbifolds. Math. Ann. **336**(1), 51–71 (2006)
7. Fantechi, B., Göttsche, L.: Orbifold cohomology for global quotients. Duke Math. J. **117**, 197–227 (2003)
8. Fulton, W., Lang, S.: Riemann-Roch algebra. Grundlehren der Mathematischen Wissenschaften, vol. 277. Springer-Verlag, New York (1985)
9. Greene, B.R., Plesser, M.R.: Duality in Calabi-Yau moduli space. Nucl. Phys. B **338**(1), 15–37 (1990)

10. Hirzebruch, F.: Topological methods in algebraic geometry. Die Grundlehren der Mathematischen Wissenschaften, Band 131, x+232 pp. Springer-Verlag, New York (1966)
11. Jarvis, T., Kaufmann, R., Kimura, T.: Pointed admissible G-covers and G-equivariant cohomological field theories. Compos. Math. **141**, 926–978 (2005)
12. Jarvis, T., Kaufmann, R., Kimura, T.: Stringy K-theory and the Chern character. Inv. Math. **168**(1), 23–81 (2007)
13. Kaufmann, R.M.: Orbifold Frobenius algebras, cobordisms, and monodromies. In: Adem, A., Morava, J., Ruan, Y. (eds.) Orbifolds in Mathematics and Physics. Contemp. Math., vol. 310, pp. 135–162. Amer. Math. Soc., Providence (2002)
14. Kaufmann, R.M.: Orbifolding Frobenius algebras. Int. J. Math. **14**, 573–619 (2003)
15. Kaufmann, R.M.: Discrete torsion, symmetric products and the Hilbert scheme. In: Hertling, C., Marcolli, M. (eds.) Frobenius Manifolds, Quantum Cohomology and Singularities. Aspects of Mathematics E 36. Vieweg (2004)
16. Kaufmann, R.M.: Second quantized Frobenius algebras. Commun. Math. Phys **248**, 33–83 (2004)
17. Kaufmann, R.M.: The algebra of discrete torsion. J. Algebra **282**, 232–259 (2004)
18. Kaufmann, R.M.: Singularities with symmetries, orbifold Frobenius algebras and mirror symmetry. Contemp. Math. **403**, 67–116 (2006)
19. Kaufmann, R.M.: Stringy multiplication for orbifold Landau–Ginzburg theories. Preprint
20. Lerche, W., Vafa, C., Warner, N.P.: Chiral rings in $N = 2$ superconformal theories. Nucl. Phys. B **324**(2), 427–474 (1989)
21. Quillen, D.: Elementary proofs of some results of cobordism theory using Steenrod operations. Adv. Math. **7**, 29–56 (1971)