Available online at www.sciencedirect.com
ScienceDirect
TOPOLOGY
Topology 46 (2007) 39–88
www.elsevier.com/locate/top

On spineless cacti, Deligne's conjecture and Connes–Kreimer's Hopf algebra

Ralph M. Kaufmann

University of Connecticut, Department of Mathematics, 196 Auditorium Road, Storrs, CT 06269, United States

Received 18 November 2004; received in revised form 12 May 2005; accepted 6 October 2006

Abstract

Using a cell model for the little discs operad in terms of spineless cacti we give a minimal common topological operadic formalism for three a priori disparate algebraic structures: (1) a solution to Deligne's conjecture on the Hochschild complex, (2) the Hopf algebra of Connes and Kreimer, and (3) the string topology of Chas and Sullivan.

MSC: 55P48; 18D50; 16E40; 17A30

Keywords: Deligne's conjecture; Little discs; Hochschild complex; Operads; Gerstenhaber algebra; Hopf algebra; Renormalization; Cacti; String topology

0. Introduction

When considering an algebraic structure there is often a topological framework which is indicative of this structure. For instance, it is well known that Gerstenhaber algebras are governed by the homology operad of the little discs operad [2,3] and that Batalin–Vilkovisky algebras are exactly the algebras over the homology of the framed little discs operad [12]. The purpose of this paper is to prove that there is a common, minimal, topological operadic formalism [17,26] for three a priori disparate algebraic structures: (1) a homotopy Gerstenhaber structure on the chains of the Hochschild complex of an associative algebra, a.k.a. Deligne's conjecture, (2) the Hopf algebra of Connes and Kreimer [7], and (3) string topology [6].

E-mail address: kaufmann@math.uconn.edu.

0040-9383/$ - see front matter
doi:10.1016/j.top.2006.10.002

To accomplish this task, we use the spineless cacti operad of [17] which is responsible for the Gerstenhaber structure of string topology [4,6,17,19,26,40]. Moreover our operad of spineless cacti is actually equivalent to the little discs operad [17]. The analysis of this operad on the chain level allows us to give a new topological proof of Deligne's conjecture. Furthermore it provides chain models for the operads whose algebras are precisely pre-Lie algebras and graded pre-Lie algebras, respectively, as well as a chain realization for the Hopf algebra of Connes and Kreimer.

As we are using cacti, this approach naturally lies within string topology on one hand and on the other hand it is embedded in the framework of a combinatorial description of the moduli space of surfaces with punctured boundaries via the $\mathcal{A}rc$ operad [17,18,26,34]. Therefore all the previous structures obtain a representation in terms of moduli spaces.

We start by giving new CW decompositions for the spaces $\mathcal{C}act^1(n)$ of normalized spineless cacti with n lobes which are homotopy equivalent to the spaces $\mathcal{C}act(n)$ of spineless cacti with n lobes, the homotopy being the contraction of n factors of $\mathbb{R}_{>0}$.

Theorem 3.5. *The space $\mathcal{C}act^1(n)$ is homeomorphic to the CW complex $K(n)$.*

As shown in [17] the spaces $\mathcal{C}act^1(n)$ form a quasi-operad whose homology is an operad isomorphic to the homology operad of cacti and hence to the homology of the little discs operad. The operad structure, however, already appears on the chain level.

Theorem 3.11. *The glueings induced from the glueings of spineless normalized cacti make the spaces $CC_*(\mathcal{C}act^1(n))$ into a chain operad. Thus $CC_*(\mathcal{C}act^1)$ is an operadic model for the chains of the little discs operad.*

Moreover, the cells of $K(n)$ are indexed by planted planar bipartite trees and the operad of cellular chains is isomorphic to a combinatorial tree dg-operad. Reinterpreting the trees as "flow charts" for multiplications and brace operations and specifying appropriate signs, we obtain an operation of the cell operad of cacti and hence a cell model of the little discs operad on the Hochschild cochains of an associative algebra. This proves Deligne's conjecture in any characteristic, i.e. over $\mathbb{Z}$.

Theorem 4.3. *Deligne's conjecture is true for the chain model of the little discs operad provided by $CC_*(\mathcal{C}act^1)$, that is $CH^*(A, A)$ is a dg-algebra over $CC_*(\mathcal{C}act^1)$ lifting the Gerstenhaber algebra structure.*

Moreover, all possible flow charts using multiplication and brace operations are realized by the operations of the cells, and these operations are exactly the set of operations which appear when studying iterations of the bracket and the product on the Hochschild cochains. In this sense our solution to Deligne's conjecture is minimal.

Deligne's conjecture has by now been proven in various ways [1,23,24,29,30,35,39] (for a full review of the history see [32]). The different approaches are basically realized by choosing adequate chain models and some more or less abstract form of homological algebra. The virtue of our approach which is in spirit close to those of [29,24] lies in its naturality and directness. It yields a new topological proof, which is constructive, transparent and economical.

Restricting our attention to the suboperad of the operad of cellular chains of normalized spineless cacti given by symmetric top-dimensional cells $CC_n^{\mathrm{top}}(n)^{\mathbb{S}}$, we obtain a chain model for the operad $\mathcal{G}Pl$ whose algebras are precisely graded pre-Lie algebras. Suitably shifting degrees in this chain operad, we obtain the operad $\mathcal{P}l$ whose algebras precisely are pre-Lie algebras.

Let L^* be a free k (or $\mathbb{Z}$) module generated by an element of degree -1.

Theorem 4.24. *The operad $CC_n^{\text{top}}(n)^{\mathbb{S}} \otimes k$ is isomorphic to the operad $\mathcal{G}Pl$ for graded pre-Lie algebras. Furthermore the shifted operad $(CC_n^{\text{top}} \otimes (L^*)^{\otimes E_w})^{\mathbb{S}}(n) \otimes k$ is isomorphic to the operad $\mathcal{P}l$ for pre-Lie algebras.*

The analogous statements also hold over $\mathbb{Z}$.

The considerations above leading to the operations of the Hochschild cochains are actually of a more general nature. To make this claim precise, we analyze meta-structures on operads and show that their structure naturally leads to pre-Lie algebras, graded pre-Lie algebras, Lie algebras and Hopf algebras. Here the same "flow-chart argument" gives these structures a cell interpretation. For another approach to relations between Hopf algebras and co-operads, see [27,28].

Specializing to the Hopf structure of the operad of the shifted top-dimensional symmetric cells above and taking $\mathbb{S}_n$-coinvariants, we obtain the renormalization Hopf algebra of Connes and Kreimer.

Proposition 6.4. *H_{CK} is isomorphic to the Hopf algebra of $\mathbb{S}_n$ coinvariants of the suboperad of top-dimensional symmetric combinations of shifted cells $((CC_*(\mathcal{C}act^1))^{\text{top}})^{\mathbb{S}} \otimes (L^*)^{\otimes E_w}$ of the shifted cellular chain operad of normalized spineless cacti $CCact \otimes (L^*)^{\otimes E_w}$.*

Our analysis thus unites the pre-Lie definition of the Gerstenhaber bracket in string topology, the arc operad and the original work of Gerstenhaber with the renormalization procedures of Connes and Kreimer in terms of natural operations on operads.

Going beyond the algebraic properties of operads, we prove that any operad with a multiplication is an algebra over the cell model of the little discs operad given by the cellular chains of normalized spineless cacti. This is a generalization of Deligne's conjecture to the operad level and realizes the program of [15] by giving a moduli space interpretation to the brace algebra structure on an operad.

Theorem 5.17. *The generalized Deligne conjecture holds. That is, the direct sum of any operad algebra which admits a direct sum is an algebra over the chains of the little discs operad in the sense that it is an algebra over the dg-operad $CC_*(\mathcal{C}act^1)$.*

The resemblance of the geometric realization of the chains defining the homotopy Gerstenhaber structure and the algebraic calculations of Gerstenhaber [11] is striking, making the case that cacti are the most natural topological incarnation of these operations. In fact, using the translation formalism developed in this paper, the topological homotopies listed in [26] are the exact geometrization of the algebraic homotopies of [11]. As an upshot, the natural boundary map present in the spineless cacti/arc description shows how the associative multiplication is related to the bracket as a degeneration. In an algebraic topological formulation this establishes that the pre-Lie multiplication is basically a $\cup_1$ operation.

It is thus tempting to say that the $\mathcal{A}rc$ operad [26] is an underlying "string mechanism" for all of the above structures.

The paper is organized as follows:

In the first section, we introduce the types of trees we wish to consider and several natural morphisms between them. This is needed to fix our notation and allows us to compare our results with the literature.

In the second section, we recall the definitions of [17] of the different types of cacti. Furthermore, we recall their quasi-operad structure from [17] and the description of spineless cacti as a semi-direct product of the normalized spineless cacti and a contractible so-called scaling operad. Finally, we recall the equivalence of the operad of spineless cacti with the operad of little discs.

In Section 3, we give a cell decomposition for the space of normalized spineless cacti. We also show that the induced quasi-operad structure on the chain level is in fact an operad structure on the cellular chains of this decomposition. This yields our chain model for the little discs operad.

Section 4 contains our new solution to Deligne's conjecture. After recalling the definition of the Hochschild complex and the brace operations, we provide two points of view of the operation of our cellular operad on the Hochschild cochains. One which is close to the operation of the chains of the $\mathcal{A}rc$ operad on itself and also to string topology and a second one which is based on a description in terms of flow charts for substitutions and multiplications among Hochschild cochains.

In the fifth section, we show that the symmetric combinations of these cells yield an operad which is the operad whose algebras are precisely graded pre-Lie algebras. We additionally show that the pre-Lie operad has a natural chain interpretation in terms of the symmetric combinations of the top-dimensional cells of our cell decomposition for normalized spineless cacti. Furthermore, we analyze the situation in which an operad that admits a direct sum also has an element which acts as an associative multiplication. In this setting, we prove a natural generalization of Deligne's conjecture which states that a chain model of the little discs operad acts on such an operad.

In the sixth section, we use our previous analysis to define pre-Lie and Hopf algebras for operads of $\mathbb{Z}$-modules or any operad leading to $\mathbb{Z}$-modules. Applying the Hopf algebra construction to our chain model operad for pre-Lie algebras and then taking $\mathbb{S}_n$-coinvariants we obtain the Hopf algebra of Connes and Kreimer.

In a final short section, we comment on the generalization to the A_∞ case and the cyclic case as well as on new developments.

Notation

We denote by $\mathbb{S}_n$ the permutation group on n letters and by C_n the cyclic group of order n.

We denote the shuffles of two ordered finite sets S and T by $Sh(S,T)$. A shuffle of two finite ordered sets $(S, \prec_S)$ and $(T \prec_T)$ is an order $\prec$ on $S \amalg T$ which respects both the order of S and that of T, i.e. for $t, t' \in T$: $t \prec t'$ is equivalent to $t \prec_T t'$ and for $s, s' \in S$: $s \prec s'$ is equivalent to $s \prec_S s'$. We also denote by $Sh'(S,T)$ the subset of $Sh(S,T)$ in which the minimal element w.r.t. $\prec$ is the minimal element of S. Finally we denote the trivial shuffle in which $s \prec t$ for all $s \in S$ and $t \in T$ by $\prec_S \amalg \prec_T$.

For any element s of an ordered finite set $(S, \prec)$ which is not the minimal element, we denote the element which immediately precedes s by $\prec(s)$. We use the same notation for a finite set with a cyclic order.

We also fix k to be a field of arbitrary characteristic.

We will tacitly assume that everything is in the super setting, that is $\mathbb{Z}/2\mathbb{Z}$ graded. For all formulas, unless otherwise indicated, the standard Koszul rules of sign apply.

We let $\mathcal{S}et, \mathcal{T}op, \mathcal{C}hain, \mathcal{V}ect_k$ be the monoidal categories of sets, topological spaces, free Abelian groups and (complexes of) vector spaces over k and call operads in these categories combinatorial, topological, chain and linear operads, respectively.

1. Trees

In the following trees will play a key role for indexing purposes and in the definition of operads and operadic actions.

1.1. General definitions

Definition 1.1. A graph Γ is a collection $(V(\Gamma), F(\Gamma), \delta : F(\Gamma) \to V(\Gamma), \imath : F(\Gamma) \to F(\Gamma))$ with $\imath^2 = id$ and no fixed points $\forall f \in F(\Gamma) : \imath(f) \neq f$. The set $V(\Gamma)$ is called the set of vertices and the set $F(\Gamma)$ is called the set of flags. We let $E(\Gamma)$ be the set of orbits of $\imath$ and call it the edges of Γ. Notice that δ induces a map $\partial : E(\Gamma) \to V(\Gamma) \times V(\Gamma)$, and that the data $(V(\Gamma), E(\Gamma), \partial)$ defines a CW complex by taking $V(\Gamma)$ to be the vertices or 0-cells and $E(\Gamma)$ to be the 1-cells and using ∂ as the attaching maps. The realization of a graph is the realization of this CW complex.

A tree is a graph whose realization is contractible.

A rooted tree is a tree with a marked vertex.

We call a rooted tree planted if the root vertex lies on a unique edge. In this case we call this unique edge the "root edge".

We usually depict the root of a planted tree τ by a small square, denote the root vertex by $\text{root}(\tau) \in V(\tau)$ and the root edge by $e_{\text{root}(\tau)} \in E(\tau)$.

Notice that an edge e of a graph or a tree gives rise to a set of vertices $\partial(e) = \{v_1, v_2\}$. In a tree the set $\partial e = \{v_1, v_2\}$ uniquely determines the edge e. An orientation of an edge is a choice of order of the two flags in the orbit. An oriented edge is an edge together with an orientation of that edge. On a tree giving an orientation to the edge e defined by the boundary vertices $\{v_1, v_2\}$ is equivalent to specifying the either ordered set (v_1, v_2) or the ordered set (v_2, v_1). If we are dealing with trees, we will denote the edge corresponding to $\{v_1, v_2\}$ just by $\{v_1, v_2\}$ and likewise the ordered edge corresponding to (v_1, v_2) just by (v_1, v_2).

An edge that has v as a vertex is called an adjacent edge to v. We denote by $E(v)$ the set of edges adjacent to v. Likewise we call a flag f adjacent to v if $\delta(f) = v$ and denote by $F(v)$ the set of flags adjacent to v. We call an oriented edge $(f, \imath(f))$ incoming to v if $\delta(\imath(f)) = v$. If $\delta(f) = v$ we call it outgoing.

Of course the oriented edges are in 1–1 correspondence with the flags by identifying $(f, \imath(f))$ with f, but it will be convenient to keep both notions.

An edge path on a graph Γ is an alternating sequence of vertices and edges $v_1, e_1, v_2, e_2, v_3, \ldots$ with $v_i \in V(\Gamma), e_i \in E(\Gamma)$, s.t. $\partial(e_i) = \{v_i, v_{i+1}\}$. Notice that since we define this notion for a general graph, we need to keep track of the vertices and edges.

Definition 1.2. Given a tree τ and an edge $e \in E(\tau)$ one obtains a new tree by contracting the edge e. We denote this tree by τ/e.

More formally let $e = \{v_1, v_2\}$, and consider the equivalence relation $\sim$ on the set of vertices which is given by $\forall w \in V(\tau) : w \sim w$ and $v_1 \sim v_2$. Then τ/e is the tree whose vertices are $V(\tau)/\sim$ and whose edges are $E(\tau) \setminus \{e\}/\sim'$ where $\sim'$ denotes the induced equivalence relation $\{w_1, w_2\} \sim' \{w'_1, w'_2\}$ if $w_1 \sim w'_1$ and $w_2 \sim w'_2$ or $w_2 \sim w'_1$ and $w_1 \sim w'_2$.

1.1.1. Structures on rooted trees

A rooted tree has a natural orientation, toward the root. In fact, for each vertex there is a unique shortest edge path to the root and thus for a rooted tree τ with root vertex root $\in V(\tau)$ we can define the function $N : V(\tau) \setminus \{\text{root}\} \to V(\tau)$ by the rule that

$$N(v) = \text{ the next vertex on unique path to the root starting at } v.$$

This gives each edge $\{v_1, v_2\}$ with $v_2 = N(v_1)$ the orientation $(v_1, N(v_1))$.

We call the set $\{(w, v) \mid w \in N^{-1}(v)\}$ the set of incoming edges of v and denote it by $In(v)$ and call the edge $(v, N(v))$ the outgoing edge of v.

Definition 1.3. We define the arity of v to be $|v| := |N^{-1}(v)|$. The set of leaves V_{leaf} of a tree is defined to be the set of vertices which have arity zero, i.e. a vertex is a leaf if the number of incoming edges is zero. We also call the outgoing edges of the leaves the leaf edges and denote the collection of all leaf edges by E_{leaf}.

CAVEAT: For $v \neq$ root: $|v| = |E(v)| - 1$. That is, $|v|$ is the number of incoming edges, which is the number of adjacent edges minus one. For the root $|v|$ is indeed the number of adjacent edges.

Remark 1.4. For a rooted tree there is also a bijection which we denote by: out : $V(\tau) \setminus \{\text{root}\} \to E(\tau)$. It associates to each vertex except the root its unique outgoing edge $v \mapsto (v, N(v))$.

Definition 1.5. An edge e' is said to be above e if e lies on the edge path to the root starting at the vertex of e' which is farther from the root. The branch corresponding to an edge e is the subtree consisting of all edges which lie above e (this includes e) and their vertices. We denote this tree by $br(e)$.

1.2. Planar trees

Definition 1.6. A planar tree is a pair (τ, p) of a tree τ together with a so-called pinning p which is a cyclic ordering of each of the sets $E(v)$, $v \in V(\tau)$.

1.2.1. Structures on planar trees

A planar tree can be embedded in the plane in such a way that the induced cyclic order from the natural orientation of the plane and the cyclic order of the pinning coincide.

The set of all pinnings of a fixed tree is finite and is a principal homogeneous set for the group

$$\mathbb{S}(\tau) := \times_{v \in V(\tau)} \mathbb{S}_{|v|}$$

where each factor $\mathbb{S}_{|v|}$ acts by permutations on the set of cyclic orders of the set $E(v)$. This action is given by an identification of the symmetric group $\mathbb{S}_{|v+1|}$ with the permutations of the set of the $|v| + 1$ edges of v and then modding out by the subgroup of cyclic permutations which act trivially on the set of induced cyclic orders of $E(v)$: $\mathbb{S}_{|v|} \simeq \mathbb{S}_{|v|+1}/C_{|v|+1}$.

1.2.2. Planted planar trees

Given a rooted planar tree (τ, root) there is a linear order on each of the sets $E(v)$, $v \in V(\tau) \setminus \{\text{root}\}$. This order is given by the cyclic order and designating the outgoing edge to be the smallest element. The root vertex has only a cyclic order, though.

Since the root of a planar planted tree has only one incoming edge and no outgoing one such a tree has a linear order at all of the vertices. Vice versa, providing a linear order of the edges of a root vertex is tantamount to planting a rooted tree by adding a new root edge which induces the given linear order. Therefore these are equivalent pictures and we will use either one of them depending on the given situation.

Furthermore, on such a tree there is an edge path which passes through all the edges exactly twice – once in each direction – by starting at the root going along the root edge and at each vertex continuing on the next edge in the cyclic order until finally terminating in the root vertex. We call this path the outside path.

By omitting recurring elements, that is counting each vertex or edge only the first time it appears, the outside path endows the set $V(\tau) \amalg E(\tau)$ with a linear order $\prec^{(\tau,p)}$. The smallest element is the root edge and the largest element in this order is the root vertex. This order induces a linear order on the subset of vertices $V(\tau)$, on the subset of all edges $E(\tau)$, as well as a linear order on each of the subsets $E(v)$ for all the vertices. In this order on $E(v)$ the smallest element is the outgoing edge. This order will be denoted by $\prec_v^{(\tau,p)}$. We omit the superscript for $\prec^{(\tau,p)}$ if it is clear from the context.

1.2.3. Labelled trees

Definition 1.7. For a finite set S an S-labelling for a tree is an injective map $L : S \to V(\tau)$. An S-labelling of a tree yields a decomposition into disjoint subsets of $V(\tau) = V_l \amalg V_u$ with $V_l = L(S)$. For a planted rooted tree, we demand that the root is not labelled: root $\in V_u$.

An n-labelled tree is a tree labelled by $\bar{n} := \{1, \dots, n\}$. For such a tree we call $v_i := L(i)$.

A fully labelled tree τ is a tree such that $V_l = V(\tau)$.

1.3. Black and white trees

Definition 1.8. A black and white graph (b/w graph) Γ is a graph together with a function $clr : V(\Gamma) \to \{0, 1\}$.

We call the set $V_w(\tau) := clr^{-1}(1)$ the set of white vertices and call the set $V_b(\tau) := clr^{-1}(0)$ the set of black vertices.

By a bipartite b/w tree we understand a b/w tree whose edges only connect vertices of different colors.

A S-labelled b/w tree is a b/w tree in which exactly the white vertices are labelled, i.e. $V_l = V_w$ and $V_u = V_b$.

For a rooted tree we call the set of black leaves the tails.

A rooted b/w tree is said to be without tails if all the leaves are white.

A rooted b/w tree is said to be stable if there are no black vertices of arity 1, except possibly the root.

A rooted b/w tree is said to be fully labelled if all vertices except for the root and the tails are white and labelled.

In a planar planted rooted b/w tree, we require that the root be black.

Definition 1.9. For a black and white bipartite tree, we define the set of white edges $E_w(\tau)$ to be the edges $\{b, N(b)\}$ with $N(b) \in V_w$ and call the elements white edges. Likewise we define $E_b = \{\{w, N(w)\} | N(w) \in V_b\}$ with elements called black edges, so that there is a partition $E(\tau) = E_w(\tau) \amalg E_b(\tau)$. White edges thus point towards white vertices and black edges towards black vertices in the natural orientation towards the root.

Notation 1.10. For a planar planted b/w tree, we understand the adjective bipartite to signify the following attributes:

(1) both of the vertices of the root edge are black, i.e. root and the vertex N^{-1}(root) are black
(2) the tree after deleting the tail vertices and their edges, but keeping the other vertices of these edges, is bipartite otherwise.

The root edge is considered to be a black edge. Also in the presence of tails, all non-white tail edges are considered to be black.

Definition 1.11. For a planar planted b/w bipartite tree τ and a white edge $e = (b, N(b))$ we define the branch of e denoted by $br(e)$ to be the planar planted bipartite rooted tree given as follows.

(1) The vertices and edges are those of the branch of e as defined in Definition 1.5.
(2) The colors of all vertices except $N(b)$ are kept and the color of $N(b)$ is changed to black. This black vertex is defined to be the root.
(3) In the case that the tree τ is also labelled, $br(e)$ is considered to be labelled by the set of labels of its white vertices—we stress that this does not include the root $N(b)$, which is the root of $br(e)$.

1.3.1. Notation I

N.B. A tree can have several of the attributes mentioned above; for instance, we will look at bipartite planar planted rooted trees. To fix the set of trees, we will consider the following notation. We denote by $\mathcal{T}$ the set of all trees and use subscripts and superscripts to indicate the restrictions. The superscript r, pp, nt will mean rooted and planar planted, without tails while the subscripts b/w, bp, st will mean black and white, bipartite, and stable, where bipartite and stable insinuate that the tree is also b/w. For example,

$\mathcal{T}^r$	The set of all rooted trees
$\mathcal{T}^{pp}_{b/w}$	The set of planar planted b/w trees
$\mathcal{T}^{pp}_{bp}$	The set of planar planted bipartite trees
$\mathcal{T}^{pp}_{st}$	The set of planar planted stable b/w trees.

Furthermore we use the superscripts fl for fully labelled trees. For example,

$\mathcal{T}^{r,fl}$ The set of all rooted fully labelled trees.

We furthermore use the notation that $\mathcal{T}(n)$ denotes the n-labelled trees and adding the sub and superscripts denotes the n-labelled trees of that particular type conforming with the restrictions above for the labelling. Likewise $\mathcal{T}(S)$ for a set S are the S-labelled trees conforming with the restrictions above for the labelling. For example,

$\mathcal{T}^{pp}_{b/w}(n)$ The set of planar planted b/w trees with n white vertices which are labelled by the set $\{1, \dots, n\}$.

1.3.2. Notation II

Often we wish to look at the free Abelian groups or free vector spaces generated by the sets of trees. We could introduce the notation Free($\mathcal{T}$, $\mathbb{Z}$) and Free($\mathcal{T}$, k) with suitable super- and subscripts, for the free Abelian groups or vector spaces generated by the appropriate trees. In the case that there is no risk of confusion, we will just denote these freely generated objects again by $\mathcal{T}$ with suitable subscripts and

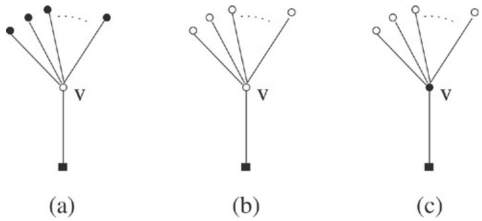

Fig. 1. (a) The n-tail tree l_n, (b) The white n-leaf tree τ_n, (c) The black n-leaf tree τ_n^b.

superscripts to avoid cluttered notation. If we define a map on the level of trees it induces a map on the level of free Abelian groups and also on the k-vector spaces. Likewise by tensoring with k a map on the level of free Abelian groups induces a map on the level of vector spaces. Again, we will denote these maps in the same way.

1.3.3. Notation III

If we will be dealing with operads of trees, we will consider the collection of the $\mathcal{T}(n)$ with the appropriate subscripts and superscripts. Again to avoid cluttered notation when dealing with operads, we also denote the whole collection of the $\mathcal{T}(n)$ just by $\mathcal{T}$ with the appropriate subscripts and superscripts.

Definition 1.12. There are some standard trees, which are essential in our study, these are the n-tail tree l_n, the white n-leaf tree τ_n, and the black n-leaf tree τ_n^b, as shown in Fig. 1.

1.4. Maps between different types of trees

1.4.1. The map cppin : $\mathcal{T}^r \to \mathcal{T}_{bp}^{pp,nt}$

First notice that there is a map from planted trees to rooted trees given by contracting the root edge. This map actually is a bijection between planted and rooted trees. The inverse map is given by adding one additional vertex which is designated to be the new root and introducing an edge from the new root to the old root. We call this map *plant*.

Secondly, there is a natural map *pin* from the free Abelian group of planted trees to that of planted planar trees. It is given by:

$$pin(\tau) = \sum_{p \in \text{Pinnings}(\tau)} (\tau, p).$$

Finally there is a map from planted planar trees to planted planar bipartite trees without tails. We call this map bp. It is given as follows. First color all vertices white except for the root vertex which is colored black, then insert a black vertex into every edge.

In total we obtain a map

$$cppin := bp \circ pin \circ plant : \mathcal{T}^r \to \mathcal{T}_{bp}^{pp,nt}$$

that plants, pins and colors and expands the tree in a bipartite way.

Using the map *cppin*, we will view $\mathcal{T}^r$ as a subgroup of $\mathcal{T}_{bp}^{pp,nt}$. The image of $\mathcal{T}^r$ coincides with the set of invariants of the actions $\mathbb{S}(\tau)$. We will call such an invariant combination a symmetric tree.

Remark 1.13. The inclusion above extends to an inclusion of the free Abelian group of fully labelled rooted trees to labelled bipartite planted planar trees: $cppin : \mathcal{T}^{r,fl}(n) \to \mathcal{T}_{bp}^{pp,nt}(n)$.

1.4.2. The map $st_\infty : \mathcal{T}_{st}^{pp} \to \mathcal{T}_{bp}^{pp}$

We define a map from the free groups of stable b/w planted planar trees to the free group of bipartite b/w planted planar trees in the following way: First, we set to zero any tree which has black vertices whose arity is greater than two. Then, we contract all edges which join two black vertices. And finally, we insert a black vertex into each edge joining two white vertices. We call this map st_∞.

Notice that st_∞ preserves the condition of having no tails and induces a map on the level of labelled trees.

This nomenclature is chosen since this map in a sense precisely forgets the trivial A_∞ structure of an associative algebra in which all higher multiplications are zero and all n-fold iterations of the multiplication agree.

1.5. An operad structure on $\mathcal{T}_{bp}^{pp}$

1.5.1. Grafting planar planted b/w trees at leaves

Given two trees, $\tau \in \mathcal{T}_{bp}^{pp}(m)$, $\tau' \in \mathcal{T}_{bp}^{pp}(n)$ and a white vertex v_i which is a leaf of τ, we define $\tau \circ_i \tau'$ by the following procedure:

First identify the root of τ' with the vertex v_i. The image of v_i and $\mathrm{root}(v_i)$ is taken to be black and unlabelled. The linear order of all of the edges is given by first enumerating the edges of τ in their order until the outgoing edge of v_i is reached, then enumerating the edges of the tree τ' in their order and finally the rest of the edges of τ in their order, the latter being all the edges following the outgoing edge of v_i in the order of τ.

Second contract the image of the root edge of τ', i.e. the image of the edge $e_{\mathrm{root}}(\tau')$ under the identification $v_i \sim \mathrm{root}(\tau')$, and also contract the image of the outgoing edge of v_i, i.e. the image after gluing and contraction of the edge $(v_i, N(v_i))$.

The root of this tree is specified to be the image of the root of τ and the labelling is defined in the usual operadic way. The labels $1, \dots, i-1$ of τ are unchanged, the labels $1, \dots, n$ of τ' are changed to $i, \dots, n+i-1$, and finally the labels $i+1, \dots, m$ of τ are changed to $i+n, \dots, m+n-1$.

1.5.2. Cutting branches

Given a tree $\tau \in \mathcal{T}_{bp}^{pp}(n)$ and a vertex v_i we denote the tree obtained by cutting off all branches at v by $\mathrm{cut}(\tau, v)$. This is the labelled subtree of τ consisting of all edges below (viz. not above) the incoming edges of v and the vertices belonging to these edges. We stress that the outgoing edge $\mathrm{out}(v)$ is a part of this tree as is v. By keeping the labels of the remaining white vertices the tree $\mathrm{cut}(\tau(v))$ becomes an S-labelled planar planted bipartite tree. Here $S \subset \{1, \dots, n\}$ is the set of labels of the subtree under consideration.

Let $br(\tau, v)$ denote the set of the branches of the incoming edges of v which is ordered by the order $\prec$ induced by $\prec_v^\tau$. That is, $br(e_i) \prec br(e_j)$ if and only if $e_i \prec_v^\tau e_j$.

$$br(\tau, v) = (\{br(e_i) | e_i \in In(v_i)\}, \prec).$$

1.5.3. Grafting branches

Fix a tree $\tau \in \mathcal{T}_{bp}^{pp}(S_0)$ and an ordered set of trees $\tau_i' \in \mathcal{T}_{bp}^{pp}(S_i) : i \in \{1, \dots, m\}$. Let $R := (\{e_{\mathrm{root}}(\tau_1'), \dots, e_{\mathrm{root}}(\tau_m')\}, \prec_R)$ be the ordered set in which $e_{\mathrm{root}}(\tau_i) \prec_R e_{\mathrm{root}}(\tau_j)$ if and only if $i < j$.

Set $E := (E(\tau) \setminus \{e_{\text{root}}\}, \prec^\tau)$ and define $v_{\text{white}} : E \to V(\tau)$ to be the map which maps each edge in E to its unique white vertex. Denote the minimal element of E by $e_{\min}$. This is the edge which immediately follows $e_{\text{root}}(\tau)$ in the linear order of τ.

Given a shuffle $\prec \in Sh'(E, R)$ such that the minimal element of the ordered set $(E \amalg R, \prec)$ is $e_{\min}$ we define the grafting of the branches $\tau'_1, \ldots, \tau'_m$ onto τ with respect to $\prec$ to be the labelled b/w tree obtained as follows:

(1) Identify the roots of the τ'_i with the white vertex of the edge immediately preceding the root edge: $\text{root}_{\tau'_i} \sim w = v_{\text{white}}(\prec (e_{\text{root}(\tau')}))$.
(2) Designate the image to be a white vertex with the label $L^{-1}(w)$.
(3) Endow this tree with the planted planar structure induced by the order $\prec$ together with the orders $\prec^{\tau'_i}$ and $\prec^\tau$. The root of this tree is the image of the root of τ.

This tree is again in $\mathcal{T}^{pp}_{bp}$ and is labelled by $S_0 \amalg S_1 \amalg \cdots \amalg S_M$. We denote it by

$$gr(\tau; \tau_1, \ldots, \tau_n; \prec).$$

1.5.4. Signs

For a shuffle of sets with weighted elements $wt_S : S \to \mathbb{N}$ $wt_T : T \to \mathbb{N}$, we define the sign of the shuffle to be the sign obtained from shuffling the elements of T past the elements of S, i.e.

$$\text{sign}(\prec) = \prod_{t \in T} (-1)^{\sum_{s \in S: t \prec s} wt_S(s) wt_T(t)}.$$

We define the weight function by

$$wt(e) = \begin{cases} 1 & \text{if } e \text{ is white} \\ 0 & \text{if } e \text{ is black and } e \text{ is not a root edge} \\ |E_w(\tau)| & \text{if } e \text{ is the root edge of } \tau. \end{cases} \tag{1.1}$$

Definition 1.14. We define the grafting of the τ_i onto τ as branches to be the signed sum over all possible graftings using the weight function (1.1):

$$gr(\tau; \tau_1, \ldots, \tau_n) := \sum_{\prec \in Sh'} \text{sign}(\prec) gr(\tau; \tau_1, \ldots, \tau_n; \prec). \tag{1.2}$$

1.5.5. An operad structure for $\mathcal{T}^{pp}_{bp}$

With the above procedures, we define operadic compositions $\tau \circ_i \tau'$ for $\mathcal{T}^{pp}_{bp}$ as follows: first cut off the branches corresponding to the incoming edges of v_i. Second graft τ' as a planar planted tree onto the remainder of τ at the vertex v_i which is now a leaf according to the grafting procedure of Section 1.5.1. Finally sum over the possibilities to graft the cut off branches onto the white vertices of the resulting planar tree which before the grafting belonged to τ'. Here one only sums over those choices in which the order of the branches given by the linear order at v_i is respected by the grafting procedure. That is, the branches after grafting appear in the same order on the grafted tree as they did in τ.

An example of such an insertion is depicted in Fig. 2.

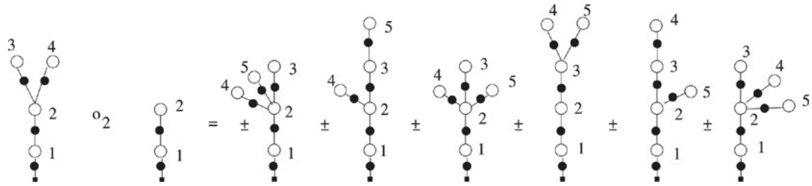

Fig. 2. Example of the insertion of a bipartite planted planar tree.

Definition 1.15. Given $\tau \in \mathcal{T}_{bp}^{pp}(m)$ and $\tau' \in \mathcal{T}_{bp}^{pp}(n)$, we define the tree $\tau \circ_i \tau' \in \mathcal{T}_{bp}^{pp}(m+n-1)$

$$\tau \circ_i \tau' := gr(\text{cut}(\tau, v_i) \circ_i \tau'; br(\tau, v_i)) \tag{1.3}$$

with the following relabelling: the labels $1, \dots, i-1$ of vertices which formerly belonged to τ remain unchanged. The labels of the vertices $1, \dots, n$ of the vertices which formerly belonged to τ' are relabelled $i, \dots, i+n-1$ and the remaining vertices of those which formerly belonged to τ which used to be labelled by $i+1, \dots, m$ are now relabelled by $i+n, \dots, m+n-1$.

1.5.6. Labelling by sets

There is a way to avoid labelling and explicit signs by working with tensors and operads labelled by arbitrary sets [8,24,32]. In this case, if S and S' are the indexing sets for τ and τ' and $i \in S$, then the indexing set of $\tau \circ_i \tau'$ is given by $S \setminus \{i\} \amalg S'$. To obtain the signs, one associates a free $\mathbb{Z}$-module (or k-vector space) generated by an element of degree minus one to each white edge. See Definition 1.24.

1.5.7. Positive signs

We define $\tau \circ_i^+ \tau'$ just as above only with the $wt(e) \equiv 0$, that is as the formal sum with the same summands and only positive coefficients.

1.5.8. Contracting trees

Given a rooted subtree $\tau' \subset \tau \in \mathcal{T}_{bp}^{pp,nt}$ with white leaves and black root, we define τ/τ' to be the tree obtained by collapsing the subtree τ' to one black edge by identifying all white and all black vertices of τ'. That is, let $v \sim_V^{\tau'} v'$ if $v = v'$ or $v, v' \in V(\tau')$ and $clr(v) = clr(v')$ then $V(\tau/\tau')/\sim_V^{\tau'}$. Likewise let $e \sim_E^{\tau'} e'$ if $e, e' \in E(\tau')$ or $e = e'$ and set $E(\tau/\tau') = E(\tau)/\sim_E^{\tau}$.

For any tree labelled tree $\tau \subset \mathcal{T}_{bp}^{pp,nt}(n)$ with labelling L, we set τ^{+i} to be the same underlying tree, but with the shifted labelling function $L^{+i} : \{i, \dots, i+n-1\} \to E_w(\tau)$ given by $L^{+i}(k) = L(k-i+1)$.

By abuse of notation we use τ' to denote a subtree $\tau' \subset \tau$ and the planted planar tree obtained from τ' by planting the root vertex while preserving the linear order already present on the original edges of τ'. Vice versa, given a planted tree τ, when identifying it with a subtree the root edge will be contracted, but the linear order of the subtree has to coincide with that of the tree considered as "free standing".

Consider $\tau' \in \mathcal{T}_{bp}^{pp,nt}(m)$. If $\tau'^{+i} \subset \tau \in \mathcal{T}_{bp}^{pp,nt}(m+n-1)$ as a labelled subtree, we write $\tau' \subset_i \tau$. In this case we label τ/τ' as follows. The labels $1, \dots, i-1$ remain unchanged, the label of the vertex representing the white vertices of the contracted τ' is set to be i and the labels $i+m, \dots, n+m-1$ are changed to $i+1, \dots, n$. Set

$$T(\tau, \tau', i) := \{\tilde{\tau} | \tau' \subset_i \tilde{\tau} \text{ and } \tilde{\tau}/\tau' = \tau\}. \tag{1.4}$$

Remark 1.16. With the above definitions, we can rewrite Eq. (1.3) as

$$\tau \circ_i \tau' := \sum_{\tilde{\tau} \in T(\tau,\tau',i)} \mathrm{sign}(\prec^{\tilde{\tau}})\tilde{\tau} \tag{1.5}$$

where $\prec^{\tilde{\tau}}$ is considered as the shuffle $(E_w(\tilde{\tau}) = E_w(\tau) \amalg E_w(\tau'), \prec)$ of the ordered sets $(E_w(\tau), \prec^\tau)$ and $(E_w(\tau'), \prec^{\tau'})$. Notice that the compatibility of the orders is automatic.

Proposition 1.17. *The gluing maps (1.3) (with or without signs) together with the symmetric group actions permuting the labels turn $\mathcal{T}_{bp}^{pp} := \{\mathcal{T}_{bp}^{pp}(n)\}$ into an operad.*

Proof. This is a straightforward calculation especially in view of the reformulation of Remark 1.16. An alternative topological way to prove the associativity of $\circ_i^+$ is given in Corollary 3.9. The signs for the maps $\circ_i$ then follow from Section 1.5.10 using Definition 1.24. □

Remark 1.18. With the above compositions $\mathcal{T}_{bp}^{pp,nt}$ is a suboperad of $\mathcal{T}_{bp}^{pp}$.

1.5.9. The differential on $\mathcal{T}_{bp}^{pp,nt}$

There is a differential on $\mathcal{T}_{bp}^{pp,nt}$ which we will now define in combinatorial terms. We will show later that it has a natural interpretation as the differential of a cell complex.

Recall that for a planted planar tree there is a linear order on all edges and therefore a linear order on all subsets of edges.

Definition 1.19. Let $\tau \in \mathcal{T}_{bp}^{pp,nt}$. We set $E_{\mathrm{angle}} = E(\tau) \setminus (E_{\mathrm{leaf}}(\tau) \cup \{e_{\mathrm{root}}\})$ and we denote by $\mathrm{num}_E : E_{\mathrm{angle}} \to \{1, \ldots, N\}$ the bijection which is induced by the linear order $\prec^{(\tau,p)}$.

Definition 1.20. Let $\tau \in \mathcal{T}_{bp}^{pp,nt}$, $e \in E_{\mathrm{angle}}$, $e = \{w, b\}$, with $w \in V_w$ and $b \in V_b$. Let $e- = \{w, b-\}$ be the edge preceding e in the cyclic order $\prec_w^\tau$ at w. Then $\partial_e(\tau)$ is defined to be the planar tree obtained by collapsing the angle between the edge e and its predecessor in the cyclic order of w by identifying b with $b-$ and e with $e-$. Formally

$$w = v_{\mathrm{white}}(e), \qquad e- = \prec_w^\tau(e), \qquad \{b-\} = \partial(e-) \cap V_b(\tau)$$
$$V_{\partial_e(\tau)} = V(\tau)/(b \sim b-), \qquad E_{\partial_e(\tau)} = E_\tau/(e \sim e-).$$

The linear order of $\partial_e(\tau)$ is given by keeping the linear order at all vertices which are not in the image $\bar{b}$ of b and $b-$ and using the linear order

$$(In(\bar{b}), \prec_{\bar{b}}^{\partial_e(\tau)}) = (In(b-) \amalg In(b), \prec_{b-}^\tau \amalg \prec_b^\tau)$$

extended to $E(\bar{b})$ by declaring the image of e and $e-$ to be the minimal element. See Fig. 3 for an example.

Definition 1.21. We define the operator ∂ on the space $\mathcal{T}_{bp}^{pp,nt}$ to be given by the following formula

$$\partial(\tau) := \sum_{e \in E_{\mathrm{angle}}} (-1)^{\mathrm{num}_E(e)-1} \partial_e(\tau). \tag{1.6}$$

Denote by $\mathcal{T}_{bp}^{pp,nt}(n)^k$ the elements of $\mathcal{T}_{bp}^{pp,nt}(n)$ with k white edges.

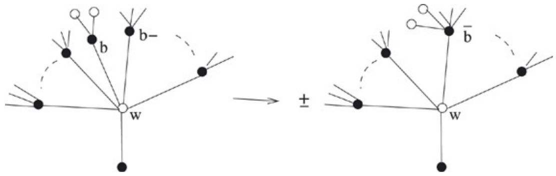

Fig. 3. The tree $\partial_e(\tau)$.

Proposition 1.22. *The map* $\partial : \mathcal{T}_{bp}^{pp,nt}(n)^k \to \mathcal{T}_{bp}^{pp,nt}(n)^{k-1}$ *is a differential for* $\mathcal{T}_{bp}^{pp,nt}$ *and turns* $\mathcal{T}_{bp}^{pp,nt}$ *into a differential operad.*

Proof. The fact that ∂ reduces the number of white edges by one is clear. The fact that $\partial^2 = 0$ follows from a straightforward calculation. Collapsing two angels in one order contributes negatively with respect to the other order. The compatibility of the multiplications $\circ_i$ is also straightforward. □

1.5.10. Other choices of signs

The way the signs are fixed in the above considerations is by giving the white edges the weight 1 and the black vertices the weight 0. Once the order of the edges for trees is fixed all signs are the standard signs obtained from permuting weighted (i.e. graded) elements. We chose the "natural" order in which the edges are enumerated with respect to $\prec^\tau$, i.e. the order derived from the embedding into the plane.

Another choice of ordering would be "operadic" in which the white edges are enumerated first according to the label of their incident white vertex and then according to their linear order at that vertex. We leave it to the reader to make the necessary adjustments in the formulae (1.2), (1.3) and (1.6) to adapt the signs to this choice.

Finally one can avoid explicitly fixing an order if one works with operads over arbitrary sets (see also Section 1.5.6).

1.5.11. Other tree insertion operads and compatibilities

There are two tree insertion operads structures already present in the literature on rooted trees $\mathcal{T}^r$ [5, 10], or to be more precise on $\mathcal{T}^{r,fl}$ and, historically the first, on planar planted stable b/w trees without tails $\mathcal{T}_{b/w}^{pp,st,nt}$ [24].

In the gluing for $\mathcal{T}^{r,fl}$ one simply omits mention of the order. And in the case of $\mathcal{T}_{b/w}^{pp,st,nt}$ one also allows gluing to the images of the black vertices of τ'. Also in the case of $\mathcal{T}^{r,fl}$ the basic grafting of trees is used (no contractions), while in the case of $\mathcal{T}_{b/w}^{pp,st,nt}$ the grafting for planted trees is used, i.e. the image of the root edge is contracted, but not the outgoing edge of v_i.

The signs for the first gluing are all plus [5] and in the second gluing they are given by associating the weight 1 to all edges except the root and weight −2 to the white vertices and weight 2 to the root vertex. The latter of course do not contribute to the signs.

Proposition 1.23. *The map* $st_\infty : \mathcal{T}_{b/w}^{pp,st,nt} \to \mathcal{T}_{bp}^{pp,nt}$ *is an operadic map. Moreover, it is a map of differential graded operads, if one reverses the grading of* $\mathcal{T}_{bp}^{pp,nt}$*, i.e. defining* $(\mathcal{T}_{bp}^{pp,nt})(n)^k$ *to have degree* $-k$*.*

Proof. The operadic properties of st_∞ over $\mathbb{Z}/2\mathbb{Z}$ follow from the fact that the summands given by gluing branches to black vertices in the composition of [24] are sent to zero by the map st_∞. In the calculation for the differential the unwanted terms disappear by the same argument. The compatibility of signs follows from the fact that the trees that survive st_∞ have trivalent black subtrees. These have an odd number of edges counting all the edges incident to the black vertices. Let the outgoing edge be the outgoing edge of the lowest vertex of the subtree. Collapsing the subtree then corresponds to assigning an odd weight to the outgoing edge and weight zero to the incoming edges, since there is no gluing to black vertices. In the case of edges connecting two white vertices, one inserts a black vertex and the weight can be transferred to the white edge, again since there is no gluing onto black vertices. Now the parity of the induced weights on the edges coincides with the weights we defined on $\mathcal{T}_{bp}^{pp,nt}$. The weights of the vertices being even play no role. Hence the signs agree. In fact taking the signs of the vertices into account, we would exactly obtain weight -1 for white edges and weight 0 for black ones. The compatibility of the degrees follows from the fact that in $\mathcal{T}_{b/w}^{pp,st,nt}$ a black vertex b will contribute degree $-(|b|-2)$ and a white vertex w will contribute degree $-|w|$ to the total degree. Thus in the case in which the tree has only binary black vertices the total degree is the sum over the $-|w|$ which is the negative of the grading in $\mathcal{T}_{bp}^{pp,nt}$. □

Definition 1.24. We define the shifted complex $S^+\mathcal{T}_{b/w}^{pp,st,nt}$ to be given by $S^+\mathcal{T}_{b/w}^{pp,st,nt}(n) := \mathcal{T}_{b/w}^{pp,st,nt}(n) \otimes L^{\otimes E_w}$ where L is a freely generated k (or $\mathbb{Z}$-module) generated by an element l of degree 1 and we used the notation of tensor products indexed by sets. This means that generators are given by $\tau \otimes l^{\otimes E_w(\tau)}$. The differential is the tree differential without signs on the trees which just collapses the angles and the multiplications on the component of trees are the $\circ_i^+$. Now the signs come from the tensor factors and their permutations as induced by the maps Section 1.5.6 and $L^{\otimes E_w(\tau)} \to L^{\otimes E_w(\partial_e(\tau))}$. Here the latter map can be induced by the multiplication map $\mu : L_e \otimes L_{e-} \to L_{\bar{e}}$ defined by $\mu(l \otimes l) = l$.

It is clear that there is an operadic isomorphism $S^+\mathcal{T}_{bp}^{pp,nt} \simeq (\mathcal{T}_{bp}^{pp,nt}, \circ_i)$.

Notice that if we shift back with the dual line L^*, i.e. set $S\mathcal{T}_{bp}^{pp,nt} = \mathcal{T}_{bp}^{pp,nt} \otimes (L^*)^{\otimes E_w}$ where now the compositions on the tree factor are given by $\circ_i$, then there is an operadic isomorphism $S\mathcal{T}_{bp}^{pp,nt} \simeq (\mathcal{T}_{bp}^{pp,nt}, \circ_i^+)$.

In an analogous way we define $\mathcal{T}^{r,fl} \otimes L^{\otimes E}$.

Proposition 1.25. *The map cppin* : $\mathcal{T}^{r,fl} \to \mathcal{T}_{bp}^{pp,nt}$ *is injective and its image are the symmetric combinations of the trees of* $\mathcal{T}_{bp}^{pp,nt}$ *with* $|E_w| = |V_w| - 1$ *which we denote by* $((\mathcal{T}_{bp}^{pp,nt})^{\mathrm{top}})^{\mathbb{S}}$. *Moreover*

(a) *cppin* : $\mathcal{T}^{r,fl} \to \mathcal{T}_{bp}^{pp,nt}$ *is an operadic embedding into the operad* $(\mathcal{T}_{bp}^{pp,nt}, \circ_i^+)$. *This corresponds to assigning weight zero to all edges in the formalism above.*
(b) *The map cppin also induces an operadic embedding of* $\mathcal{T}^{r,fl} \otimes L^{\otimes E}$ *into the* $(\mathcal{T}_{b/w}^{pp,st,nt}, \circ_i)$, *via*

$$\mathcal{T}^{r,fl} \otimes L^{\otimes E} \xrightarrow{cppin \otimes id^{\otimes |E(\tau)|}} \mathcal{T}_{bp}^{pp,nt} \otimes L^{\otimes E_w} = S^+\mathcal{T}_{bp}^{pp,nt}.$$

We will also denote this map by cppin.

Proof. For rooted trees the composition defined in [5] tells us to cut off the branches, glue the second tree to the truncated tree and redistribute the branches. Now the map *cppin* associates to each tree a sum whose summands are uniquely determined by a linear order on the underlying rooted tree, which is obtained by forgetting the linear order, contracting the root edge and all black edges. The possible linear orders on the composed tree are naturally in a 1–1 correspondence between the linear orders on the truncated tree, the tree which is grafted on and a compatible order of the re-grafted branches. These are the terms appearing in the gluing of the planar planted trees. Before and after the embedding, the symmetric group actions produces no signs. The second statement follows clearly from the first. □

2. Species of cacti and their relations to other operads

2.1. Spineless cacti

In this section, we review the spaces of spineless cacti, normalized spineless cacti, their relation to each other and the little discs operad. These spaces were introduced in [17,26], but in order to facilitate the reading we reiterate one definition of the sets corresponding to spineless cacti and normalized spineless cacti and define operadic respectively quasi-operad maps on these sets. In order to not disrupt the flow of the paper too much, we relegate some of the technical details to the Appendix. To be completely self-contained, we also define a topology on these sets which makes spineless cacti into an operad of spaces and normalized spineless cacti into a homotopy associative quasi-operad. After this we briefly recall other equivalent ways of giving a topology.

2.1.1. Background on spineless cacti and cacti

In [17] we introduced the operad of spineless cacti. It is a suboperad of the operad of cacti which was introduced by Voronov in [40] descriptively as treelike configurations of circles in the plane. One precise definition of the topology of cacti was given in [26] via an operadic embedding of cacti into the operad $\mathcal{DA}rc$ which is a deprojectivized version of the $\mathcal{A}rc$ operad [26]. By definition $\mathcal{DA}rc = \mathcal{A}rc \times \mathbb{R}_{>0}$. It carries the obvious action of $\mathbb{R}_{>0}$ acting freely on the right factor. This action is a global re-scaling. The spaces $\mathcal{A}rc(n)$ of the $\mathcal{A}rc$ operad are open subsets of a cell complex [26]. This realizes cacti as a subset of a cell complex crossed with $\mathbb{R}_{>0}$ whence it inherits the subspace topology. Equivalently, using the map $\mathcal{L}oop$ of [26] one arrives at a description of the set of cacti as marked treelike ribbon graphs with a metric. Again, one obtains a subspace topology from that of marked metric ribbon graphs. A marking is the choice of a point on each cycle. In this setting treelike means that the ribbon graph is of genus zero and that there is one marked cycle which passes through all of the unoriented edges. A very brief summary is given below. A short but detailed summary of both these constructions is given in Appendix B of [17].

2.1.2. Background on the normalized versions

In [17] we also introduced normalized spineless cacti and normalized cacti. The condition of normalization means in the description via configurations of circles that all the circles are of length one. The normalized versions are homotopy equivalent to the non-normalized versions as spaces. They are not operads, but they can be endowed with a quasi-operad structure which is quasi-isomorphic to the operad structure of the non-normalized version, i.e. it agrees with the operad structure of cacti respectively spineless cacti on the level of homology [17].

The main result of the next section is that there is a cell decomposition for normalized spineless cacti, such that the induced quasi-operad structure on the level of cellular chains is an operad structure, and hence gives an operadic chain model for spineless cacti.

We start by recalling the definition of spineless cacti and then recast their definition in terms of trees which allows us to give a new cell decomposition.

2.2. Spineless cacti as treelike ribbon graphs

We recall from [17,19] the definition of normalized spineless cacti and spineless cacti in terms of treelike ribbon graphs.

2.2.1. Ribbon graphs

A ribbon graph is a graph Γ together with a cyclic order $\prec_v$ of each of the sets $F(v)$. On a ribbon graph, there is a natural map N from flags to flags given by associating to a flag f the flag following $\imath(f)$ in the cyclic order $\prec_{\partial(\imath(f))}$. The orbits of the map N are called the cycles. We say that a vertex lies on a cycle, if there is a flag in the cycle which is adjacent to this vertex.

If a graph is a ribbon graph, the knowledge of the map N is equivalent to the knowledge of the cyclic orders $\prec_v$, since the successor of a flag f is given by $N(\imath(f))$.

Every ribbon graph has a genus which is defined by $2-2g = \#\text{vertices} - \#\text{edges} + \#\text{cycles}$.[1]

A *metric* on a graph is a function $\mu : E(\Gamma) \to \mathbb{R}_{\geq 0}$. A graph with a metric is called a metric graph.

Definition 2.1. A marked spineless treelike ribbon graph is a ribbon graph of genus 0 together with a distinguished flag f_0, such that

(i) if c_0 is the unique cycle that contains the flag f_0, then for each flag f either $f \in c_0$ or $\imath(f) \in c_0$ and
(ii) if $v_0 = \delta(f_0)$, then $|F(v_0)| \geq 2$ and for $v \neq v_0 : |F(v)| \geq 3$.

We will fix the notation c_0 for the cycle containing f_0 and call it the distinguished cycle. Likewise we fix the notation $v_0 := \delta(f_0)$ and call it the root or global zero.

2.2.2. Spineless cacti

Cacti without spines are the set of metric marked treelike ribbon graphs.

Definition 2.2. We define $\mathcal{C}act(n)$ to be the set of metric marked treelike ribbon graphs which have $n+1$ cycles together with a labelling of the cycles by $\{0, \dots, n\}$ such that c_0 is labelled by 0. We call the elements of $\mathcal{C}act(n)$ spineless cacti with n lobes.

2.2.3. $\mathbb{S}_n$-action

Notice that $\mathbb{S}_n$ acts via permuting the labels $\{1, \dots, n\}$.

2.2.4. Scaling

There is an action of $\mathbb{R}_{>0}$ on $\mathcal{C}act(n)$ which simply scales the metric. That is if μ is the metric of the cactus, the metric scaled by $\lambda \in \mathbb{R}_{>0}$ is defined by $(\lambda\mu)(e) = \lambda\mu(e)$.

2.2.5. Cactus terminology

Since the notion of cacti comes with a history, we set up the usual terminology that is used in the literature to describe these objects. Given a spineless cactus with n lobes, we use the alternate name

[1] It is well known that a ribbon graph can be fattened to a surface with boundary, such that it is the spine of this surface and the cycles correspond to the boundary components. The genus is the genus of the corresponding surface without boundary obtained by contracting the boundaries to points (or equivalently gluing in discs).

"arc" for "edge" and call v_0 the root. Also, we will use the terminology "special points" for the vertices and call the vertices v with $|F(v)| \geq 3$ the intersection points. Sticking with this theme the arc length of an arc e of a metric spineless treelike ribbon graph will be simply $\mu(e)$ where μ is the metric. The lobes will be the cycles labelled by $\{1, \dots, n\}$ and the cycle c_0 will be called the perimeter or outside circle. The length or radius of a lobe or the perimeter is the sum of the lengths of the underlying edges of the oriented edged belonging to the cycle.

2.3. Lobes and base points

Notice that in a spineless cactus, every cycle has a distinguished flag. For this enumerate the flags of c_0 starting with f_0. Now for each cycle c_i other that c_0, there is a first flag f of this linearly ordered set c_0 such that $\imath(f)$ is an element of the cycle c_i. The distinguished flag is then defined to be flag $N(\imath(f))$. Notice that this flag and f share the same vertex, the distinguished flag of the cycle. In cactus terminology these vertices are called the local zeros and $v_0 = \delta(f_0)$ is sometimes referred to as the global zero.

Also notice that the cycle c_0 gives a linear order to all the edges of a cactus. This linear order is given by fixing the edge $\{f_0, \imath(f_0)\}$ to be the first edge. In the same fashion, there is a linear order on all edges belonging to the other cycles by letting the first edge be the one containing the distinguished flag discussed above.

2.3.1. Dual black and white graph

Given a marked spineless treelike metric ribbon graph, we can associate to it a dual graph. This is a graph with two types of vertices, white and black. The first set of vertices is given by replacing each cycle except c_0 by a white vertex. The second set of vertices, the set of black vertices, is given by the vertices of the original graph. The set flags of the dual graph is taken to be equal to the set of flags of the original graph. The maps $\imath$ and δ are defined in such a way, that the edges run only from white to black vertices, where we two such vertices are joined if the vertex of the ribbon graph corresponding to the black vertex lies on the cycle represented by the white vertex. See the Appendix for the precise combinatorial description of this construction.

Notice that this is a planted planar bipartite tree. It is a planar tree, since the underlying ribbon graph had genus zero, and the pinning is the one induced by the cyclic order of the flags of the original graph (see Appendix). It is planted by defining the linear order at $v_0 := \partial(f_0)$ by letting f_0 be the first flag. Moreover, the set of edges of this tree is in 1–1 correspondence with the set of edges of the ribbon graph and the two enumerations of the edges by the distinguished cycles agree. For details, see the Appendix.

Definition 2.3. The topological type of a spineless cactus in $\mathcal{C}act^1(n)$ is defined to be the tree $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$ which is its dual b/w graph together with the labelling induced from the labels of the cactus and the linear order induced on the edges by the embedding into the plane and the position of the root.

For an example of a spineless cactus and its topological type, see Fig. 4.

2.4. The marked treelike spineless ribbon graph obtained from a tree

The inverse construction to the dual graph is combinatorially a little tricky and it can be found in the Appendix.

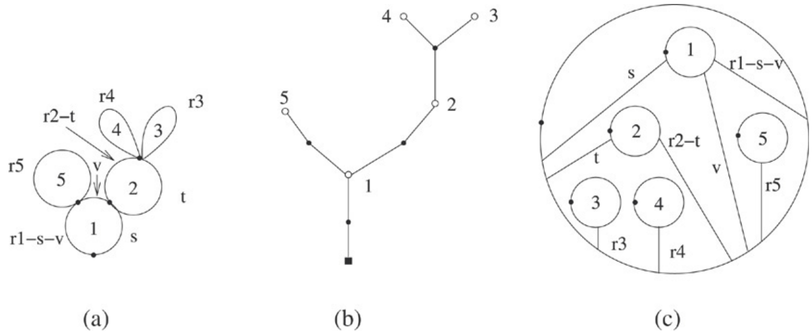

Fig. 4. (a) A cactus without spines with its arc lengths. (b) Its topological type as a labelled planted planar bipartite tree. (c) Its depiction as a surface with boundary and arcs in $\mathcal{DA}rc$.

A good geometric picture is the following. Given a b/w planar planted bi-partite tree τ. Draw the tree in the plane, where we do not draw a root edge, but rather mark the first flag at the root vertex. The first part of the construction is to expand each white vertex to a circle. This can be done, since we have a cyclic order at each edge. This graph has twice the number of edges as τ, the new edges and the edges of τ. It is a genus zero ribbon graph and the cycle of f_0 contains all edges. The new edges are in 1–1 correspondence with the edges of τ, by assigning to each edge of τ the edge which is the next edge in the cyclic order given by c_0. To obtain the marked ribbon graph contract all the edges which were formerly the edges of τ. We choose a marking by fixing the flag which is the successor of the image of the originally marked flag in the unique cycle of this graph. The result of this operation is a marked spineless ribbon graph whose topological type is that of τ and whose edges are in 1–1 correspondence with those of τ. The precise graph theoretic proof of these statements can be found in the Appendix.

Lemma 2.4. *A spineless cactus $c \in \mathcal{C}act(n)$ is uniquely determined by its topological type $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$ (as defined in Definition 2.3) and the lengths of its arcs. Moreover the arcs are in 1–1 correspondence to the edges of it topological type.*

Proof. By the above results, it follows that each spineless cactus gives rise to the corresponding data. Vice versa, given $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$, it gives rise to a marked spineless ribbon graph with both constructions being inverses. Since the set of edges of both the ribbon graph and the tree can be identified, the claim follows. □

2.4.1. Lobes as circles

To simplify the exposition, we now explain how to think of a cycle as a parameterized circle. Each cycle is a sequence of arcs with a given length. So by simply combining these arcs and viewing the result as a CW complex, we obtain a CW decomposition of S^1 and a metric on this space. In effect the radius of this metric S^1 is the radius of the lobe or the outside circle. Moreover there is a distinguished point on this S^1 given by the vertex of the distinguished flag and there is an orientation induced by the order of the cycle. Thus we can think of a cycle as a parameterized S^1 of radius r where r is the length of the cycle. The technical purely combinatorial version of this construction is relegated to the Appendix.

A realization of a cactus with n lobes as a CW complex can then be thought of as a topological space with $n+1$ maps ϕ_i from $S^1_{r_i}$ to this space, so that the i-th map is bijective onto the i-th lobe for $i > 0$

and r_i is the radius of the i-th lobe. For $i = 0$ the map is surjective. In this way we can think of a cactus in $\mathcal{C}act(n)$ as a collection of $S^1_{r_i}$'s which intersect in a treelike fashion.

Moreover, removing the special points, we are left with a 1-manifold with many components, one for each edge, which is naturally a subset of the realization of the cactus. We can think of a lobe as the closure of its components which presents it as a metric space.

2.4.2. Gluing for cacti without spines

In this section, we recall how to define the operations

$$\circ_i : \mathcal{C}act(n) \times \mathcal{C}act(m) \to \mathcal{C}act(n+m-1). \tag{2.1}$$

In plain words: given two cacti without spines let r_i be the length of the i-th lobe of the first cactus. First scale, such that the outside circle of the second cactus has length r_i, then glue in the second cactus by identifying the outside circle of the second cactus with the i-th circle of the first cactus.

This gluing is naturally understood as a gluing of the CW complexes. For this realize both graphs, then consider the i-th lobe and the outside circle as a S^1s as in Section 2.4.1 and then glue the two CW complexes by identifying points using the maps of S^1 to both spaces as gluing data. The topologically most satisfying way to define the gluing is via the Arc operad as it is done in [26]. A detailed recollection of these results would be too much of a detour, however, to be self-contained, we give the somewhat technical combinatorial description corresponding to the graph theoretic definition of spineless cacti in the Appendix.

It is easily checked that the operations (2.1) turn $\mathcal{C}act = \{\mathcal{C}act(n)\}$ into an operad [17,26], albeit for the moment an operad of sets. We will deal with the topology shortly.

2.4.3. Normalized cacti without spines

Normalized cacti without spines are the subset of spineless cacti whose lobes all have length one.

Definition 2.5. The set of normalized cacti without spines with n lobes is the subset $\mathcal{C}act^1(n) \subset \mathcal{C}act(n)$ of spineless cacti with n lobes all of whose length is 1.

2.4.4. Gluing for normalized cacti without spines

We define the operations

$$\circ_i : \mathcal{C}act^1(n) \times \mathcal{C}act^1(m) \to \mathcal{C}act^1(n+m-1) \tag{2.2}$$

basically by the following procedure: given two normalized cacti without spines we re-parameterize the i-th component circle of the first cactus to have length m and glue in the second cactus by identifying the outside circle of the second cactus with the i-th circle of the first cactus. The rigorous combinatorial gluing operation is in the Appendix.

Notice that this gluing differs from the one for spineless cacti, since now just a lobe and not a whole cactus is re-scaled.

These maps do not endow the normalized spineless cacti with the structure of an operad, since they are not associative. But they induce the slightly weaker structure of a quasi-operad.

Definition 2.6 (*[17]*). A quasi-operad is an operad where the associativity need not hold. An operad in the category of topological spaces is called homotopy associative if the compositions are associative up to homotopy.

2.5. Scaled cacti and other variations

2.5.1. Projective cacti

Spineless cacti come with a universal scaling operation of $\mathbb{R}_{>0}$ which simultaneously scales all radii by the same factor $\lambda \in \mathbb{R}_{>0}$. This action is a free action and the gluing descends to the quotient by this action. We call the resulting operad projective spineless cacti and denote its spaces by $\mathbb{P}\mathcal{C}act(n) := \mathcal{C}act(n)/\mathbb{R}_{>0}$.

It is clear that $\mathcal{C}act(n) = \mathbb{P}\mathcal{C}act(n) \times \mathbb{R}_{>0}$.

Remark 2.7. Notice that the topological type as defined in Definition 2.3 is invariant under the global re-scaling action of $\mathbb{R}_{>0}$ so that a projective spineless cactus also has a well defined topological type.

2.5.2. Left, right and symmetric cacti operads

For the gluing maps (2.1) one has three basic possibilities to scale in order to make the size of the outer loop of the cactus that is to be inserted match the size of the lobe into which the insertion should be made.

(1) Scale down the cactus which is to be inserted. This is the original version—we call it the right scaling version.
(2) Scale up the cactus into which will be inserted. We call it the left scaling version.
(3) Scale both cacti before gluing. Let r_i be the length of the lobe i of c and let $R = \sum_j r'_j$ be the length of the outside circle of c'. Now to define $c \circ_i c'$ first scale c by R and c' by r_i. Then identify the outside loop of c' with the lobe i of c which now both have length Rr_i. We call this the symmetric scaling version.

All of these versions turn out to be homotopy equivalent. In particular passing to the quotient operad $\mathbb{P}\mathcal{C}act$ they all descend to the same operations.

The advantages of the different versions are as follows: version (1) is the original one and inspired by the re-scaling of loops, i.e. the size of the outer loop of the first cactus is constant. Version (2) has the advantage that cacti whose lobes have integer sizes are a suboperad. And version (3) is the one which needs to be used to embed into the operad $\mathcal{DA}rc$ of [26]. In this version there is an embedding of the operad of spineless cacti operad into the cyclic operad $\mathcal{DA}rc$. Projective spineless cacti embed into the cyclic operad $\mathcal{A}rc$. The equivalent statements for the larger operad of cacti which contains the operad of spineless cacti under consideration as a suboperad also hold true.

2.6. The topology

In this section, we give a short account of how to put a topology on the set of spineless cacti. As discussed above, the quickest way would be to give the topology to the spaces $\mathcal{C}act^1$ as subspaces of the operad $\mathcal{DA}rc$ as defined in [26], see also [17]. We will also give an equivalent way to define the topology in the next section by identifying normalized spineless cacti with a CW complex. This description is intrinsic and the most adapted to cacti. This is the reason why we decided to define the topology for spineless cacti and cacti using the CW complex approach of this article in [17] rather than using one of the other equivalent descriptions. Endowed with a topology spineless cacti become a topological operad and normalized spineless cacti a homotopy associative topological quasi-operad.

2.6.1. Cacti glued from open cells

For definiteness, we give one construction of the topology which is tantamount to defining the topology of $\mathbb{P}\mathcal{C}act$ as an open subset of $\mathcal{A}rc$ without referring the reader to [26].

Notation 2.8. Let Δ^n denote the standard n-simplex and $\dot{\Delta}^n$ its interior.

For $\tau \in \mathcal{T}_{bp}^{pp,nt}$, set $S(\tau) = \Delta^{|E_w(\tau)|}$ and denote its interior by $S(\dot{\tau})$.

Lemma 2.9. *The projective spineless cacti of a fixed topological type τ are in bijection with points of the interior $S(\dot{\tau})$ of the simplex $S(\tau)$. Moreover*

$$\mathbb{P}\mathcal{C}act(n) = \coprod_{\tau \in \mathcal{T}_{bp}^{pp,nt}(n)} S(\dot{\tau}).$$

Proof. First notice that in each class of a projective spineless cactus there is a unique representative whose arc lengths sum up to one. Using barycentric coordinates on the simplex $S(\tau)$ we can thus identify the projective spineless cacti of the given topological type τ with points in the open simplex $S(\dot{\tau})$. Hence the first bijection follows from Lemma 2.4. It is clear that every topological type occurs, so the second statement follows. □

Definition 2.10. We define the degeneration of a spineless cactus c with respect to an arc a which is not an entire lobe to be the spineless cactus obtained by contracting the arc a. If the root was on the boundary of a, the image of a is the new root. The marked lobe is the image of the lobe to which the arc immediately following a around the outside circle belonged.

Remark 2.11. Notice that if τ is the topological type of a spineless cactus c and $e = \{w, b\}$ is the edge corresponding to the arc a in the terminology of Section 2.4 then the topological type of the degeneration of the spineless cactus c with respect to the arc a is $\partial_e(\tau)$.

Definition 2.12. We give $\mathbb{P}\mathcal{C}act(n)$ the topology induced by identifying the e-th open face of $S(\tau)$ with $S(\dot{\partial_e}(\tau))$ for any $e \in E_{\text{angle}}(\tau)$.

We define the topology for $\mathcal{C}act(n) = \mathbb{P}\mathcal{C}act(n) \times \mathbb{R}_{>0}$ to be the product topology and endow $\mathcal{C}act^1(n) \subset \mathcal{C}act(n)$ with the subset topology.

This identifies the e-th open face of $S(\tau)$ with the degenerations of spineless cacti in $S(\tau)$ with respect to the arc a that corresponds to e.

Lemma 2.13. *As topological spaces $\mathcal{C}act(n) = \mathcal{C}act^1(n) \times \mathbb{R}^n_{>0}$ and $\mathbb{P}\mathcal{C}act(n) = \mathcal{C}act(n)/\mathbb{R}_{>0}$.*

Proof. The first statement follows by identifying the factors $\mathbb{R}^n_{>0}$ with the sizes of the lobes. The second statement follows from the observation that the $\mathbb{R}_{>0}$ action is free and continuous. □

Proposition 2.14. *The gluing maps (2.1) endow the spaces $\mathcal{C}act(n)$ with the structure of a topological operad.*

Proof. Straightforward. □

Proposition 2.15 (*[17]*). *The glueings of Section 2.4.4 together with the permutation action of $\mathbb{S}_n$ on $\mathcal{C}act^1(n)$ turn $\mathcal{C}act^1(n)$ into a topological quasi-operad which is homotopy associative.*

Proof. Straightforward. □

2.7. Other approaches to the topology

In this subsection we give a brief non-technical summary of alternative approaches to give spineless cacti a topology by identifying them with other well known objects. Although the approaches seem vastly different they all lead to the same topology. For a detailed but still concise summary see [17].

2.7.1. Topology of cacti as special types of ribbon graphs

A topology on this set of cacti could alternatively be given as follows: given by the metric and the following convention. If the length of an arc goes to zero, the respective edge is contracted. If one of the two vertices of the edge corresponded to a root, the new root marking will be the vertex corresponding to the contracted edge. The cycle of the root marking is defined as the image of the cycle to which the edge immediately preceding the contracted edge in the order of the distinguished cycle belonged. Vice versa, the realization of a graph as above defines a unique normalized spineless cactus and hence the set of normalized spineless cacti is topologized.

2.7.2. Cacti–ribbon graphs as surfaces with weighted arcs

To get an element of $\mathcal{DArc}$ take the ribbon graph and consider a surface of which it is the spine. This surface will be of genus zero and can be realized as a disc with holes. Run arcs from the inside boundaries (viz. the “holes”) to the boundary, one for each edge, such that it crosses this edge transversally and has no other crossings with edges or arcs. In other words the ribbon graph is the dual graph on the surface to the graph given by the arcs and the boundaries. Mark a point on each boundary, which is not the endpoint of an arc, so that the linear order on the cycles of the ribbon graph agrees with the linear order on the arcs induced by going around the boundary in the orientation of the disc starting at the marked point. Each arc carries a weight given by the value of the metric on the respective edge to it. This gives the unique element of $\mathcal{DArc}$ representing the spineless cactus. In the limit where a weight goes to zero, the respective arc is simply erased [26].

For an example, see Fig. 4.

2.8. The relation between spineless cacti and normalized spineless cacti

In this section we briefly review the relationship between spineless cacti and their normalized version. For the full details, we refer to [17].

2.8.1. The scaling operad

We define the scaling operad $\mathcal{R}_{>0}$ to be given by the spaces $\mathcal{R}_{>0}(n) := \mathbb{R}^n_{>0}$ with the permutation action by $\mathbb{S}_n$ and the following products

$$(r_1, \ldots, r_n) \circ_i (r'_1, \ldots, r'_m) = \left(r_1, \ldots r_{i-1}, \frac{r_i}{R} r'_1, \ldots, \frac{r_i}{R} r'_m, r_{i+1}, \ldots r_n\right)$$

where $R = \sum_{k=1}^{m} r'_k$.

On one hand the scaling operad keeps track of the sizes of the lobes of a cactus. On the other hand the difference between the compositions in normalized and non-normalized spineless cacti is given by an action of the following type:

$$\rho_i : Cact^1(n) \times \mathcal{R}_{>0}(m) \times Cact^1(m) \to Cact^1(n). \tag{2.3}$$

The effect of the action is to move the intersection points of the lobes incident to the lobe i of the first normalized spineless cactus around that lobe according to the outside circle of spineless cactus to be inserted. The action is defined so that after the displacement of the lobes the composition of the perturbed normalized spineless cactus with the other normalized spineless cactus as normalized spineless cacti coincides with the composition obtained by gluing the two normalized spineless cacti simply as spineless cacti and then scaling back each lobe of the resulting cactus to length one. For the explicit formulas, we refer to [17]. Using this action, we can perturb the multiplications of normalized spineless cacti to fit with those of spineless cacti.

$$\begin{aligned}\circ_i^{\mathcal{R}_{>0}} &: \mathcal{C}act^1(n) \times \mathcal{R}_{>0}(m) \times \mathcal{C}act^1(m)\\ &\stackrel{id\times id\times\Delta}{\longrightarrow} \mathcal{C}act^1(n) \times \mathcal{R}_{>0}(m) \times \mathcal{C}act^1(m) \times \mathcal{C}act^1(m)\\ &\stackrel{\rho_i\times id}{\longrightarrow} \mathcal{C}act^1(n) \times \mathcal{C}act^1(m) \stackrel{\circ_i}{\longrightarrow} \mathcal{C}act^1(n+m-1)\end{aligned} \tag{2.4}$$

to get a continuous map

$$(c, \vec{r}', c') \mapsto c \circ_i^{\vec{r}'} c'.$$

Due to the nature of the maps above it is possible to continuously "undeform" the deformed product while staying in the category of quasi-operads.

Theorem 2.16 (*[17]*). *The operad of spineless cacti is isomorphic to the operad given by the semi-direct product of its normalized version with the scaling operad. The latter is homotopy equivalent (through quasi-operads) to the direct product as a quasi-operad. The direct product is in turn equivalent as a quasi-operad to* $\mathcal{C}act^1$.

$$\mathcal{C}act \cong \mathcal{R}_{>0} \ltimes \mathcal{C}act^1 \sim \mathcal{C}act^1 \times \mathcal{R}_{>0} \simeq \mathcal{C}act^1 \tag{2.5}$$

where the semi-direct product compositions are given by

$$(\vec{r}, c) \circ_i (\vec{r}', c') = (\vec{r} \circ_i \vec{r}', c \circ_i^{\vec{r}'} c'). \tag{2.6}$$

From this description one obtains several useful corollaries [17]. The ones relevant to the present discussion are listed below.

Corollary 2.17. *The quasi-operad of normalized spineless cacti is homotopy associative and thus its homology quasi-operad is an operad.*

Moreover as quasi-operads normalized spineless cacti and spineless cacti are homotopy equivalent via a homotopy of quasi-operads.

And finally:

Corollary 2.18. *Normalized spineless cacti are operadically quasi-isomorphic spineless cacti. That is, their homology operads are isomorphic.*

2.9. Spineless cacti and the little discs operad

The most important result of [17] which we will use is:

Theorem 2.19 (*[17]*). *The operad* $\mathcal{C}act$ *is equivalent to the little discs operad.*

3. A cell decomposition for normalized spineless cacti

3.1. The cell complex

Remark 3.1. For a normalized spineless cactus the lengths of the arcs have to sum up to the radius of the lobe and the number of arcs on a given lobe represented by a vertex v is $|v|+1$. Hence the lengths of the arcs lying on the lobe represented by a vertex v are in 1–1 correspondence with points of the simplex $|\Delta^{|v|}|$. The coordinates of $|\Delta^{|v|}|$ naturally correspond to the arcs of the lobe v on one hand and on the other hand to the incident edges to v in the dual b/w graph.

Definition 3.2. We define $\mathcal{T}_{bp}^{pp,nt}(n)^k$ to be the elements of $\mathcal{T}_{bp}^{pp,nt}(n)$ with $|E_w| = k$.

Definition 3.3. For $\tau \in \mathcal{T}_{bp}^{pp,nt}$ we define

$$\Delta(\tau) := \times_{v \in V_w(\tau)} \Delta^{|v|}. \tag{3.1}$$

We define $C(\tau) = |\Delta(\tau)|$. Notice that $\dim(C(\tau)) = |E_w(\tau)|$.

Given $\Delta(\tau)$ and a vertex x of any of the constituting simplices of $\Delta(\tau)$ we define the x-th face of $C(\tau)$ to be the subset of $|\Delta(\tau)|$ whose points have the x-th coordinate equal to zero.

Definition 3.4. We let $K(n)$ be the CW complex whose k-cells are indexed by $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)^k$ with the cell $C(\tau) = |\Delta(\tau)|$ and the attaching maps e_τ defined as follows. We identify the x-th face of $C(\tau)$ with $C(\tau')$ where $\tau' = \partial_e(\tau)$ is the topological type of the spineless cactus c' which is the degeneration of a spineless cactus c of topological type τ with respect to the arc a that simultaneously represents the vertex x of $\Delta(\tau)$ and the edge e of τ (see Section 2.4).

We denote by $\dot{e}_\tau$ the restriction of e_τ to the interior of $\Delta(\tau)$. Notice that $\dot{e}_\tau$ is a bijection.

Theorem 3.5. *The space $\mathcal{C}act^1(n)$ is homeomorphic to the CW complex $K(n)$.*

Proof. The map from $\mathcal{C}act^1(n)$ to $K(n)$ is given by Lemma 2.4 and Remark 3.1. Vice versa given an element on the right hand side, the unique open cell it belongs to determines the topological type and it is obvious that any tree in $\mathcal{T}_{bp}^{pp,nt}$ can be realized. Then the barycentric coordinates assign weights to the arcs via the correspondence of the vertices of the factors of $\Delta(\tau)$ with arcs of the cactus.

For the homeomorphism, we notice that the bijection restricted to the insides of the top-dimensional cells is obviously a homeomorphism. This is seen by slightly perturbing the non-zero lengths of the arcs. The limit where one of the lengths of the arcs goes to zero is given by passing to the corresponding face of the corresponding simplex factor of $C(\tau)$. The resulting tree, which is the topological type of the limit cactus, will be the tree which was used in the definition of the attaching map. Thus the contraction of the arc corresponds to the projection map sending the respective coordinate to zero. This agrees with the limit in $S(\tau)$. Therefore in the case of a degeneration the limits also agree and the result follows. □

3.1.1. Chains for $\mathcal{C}act$

Since the factors of $\mathbb{R}_{>0}$ are contractible, it is clear that $CC_*(\mathcal{C}act^1)$ is a chain model for $\mathcal{C}act$. Furthermore, by Theorem 3.11 below, we will see that $CC_*(\mathcal{C}act^1)$ is even an operadic chain model for $\mathcal{C}act$ and hence by Theorem 2.19 for the little discs operad.

Instead of using purely the cells of $\mathcal{C}act^1$ as a chain model for the little discs operad one can choose any operadic chain model $\mathrm{Chain}(\mathcal{R}_{>0})$ for the scaling operad and then use the mixed chains for $\mathcal{C}act$, i.e. $CC_*(\mathcal{C}act^1) \otimes \mathrm{Chain}(\mathcal{R}_{>0})$. It follows that the inclusion of the cellular chains of $\mathcal{C}act^1$ into the mixed chains is an inclusion of operads up to homotopy.

Given an operation of the $CC_*(\mathcal{C}act^1)$ we can let the mixed chains of $\mathcal{C}act$ act by letting the action of the mixed chains of bi-degree $(n, 0)$ be that of the component of $\mathcal{C}act^1$ and setting the action of all the other chains to zero.

In any case, $CC_*(\mathcal{C}act^1)$ is chain equivalent to any form of chain complex of mixed chains $CC_*(\mathcal{C}act^1) \otimes \mathrm{Chain}(\mathcal{R}_{>0})$.

3.1.2. Pseudo-cells for $\mathbb{P}\mathcal{C}act$

From Theorem 3.5 and Lemma 2.13, we obtain a decomposition for $\mathbb{P}\mathcal{C}act$ as

$$\mathbb{P}\mathcal{C}act(n) = \coprod_{\tau \in \mathcal{T}_{bp}^{pp,nt}(n)} \tilde{C}(\tau), \quad \tilde{C}(\tau) = \Delta(\tau) \times \dot{\Delta}^n.$$

This is not a cell decomposition since this $\tilde{C}(\tau)$ are not really cells, but this decomposition will be useful in the following.

Remark 3.6. One could formally close the $\tilde{C}(\tau)$ to cells and glue them by forgetting the lobes whose lengths goes to zero as described. This type of forgetting map is a quasi-fibration as shown in [17]. The gluing would yield a CW space which is the union over n of all $\mathbb{P}\mathcal{C}act(n)$. We will not pursue this construction further here.

3.2. Orientations of chains

To fix the generators and thereby the signs for the chain operad we have several choices, each of which is natural and has appeared in the literature.

To fix a generator $g(\tau)$ of $CC_*(\mathcal{C}act^1)$ corresponding to the cell indexed by $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$ we need to specify an orientation for it, i.e. an explicit parametrization or equivalently an order of the white edges of the tree it is represented by.

The first orientation which we call *Nat* is the orientation given by the natural orientation for a planar planted tree. That is, fixing the order of the white edges to be $\prec^\tau$. N.B. this actually coincides with the natural orientation of the cells of $\mathcal{A}rc$, see [26].

We will also consider the orientation *Op* which is the enumeration of the white edges which is obtained by starting with the incoming edges of the white vertex labelled one, in the order $\prec_{v_1}^\tau$, then continuing with the incoming white edges of the vertex two, etc. until the last label is reached.

Finally, for top-dimensional cells, we will consider the orientation of the edges induced by the labels, which we call *Lab*. It is obtained from *Nat* as follows: for $\tau \in \mathcal{T}_{b/w}^{pp,nt,fl}$ let $\sigma \in \mathbb{S}_n$ be the permutation which permutes the vertices $v_1, \ldots, v_n$ to their natural order induced by the order $\prec^\tau$. Then let the enumeration of E_w be $\sigma(Nat)$, where the action of σ on E_w is given by the correspondence out and the correspondence between black and white edges via $(v, N(v)) \mapsto (N(v), N^2(v))$ for top-dimensional cells.

To compare with the literature it is also useful to introduce the orientations $\overline{Nat}$, $\overline{Lab}$, and $\overline{Op}$ which are the reversed orientation of *Nat*, *Lab* and *Op*, i.e. reading the respective orders from right to left.

Lemma 3.7. *In the orientation Nat the induced quasi-operad structure on the cellular chains of $CC_*(\mathcal{C}act^1)$ is given by*

$$C(\tau) \circ_i C(\tau') = C(\tau \circ_i \tau')$$

where we understand the right hand side to be given by the natural extension to $\mathbb{Z}$-modules, i.e. $C(\sum_i z_i \tau) := \sum_i z_i C(\tau)$.

Proof. Let $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$ and $\tau' \in \mathcal{T}_{bp}^{pp,nt}(m)$. If we glue a cactus from $C(\tau')$ into $C(\tau')$ at the lobe i, its topological type will be one of the trees in the sum $\tau \circ_i \tau'$. Conversely, fix a tree $\tilde{\tau}$ which appears in the sum $\tau \circ_i \tau'$. Now any element $\tilde{c}$ of $C(\tilde{\tau})$ can be uniquely decomposed as $c \circ_i c'$ with $c \in C(\tau)$ and $c' \in C(\tau')$ as follows. Since the topological type $\tilde{\tau}$ is one of the summands of $\tau \circ_i \tau'$ if follows that the lobes i through $i + n$ are connected. Let c' be the normalized spineless subcactus which consists of the lobes i through $i + m - 1$ and has the local zero of the lobe i as the root marking. The cactus c will be the cactus constituted by the lobes 1 through $i - 1$ and $i + n$ through $n + m - 1$ together with a lobe marked i which is the outside circle of c' re-parameterized to the length one. Its root will be the root of $\tilde{c}$ if it does not lie on c'. And if it does, it will necessarily be the local zero of the i-th lobe of $\tilde{c}$ and the root of c will be this point thought of as the zero of the outside circle of c' which is the i-th lobe of c.

To give the coordinates of this construction let τ' be the connected subtree of $\tilde{\tau}$ whose white vertices are the white vertices labelled i through $i + m - 1$ and whose black vertices are the $N(v_j)$, $j \in \{i, \dots, i + m - 1\}$ together with the induced structure of planar planted tree. Let $\tau := \tilde{\tau}/\tau'$ be the planar planted tree obtained from $\tilde{\tau}$ by contracting τ'. Let the coordinates of $\tilde{c}$ in $C(\tilde{\tau})$ be $v = (\mathbf{v}_1, \dots \mathbf{v}_{n+m-1})$, were we use the short hand notation $\mathbf{v}_i = (v_{v_i,1}, \dots, v_{v_i,|v_i|+1}) \in |\Delta^{|v_i|}|$. Then c' is the cactus with coordinates $(\mathbf{v}_i, \dots, \mathbf{v}_{n+m-1})$ in $C(\tau')$. To define c let $(d_1, \dots, d_k)$ be the non-zero distances along the outside circle between the lobes of $\tilde{\tau}$ meeting the subtree τ' which are not part of τ'. Now c is the cactus with coordinates $v = (\mathbf{v}_1, \mathbf{v}_{i-1}, \bar{\mathbf{v}}, \mathbf{v}_{i+m-1}, \dots \mathbf{v}_{n+m-1})$ in $C(\tau)$ where $\bar{\mathbf{v}} = \frac{1}{m}(d_1, \dots, d_k)$. It is clear that (c, c') will be the only pre-image of $\tilde{c}$. Hence the sets are in bijection and therefore the sum contains the correct summands with the correct multiplicity up to a sign.

If one wishes to construct a representing configuration one just has to resolve the intersection points of c'. This can be done by for instance choosing a thickening of the cactus to a surface taking the boundary corresponding to the outside circle and attaching the lobes at the distances around this circle as dictated by the cactus $\tilde{c}$.

To verify the signs, we have to check that the respective orientations agree. For this we notice that the coordinates in the orientation *Nat* correspond to the white edges enumerated in the natural order of the given tree and that the permutation induced on the coordinates is exactly the sign incorporated into the definition of $\tau \circ_i \tau'$. Thus the signs agree as stated. □

Lemma 3.8. *On the level of sets let $\tilde{C}(\tau \circ_i^+ \tau') = \amalg_{\tilde{\tau}} \tilde{C}(\tilde{\tau})$ where $\tilde{\tau}$ is a summand of $\tau \circ_i^+ \tau'$ then the map $\circ_i : \tilde{C}(\tau) \times \tilde{C}(\tau') \to \tilde{C}(\tau \circ_i^+ \tau')$ is a bijection.*

Proof. As above, for the equation $\tilde{c} = c \circ_i c'$, identify c' and c with the respective spineless subcactus and quotient cactus, yielding a unique pre-image. □

This gives a topological proof for the associativity of the $\circ_i^+$.

Corollary 3.9. *The operations $\circ_i^+$ give $\mathcal{T}_{bp}^{pp,nt}$ the structure of an operad.*

Proposition 3.10. *For the choice of orientation Nat and the induced operad structure* $\circ_i$ *the map* $\tau \mapsto g(\tau)$ *where* $g(\tau)$ *is the generator corresponding to* $C(\tau)$ *fixed in Section 3.2 is a map of differential graded operads which identifies* $\mathcal{T}_{bp}^{pp,nt}(n)^k$ *with* $CC_k(\mathcal{C}act^1(n))$*, where* CC_k *are the dimension-k cellular chains.*

The same holds true for the orientation Op with the appropriate changes to the signs of the operad $\mathcal{T}_{bp}^{pp,nt}$ *discussed in Section 1.5.10. Finally the analogous statement holds true when passing to operads indexed by sets for both* $\mathcal{C}act$ *and* $\mathcal{T}_{bp}^{pp,nt}$*.*

Proof. This statement for the orientation *Nat* follows from the Lemma 3.7 above. For the other orientations the signs will agree, since they are forced onto the combinatorial operad by definition. □

Theorem 3.11. *The glueings induced from the glueings of spineless normalized cacti make the spaces* $CC_*(\mathcal{C}act^1(n))$ *into a chain operad. Thus* $CC_*(\mathcal{C}act^1)$ *is an operadic model for the chains of the little discs operad.*

Proof. The first statement follows from Proposition 1.17 together with the Proposition 3.10. The second part of the statement follows from Theorem 2.19 and Corollary 2.18. □

3.3. Operadic action of $\mathcal{T}_{bp}^{pp,nt}$

Given an assignment of operations on a complex $(\mathcal{O}, \delta)$ to the trees l_n and τ_n^b of Fig. 1, a natural way to let a tree $\tau \in \mathcal{T}_{bp}^{pp,nt}$ act on the complex $(\mathcal{O}, \delta)$ is given as follows. Formally read off the operation of τ by decorating the white vertices with elements of $\mathcal{O}$ and interpreting the tree as a flow chart, assigning to each white vertex w the operation of $l_{|w|}$ and to each black vertex v the operation of $\tau_{|v|}^b$. If the operations of l_n and τ_n^b are also compatible with the differentials, then one can obtain a dg-action of $\mathcal{T}_{bp}^{pp,nt}$ by regarding a mixed complex $CC_*(\mathcal{C}act^1) \otimes \underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{O})$ in which the order of the tensor factors of homogeneous elements of the tensor product is dictated by the trees $\tau \in \mathcal{T}_{bp}^{pp,nt}$. For this one identifies the white edges of a tree with the coordinates from $\mathcal{C}act^1$ and the vertices with elements from $\underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{O})$ and builds a differential from the tree differential and the internal differential of $\mathcal{O}$.

In the graded case, a sign for these operations and the differentials has to be included. A way for implementing this idea compatibly is as discussed in the following.

3.3.1. Tensor orders

For an action of the operad $\mathcal{T}_{bp}^{pp,nt}$ on a collection of objects $O(n_i)$ of a monoidal category $\mathcal{C}$, we will consider maps

$$\begin{aligned} \rho : \mathcal{T}_{bp}^{pp,nt}(k) &\to \mathrm{Hom}(O(n_1) \otimes \cdots \otimes O(n_k), O(m)) \\ \tau &\mapsto [f_1 \otimes \cdots \otimes f_k \mapsto \tau(f_1 \otimes \cdots \otimes f_k)]. \end{aligned} \tag{3.2}$$

Actually, $\tau(f_1 \otimes \cdots \otimes f_k)$ will be zero unless $m = \sum_{v \in E_w(\tau)} (n_{L^{-1}(v)} - |v|)$.

In the cases of interest for the present considerations, one can furthermore associate an object of $\mathcal{C}$ to each $\tau \in \mathcal{T}_{bp}^{pp,nt}$ which is usually of the form $\bigotimes_{e \in E_w(\tau)} L$ with L of dimension or degree plus or minus one. We call this the decomposable case.

Alternatively one can sometimes associate an object of the form $\bigotimes_{v \in E_w(\tau)} D_{|v|}$ to τ where the dimension or degree of $D_{|v|}$ is $|v|$, e.g. the simplex $\Delta^{|v|}$. We call this the operadic case.

If this is not possible, one can still consider a mixed complex made up of tensors of objects $C(\tau)$ and the $O(n_i)$.

In all these descriptions if the operad has a differential there is a natural differential obtained by using a combination of the tree differential and the operadic differential on the various factors. We now make these constructions precise.

3.3.2. Tensor order for the decomposable case

Fix $\mathcal{C}$ to be $\mathcal{C}hain$ or $\mathcal{V}ect_k$. In this case we wish to consider the maps (3.2) formally as maps

$$\bigotimes_{e\in E_w(\tau)} L \otimes \bigotimes_{v\in V_w(\tau)} O_{n_{L^{-1}(v)}} \to O_m. \tag{3.3}$$

In the graded case, we have to fix the order of (3.3). We do this by using $\prec^\tau$ to give the tensor product the natural operadic order. This amounts to formally inserting $O(n_i)$ into the vertex v_i. For this let $N := |V_w(\tau) \amalg E_w(\tau)|$ and again let num : $V(\tau) \amalg E(\tau) \to \{1,\dots,N\}$ be the bijection which is induced by $\prec^\tau$. We fix L to be a "shifted line", i.e. a free $\mathbb{Z}$-module or k-vector space generated by an element of degree plus one (or minus one). Now set

$$W_i := \begin{cases} O(n_j) & \text{if } \mathrm{num}^{-1}(i) = v_j \\ L & \text{if } \mathrm{num}^{-1}(i) \text{ is a white edge.} \end{cases} \tag{3.4}$$

We then define the order on the tensor product on the l.h.s. of the expression (3.3) to be given by

$$W := W_1 \otimes \cdots \otimes W_N.$$

Another way to add the necessary signs but circumnavigating the use of maps of the type (3.3) would be to define the operation ρ to include the sign which is the sign of the shuffle of the two ordered sets on the l.h.s. of (3.3) ordered by the orders on the edges and vertices into the order given by W. This has the drawback of using a new convention for each ρ.

3.3.3. The operadic case

In case we can associate objects D_n in $\mathcal{C}$ to each n-ary white vertex, we would consider the maps (3.2) formally as maps

$$\bigotimes_{v\in V_w(\tau)} (O_{n_{L^{-1}(v)}} \otimes D_{|v|}) \to O_m \tag{3.5}$$

where the order for the tensor product on the l.h.s. is defined to be the one given by $\prec^\tau$ on the white vertices.

3.3.4. The action of the symmetric group

The action of the symmetric group on the maps ρ is induced by permuting the labels and permuting the elements O_{n_i}. This induces signs by pulling back the permutation onto the factors of W. These signs should be included in the definition of $\mathbb{S}_n$-equivariance of the operadic action.

Remark 3.12. This treatment of the signs is essential if one is dealing with operads versus non-Σ operads and wishes to obtain equivariance with respect to the symmetric group actions. In general the symmetric group action on the endomorphism operads will not produce the right signs needed in the description of the iterations of the universal concatenation $\circ$ of Section 5. In particular this is the case for Gerstenhaber's product on the Hochschild cochains. The above modification however leads to an agreement of signs for the action of the symmetric group for the subcomplex of the Hochschild complex generated by products and the brace operations, see Section 4. Another approach is given by viewing the operations not as endomorphisms of the Hochschild cochains but rather as maps of the Hochschild cochains to Hochschild cochains twisted by tensoring with copies of the dual line L^*.

If one is not concerned with the action of the symmetric group, then one can forgo this step.

3.4. Examples

An example of the type of action described above as the operadic case is given in [26] by the action of Chain($\mathcal{A}rc$) on itself. For the homotopy Gerstenhaber structure we should consider the chains $CC_*(\mathcal{C}acti^1)$ and any choice of chain model for $\mathcal{A}rc$ or any of the suboperads which are stable under the action of the linear trees suboperad which is the image of spineless cacti.

The operadic ordering was also used in [6] to define the action of string topology on the chains of the free loop space of a compact manifold.

We get agreement with the usual signs and conventions of Gerstenhaber's original results [11] upgraded to operads (see Section 5) with those of the operations of $\mathcal{A}rc$ and those of string topology [6], if we denote the action of τ_1 as $*^{op}$ and τ_2^b as $\cdot$; see [26] for the operations and Fig. 1 for the definitions of the trees.

4. Spineless cacti as a natural solution to Deligne's conjecture

4.1. The Hochschild complex, its Gerstenhaber structure and Deligne's conjecture

Let A be an associative algebra over a field k. We define $CH^*(A,A) := \bigoplus_{q\geq 0} CH^q(A,A)$ with $CH^q(A,A) = \mathrm{Hom}(A^{\otimes q}, A)$.

There are two natural operations

$$\circ_i : CH^p(A,A) \otimes CH^q(A,A) \to CH^{p+q-1}(A,A)$$
$$\cup : CH^n(A,A) \otimes CH^m(A,A) \to CH^{m+n}(A,A)$$

where the first morphism is for $f \in CH^p(A,A)$ and $g \in CH^q(A,A)$

$$f \circ_i g(x_1,\dots,x_{p+q-1}) = f(x_1,\dots,x_{i-1}, g(x_i,\dots,x_{i+q-1}), x_{i+q},\dots,x_{p+q-1}) \tag{4.1}$$

and the second is given by the multiplication

$$f(a_1\dots,a_m) \cup g(b_1,\dots,b_n) = f(a_1\dots,a_m)g(b_1,\dots,b_n). \tag{4.2}$$

4.1.1. The differential on CH^*

The Hochschild complex has a differential which is derived from the algebra structure.

Given $f \in CH^n(A,A)$ then

$$\partial(f)(a_1,\dots,a_{n+1}) := a_1 f(a_2,\dots,a_{n+1}) - f(a_1a_2,\dots,a_{n+1}) + \dots + (-1)^{n+1} f(a_1,\dots,a_n a_{n+1}) + (-1)^{n+2} f(a_1,\dots,a_n)a_{n+1}. \tag{4.3}$$

Definition 4.1. The Hochschild complex is the complex (CH^*, ∂), its cohomology is called the Hochschild cohomology and denoted by $HH^*(A, A)$.

4.1.2. The Gerstenhaber structure

Gerstenhaber [11] introduced the $\circ$ operations: for $f \in CH^p(A, A)$ and $g \in CH^q(A, A)$

$$f \circ g := \sum_{i=1}^{p} (-1)^{(i-1)(q+1)} f \circ_i g \tag{4.4}$$

and defined the bracket

$$\{f, g\} := f \circ g - (-1)^{(p-1)(q-1)} g \circ f \tag{4.5}$$

and showed that this indeed induces what is now called a Gerstenhaber bracket, i.e. an odd Poisson bracket for $\cup$, on $HH^*(A, A)$. Here odd Poisson bracket means odd Lie bracket and the derivation property of the bracket with shifted (odd) signs.

4.2. Contents of Deligne's conjecture

Since $HH^*(A, A)$ has the structure of a Gerstenhaber algebra one knows from general theory [2,3] that thereby $HH^*(A, A)$ is an algebra over the homology operad of the little discs operad D_2. Now the Gerstenhaber structure on $HH^*(A, A)$ actually stems from the cochain level and the operadic structure of $H_*(D_2)$ even originates from the level of topological operads. The question of Deligne was:

Question 4.2 (*[9]*). Can one lift the action of the homology of the little discs operad to the chain respectively cochain level? Or in other words: is there a chain model for the little discs operad that operates on the Hochschild cochains which reduces to the usual action on the homology/cohomology level?

This question has an affirmative answer in many ways by picking a suitable chain model for the little discs operad [1,23,24,29,30,35,39], see also [32] for a review of these constructions. We will provide a new, natural, transparent and minimal positive answer to this question, by giving an operation of $CC_*(\mathcal{C}act^1)$ on the Hochschild cochains.

There is a certain minimal set of operations necessary for the proof of such a statement which is given by iterations of the operations $\cup$ and $\circ_i$. These are, as we argue below, in bijective correspondence with trees in $\mathcal{T}_{bp}^{pp,nt}$. Our model for the chains of the little discs operad $CC_*(\mathcal{C}act^1)$ has cells which are exactly indexed by these trees hence the operation of the cellular operad contains only the minimal number of operations and all operations are non-zero. Furthermore, the top-dimensional cells which control the bracket constitute the universal concatenation operad and hence yield operations for any operad, see Section 5.

Finally, we will show that the differential which contracts arcs of the cactus can be seen as a topological version of the Hochschild differential. This makes our new topological solution natural and minimal.

The main statement of this section is

Theorem 4.3. *Deligne's conjecture is true for the chain model of the little discs operad provided by $CC_*(\mathcal{C}act^1)$, that is $CH^*(A, A)$ is a dg-algebra over $CC_*(\mathcal{C}act^1)$ lifting the Gerstenhaber algebra structure.*

Proof. This follows from Theorem 3.11 together with either Proposition 4.10 or equivalently Proposition 4.21. □

Remark 4.4. We will give two proofs using the cell model $CC_*(\mathcal{C}act^1)$ for the chains of the little disc operad by defining actions of its operadically isomorphic model $\mathcal{T}_{bp}^{pp,nt}$. First the trees act naturally on the Hochschild complex by considering a tree as a flow chart of brace operations and multiplications; the signs being fixed by one of the schemes discussed below. Going one level deeper, instead of flow charts for brace and multiplication operations we will use a so-called foliage operator to reduce the brace operations to those of insertion.

4.3. The brace operations

The following operations appear naturally when considering the iterations of Gerstenhaber's operation $\circ$. They were first described by Getzler [13,16] and are called brace operations. For homogeneous f, g_i of degrees $|f|$ and $|g_i|$, $N = |f| + \sum_i |g_i| - n$

$$f\{g_1, \dots, g_n\}(x_1, \dots, x_N) := \sum_{\substack{1 \le i_1 \le \dots \le i_n \le |f|:\\ i_j + |g_j| \le i_{j+1}}} \pm f(x_1, \dots, x_{i_1-1}, g_1(x_{i_1}, \dots, x_{i_1+|g_1|}), \dots, \dots, x_{i_n-1}, g_n(x_{i_n}, \dots, x_{i_n+|g_n|}), \dots, x_N) \tag{4.6}$$

where the sign is the sign of the shuffle of the g_j and x_i which is determined by assigning shifted degrees to the x_i and g_j; namely the x_i are considered to be of degree 1 and the g_j are considered to be of degree $|g_j| + 1$.

Notice that $f\{g\} = f \circ g$.

4.3.1. The suboperad generated by the brace operations

It is well known ([13,14,16] see also [32] for the history of the brace operations and Deligne's conjecture) that the set of concatenations of multiplications and brace operations form a suboperad of the endomorphism operad of the Hochschild complex. We will call it $\mathcal{B}race$.

The generators of this suboperad are in 1–1 correspondence with elements of $\mathcal{T}_{bp}^{pp,nt}$. Such a tree represents a flow chart. The functions to be acted upon are to be inserted into the white vertices. A black vertex signifies the multiplication of the incoming entities. A white vertex represents the brace operation of the element attached to that vertex outside the brace and the elements obtained by performing the operations associated to the incoming branches inside the braces.

Remark 4.5. Notice that in the flow chart of an expression of the type $f\{(g_1), (g_2 \cdot g_3 \cdot g_4\{h_1, h_2\})\}$ the symbols "{" and "," correspond to the white edges when reading off the operation from a labelled tree $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$ in the order $\prec^\tau$.

Proposition 4.6. *The association of a flow chart is a non-Σ operadic isomorphism between $\mathcal{B}race$ and $\mathcal{T}_{bp}^{pp,nt}$ of operads with a differential.*

Proof. The fact that the association of a flow chart is a bijection on the generators was already mentioned. It is straightforward to check the combinatorics of inserting a formula made up out of braces and multiplications into a brace or a multiplication leads to exactly the behavior described by our operad structure on the trees $\mathcal{T}_{bp}^{pp,nt}$. The checking of the compatibility of the signs is tedious but straightforward. We would like to remark that for the signs, in the brace formalisms the sums can be viewed as being parameterized over a discretized simplex, in the sense that the size of the gaps (number of variables between function insertions) parameterize the summands in the formula (4.6) and the sum of all the sizes of the gaps is fixed. This formalism also yields the agreement of signs. The compatibility of the differential can then be seen by a either a straightforward calculation, a comparison with [11,14] or the above formalism of discretized simplices. □

4.4. Signs for $\mathcal{B}race$

Definition 4.7. We define an action of the symmetric group on $\mathcal{B}race$, by considering the symbols "{" and "," to be each of degree one.

Proposition 4.8. *With the above action of the symmetric group on $\mathcal{B}race$ the isomorphism of Proposition 4.6 is an isomorphism of operads.*

Proof. This follows from the fact that the white edges correspond to the symbols "{" and ",". It can also be seen directly by comparing to the formulas of [11,14]. □

4.5. The operation of $CC_*(\mathcal{C}act^1)$ on $\mathcal{H}om_{CH}$

For $\mathcal{O} = \mathcal{H}om_{CH}$ the endomorphism operad of $CH^*(A, A)$, we define the map ρ of Eq. (3.2) to be given by the operadic extension of the maps which send the tree τ_n of Fig. 1 to the non-intersecting brace operations. We let τ_0 act as the identity and τ_n^b as multiplication.

As discussed above, in order to make signs match for the symmetric group actions, we consider the action as described in Section 3.3. To get complete agreement with the signs of [11], we will have to consider the opposite orientation for tensor factors in (3.3) to that of W, i.e. we use the order $\overline{W} := W_N \otimes \cdots \otimes W_1$ for the tensor factors. To implement this change of sign we define $\mathrm{sign}^W(\tau)$ to be the sign obtained by shuffling the graded tensor factors from the order W to the order $\overline{W}$. This basically means that in the orientation given by W one would regard the operations $\circ^{op}$ and $\cup^{op}$ on the Hochschild complex, where $f \circ^{op} g = (-1)^{pq+p+1} g \circ f$ and $f \cup^{op} g := (-1)^{pq} g \cup f$.

The action of the tree τ_n is given by:

$$f \otimes L_{e_1} \otimes g_1 \otimes \cdots \otimes L_{e_n} \otimes g_n \mapsto (-1)^{\mathrm{sign}^W(\tau)} f\{g_1, \ldots, g_n\}. \tag{4.7}$$

Here we used the notation L_{e_1} to mean L in the position of e_1 in the tensor product of the form (3.3).

The action of τ_n^b is given by

$$g_1 \otimes \cdots \otimes g_n \mapsto (-1)^{\mathrm{sign}^W(\tau)} g_1 \cup \cdots \cup g_n. \tag{4.8}$$

The operadic extension means that we read the tree as a flow chart: at each black vertex $|v|$ the operation $\tau_{|v|}^b$ is performed and at each white vertex the operation $\tau_{|v|}^w$ is performed. The $\mathbb{S}_n$-action is given by permutations and indeed induces the right signs on the Hochschild complex as seen by straightforward calculation.

Proposition 4.9. *The above procedure gives an operation of* $CC_*(\mathcal{C}act^1)$ *on* $CH^*(A, A)$ *and an operadic isomorphism between* $\mathcal{B}race$ *and* $CC_*(\mathcal{C}act^1)$*.*

Proof. This follows directly from Proposition 3.10, the previous section and the considerations above. □

4.5.1. The differential

Denote the differential on $CC_*(\mathcal{C}act^1)$ by ∂ and the differential of $CH^*(A, A)$ by δ. On the space W there is a natural differential ∂_W which is induced by $\delta + \partial$. The differential on W is induced by the respective tree differential which equivalently collapses the arcs of the spineless cactus or removes factors of W. Then the action of $CC_*(\mathcal{C}act^1)$ on $\mathcal{H}om_{CH}$ commutes with the differential in the following sense.

Proposition 4.10.

$$\rho \circ (\partial_W) = \delta \circ \rho. \tag{4.9}$$

The Hochschild cochains $CH^*(A, A)$ *are a dg-algebra over* $CC_*(\mathcal{C}act^1)$ *and there is an operadic isomorphism of the differential operads* $\mathcal{B}race$ *and* $CC_*(\mathcal{C}act^1)$*.*

Proof. The verification of the compatibility of the grading and differentials is a straightforward computation. □

4.6. A second approach to the operation of $CC_*(\mathcal{C}act^1)$

Another way to make $CC_*(\mathcal{C}act^1)$ or $\mathcal{T}_{bp}^{pp,nt}$ act is by using the foliage operator, see below Section 4.7. This approach was first taken in [24]. It stresses the fact that a function $f \in CH^n(A, A)$ is naturally depicted by τ_n. Notice for instance the compatibility of the differentials. The tree for $\partial(f)$ is $\partial(\tau_n)$ where ∂ is the differential on trees with tails given in Section 4.7.1.

4.6.1. Natural operations on CH^* *and their tree depiction*

Given elements of the Hochschild cochain complex there are two types of natural operations which are defined for them. Suppose f_i is a homogeneous element, then it is given by a function $f : A^{\otimes n} \to A$. Viewing the cochains as functions, we have the operation of insertion. The second type of operation comes from the fact that A is an associative algebra; therefore, for each collection $f_1, \dots, f_n \in CH^*(A, A)$ we have the n! ways of multiplying them together.

We can encode the concatenation of these operations into a black and white bipartite tree as follows: Suppose that we would like to build a cochain by using insertion and multiplication on the homogeneous cochains $f_1, \dots, f_n$. First we represent each function f_i as a white vertex with $|f_i|$ inputs and one output with the cyclic order according to the inputs $1, \dots, |f_i|$ of the function. For each insertion of a function into a function we put a black vertex of arity one having as input edge the output of the function to be inserted and as an output edge the input of the function into which the insertion is being made. For a multiplication of $k \geq 2$ functions we put a black vertex whose inputs represent the factors in the order of their multiplication. Finally, we add tails to the tree by putting a black vertex at the end of each input edge which has not yet been given a black vertex, and we decorate the tails by variables $a_1, \dots, a_s$ according to their order in the total order of the vertices of the rooted planted planar tree. It is clear that this determines a black and white bipartite tree.

4.6.2. Operations on Hochschild from trees with tails

A rooted planted planar bipartite black and white tree whose tails are all black and decorated by variables $a_1, \ldots, a_s$ and whose white vertices are labelled by homogeneous elements $f_v \in CH^{|v|}(A, A)$ determines an element in $CH^s(A, A)$ by using the tree as a flow chart. This means the operation of insertion for each black vertex of arity one and multiplication for each black vertex of higher arity. Notice that, since the algebra is associative, given an ordered set of elements $a_i : i \in \{1, \ldots, n\}$ there is a unique multiplication $\prod_{i=1}^n a_i$.

Remark 4.11. Using the above procedures, the possible ways to compose k homogeneous elements of $CH^*(A, A)$ using insertion and cup product are bijectively enumerated by black and white bipartite planar rooted planted trees with tails and k white vertices labelled by k functions whose degree is equal to the arity of the vertex.

Notation 4.12. We will fix A and use the short hand notation $CH := CH^*(A, A)$. For an element $f \in CH$, we write $f^{(d)}$ for its homogeneous component of degree d.

Definition 4.13. For $\tau \in \mathcal{T}_{bp}^{pp}(n)$ and $f_1, \ldots, f_n \in CH$ we let $\tau(f_1, \ldots, f_n)$ be the operation obtained in the above fashion by decorating the vertex v_i with label i with the homogeneous component of $f_i^{(|v_i|)}$. Notice that the result is zero if any of the homogeneous components $f_i^{(|v_i|)}$ vanishes.

Remark 4.14. Up to the signs which are discussed below this gives an operation of $CC_*(\mathcal{C}act^1)$ on the Hochschild complex.

4.7. The operation of $\mathcal{T}_{bp}^{pp,nt}$

In order to define the operation we need foliation operators in the botanical sense, i.e. operators that add leaves. To avoid confusion with the mathematical term “foliation”, we choose to abuse the English language and call these operations “foliage” operators.

Definition 4.15. Let l_n be the tree in $\mathcal{T}_{bp}^{pp}$ with one white vertex labelled by v and n tails as depicted in Fig. 1. The foliage operator $F : \mathcal{T}_{b/w}^{pp,st,nt} \to \mathcal{T}_{b/w}^{pp,st}$ is defined by the following equation

$$F(\tau) := \sum_{n \in \mathbb{N}} l_n \circ_v \tau.$$

Remark 4.16. Notice that the right hand side is infinite, but $\mathcal{T}_{b/w}^{pp,st}$ is graded by the number of leaves, and $F(\tau)$ is finite for a fixed number of black leaves so that the definition does not pose any problems. Furthermore, one could let F take values in $\mathcal{T}_{b/w}^{pp,st}[[t]]$ where t keeps track of the number of tails which would make the grading explicit.

Also notice that $F : \mathcal{T}_{bp}^{pp,nt} \to \mathcal{T}_{bp}^{pp}$ and $F : \mathcal{T}_{b/w}^{pp,nt,fl} \to \mathcal{T}_{b/w}^{pp,fl}$.

Definition 4.17. For a tree $\tau \in \mathcal{T}_{bp}^{pp,nt}(n)$ with n white vertices we define a map $op(\tau) \in \mathrm{Hom}(CH^{\otimes n}, CH) = \mathcal{H}om_{CH}(n)$ by

$$op(\tau)(f_1, \ldots, f_n) := \pm(F(\tau))(f_1, \ldots, f_n)$$

here the signs are as discussed in Section 3.2 and the right hand side is well defined since it only has finitely many non-zero terms.

Remark 4.18. The considerations of this section naturally lead to brace operations in a far more general setting. This is explained in detail in Section 5.

4.7.1. Differential on trees with tails

Definition 4.19. For a tree τ with tails in $\mathcal{T}_{bp}^{pp}$ and vertex $v \in V_b \setminus \{v_{\text{root}}\}$ we define τ_v^+ to be the b/w tree obtained by adding a black vertex $b+$ and an edge $e^+ := \{b+, v\}$, if $|v| \neq 0$ and if $|v| = 0$, the tree obtained by adding two vertices $b+$ and b_{st} and two edges $e+ = \{b+, v\}$ and $e_{st} = \{b_{st}, v\}$ to τ.

We call a linear order $\prec'$ on τ_v^+ compatible with the order $\prec$ on τ if (a) $e+ \prec' e_{st}$ if applicable and (b) the order induced on τ by $\prec'$ by contracting e^+ and e_{st} (if applicable) coincides with $\prec$. We define E_{w-int} to be the white internal edges, i.e. white edges which are not leaves and set $E_{b-\text{angle}} : E(\tau) \setminus E_{w-int}$. For a linear order $\prec'$ on τ_v^+

$$\text{sign}(\prec') := (-1)^{|\{e|e \in E_{b-\text{angle}}, e \prec' v\}|}$$

and set

$$\partial_v(\tau) := \sum_{\text{compatible } \prec'} \text{sign}(\prec')(\tau_v^+, \prec').$$

We recall that tail edges are considered to be black. Finally, we define

$$\partial(\tau) := \sum_{v \in V_b \setminus \{v_{\text{root}}\}} \partial_v(\tau). \tag{4.10}$$

Remark 4.20. In the tree depiction for the operations of inserting and cup product Section 4.6.1, the tree differential amounts to inserting $\cup$ products into the "slots" represented by black vertices. Using this interpretation and the tree notation for the calculations of [11,14] it is straightforward to check that the tree differential (4.10) defined above agrees with the differential on the Hochschild cochains.

Proposition 4.21. *The Hochschild cochains are a dg-algebra over the operad $\mathcal{T}_{bp}^{pp,nt}$.*

Proof. The properties of an operad follow in a straightforward way from Definition 4.17 and the definition of the differentials. □

4.8. The top-dimensional cells of spineless cacti and pre-Lie operad

We denote the top-dimensional cells of $CC_n(\mathcal{C}act^1(n))$ by $CC_n^{\text{top}}(n)$. These cells again form an operad and they are indexed by trees with black vertices of arity one. Furthermore, the symmetric combinations of these cells which are the image of $\mathcal{T}^{r,fl}$ under the embedding *cppin* form a suboperad, see Proposition 1.25. For one choice of orientation we make the signs explicit in the next lemma.

Lemma 4.22. *In the orientation $\overline{Lab}$ for the top-dimensional cells for $\tau \in \mathcal{T}^{r,fl}(n)$ and $\tau' \in \mathcal{T}^{r,fl}(m)$*

$$cppin(\tau) \circ_i cppin(\tau') = \pm cppin(\tau \circ_i \tau')$$

where the sign is $(-1)^{(i-1)(n-1)}$ if the root vertex of τ has a label which is less than i and $(-1)^{i(n-1)}$ if the root vertex of τ has a label which is bigger than i.

Proof. First notice that by Proposition 1.25, we have that the right hand side contains the terms indexed by the trees appearing in the embedding. Furthermore, notice that under the choice of signs induced by the orientation $\overline{Lab}$ the signs do not depend on the particular structure of the tree and are only dictated by the labels. In the composition the labels are such that the n labels of the second tree are permuted to the i-th label of the first tree. Therefore there is a universal sign which is given by $(-1)^{(i-1)(n-1)}$ if the root vertex of τ has a label which is less than i and by $(-1)^{i(n-1)}$ if the root vertex of τ has a label which is bigger than i. □

Definition 4.23. Let $\mathcal{G}Pl$ be the quadratic operad in the category $Vect_{\mathbb{Z}}$ obtained as the quotient of a free operad $\mathcal{F}$ generated by the regular representations of $\mathbb{S}_2$ by the quadratic relations defining a graded pre-Lie algebra. Let $\mathcal{R}$ be the ideal generated by the graded $\mathbb{S}_3$-submodule generated by the relation

$$r = (x_1x_2)x_3 - x_1(x_2x_3) - (-1)^{|x_2||x_1|}((x_1x_3)x_2 - x_1(x_3x_2)).$$

Then $\mathcal{G}Pl = \mathcal{F}/\mathcal{I}$. Here $\mathcal{F}$ and $\mathcal{R}$ are considered to be graded by assigning the degree $n-1$ to $\mathcal{F}(n)$.

This operad is the operad for graded pre-Lie algebras in the sense that any algebra over this operad is a graded pre-Lie algebra and vice versa any graded pre-Lie algebra is an algebra over this operad:

Theorem 4.24. *The operad $CC_n^{\text{top}}(n)^{\mathbb{S}} \otimes k$ is isomorphic to the operad $\mathcal{G}Pl$ for graded pre-Lie algebras. Furthermore the shifted operad $(CC_n^{\text{top}} \otimes (L^*)^{\otimes E_w})^{\mathbb{S}}(n) \otimes k$ is isomorphic to the operad $\mathcal{P}l$ for pre-Lie algebras.*

The analogous statements also hold over $\mathbb{Z}$.

Proof. In view of Lemma 4.22 and Proposition 1.25, the second statement follows from the operadic isomorphism of $\mathcal{T}^{r,fl}$ and the pre-Lie operad $\mathcal{P}l$ of [5]. This also proves the first statement up to signs. The matching of the signs is guaranteed by the shift, see Definition 1.24. The fact that the relation r holds and generates the respective ideal was verified in the presentation of Gerstenhaber structure for spineless cacti [17] by a translation from the explicitly given relation on the chains of the arc operad [26]. □

Corollary 4.25. *The pre-Lie algebra of $\mathbb{S}_n$-coinvariants $\bigoplus_n((CC_n^{\text{top}} \otimes (L^*)^{\otimes E_w})^{\mathbb{S}}(n))_{\mathbb{S}_n}$ is isomorphic to the free pre-Lie algebra in one generator.*

Likewise the graded pre-Lie algebra of $\mathbb{S}_n$-coinvariants $\bigoplus_n((CC_n^{\text{top}})^{\mathbb{S}}(n))_{\mathbb{S}_n}$ is isomorphic to the graded free pre-Lie algebra in one generator.

Proof. The first statement follows from [5] and thus so does the second up to signs. These are guaranteed to conform by the shifting procedure and Theorem 4.24. □

5. Structures on operads and meta-operads

In this section, we discuss how Deligne's conjecture and the Gerstenhaber structure for the Hochschild complex can in fact be generalized to structures on operads. This helps to explain some choices of signs and explains the naturality of the construction of insertion operads which gives a special role to spineless cacti as their topological incarnation as well as to $\mathcal{A}rc$ as a natural generalization.

This analysis also enables us surprisingly to relate spineless cacti to the renormalization Hopf algebra of Connes and Kreimer [7], see the next section.

5.1. The universal concatenations

Given any operad there are certain universal operations, i.e. maps of the operad to itself. We will first ignore possible signs and comment on them later.

Any operad possesses the operations given by its structure maps

$$\circ_i : O(m) \otimes O(n) \to O(m+n-1).$$

Any concatenations of these maps will also yield operations on the operad. All possible concatenations of the structure map are described by their flow charts. These charts are in turn given by trees $\tau \in \mathcal{T}_{b/w}^{pp,fl}$ as we now explain.

Any concatenation of k objects $op_i \in O(n_i)$ using the structural maps $\circ_i$ will be given by

$$(\ldots((op_{\sigma(1)} \circ_{i_1} op_{\sigma(2)}) \circ_{i_2} op_{\sigma(3)} \circ_{i_3}) \ldots) \circ_{i_{k-1}} op_{\sigma(k)} \tag{5.1}$$

with $i_1 < i_2, \ldots < i_k$ and some permutation $\sigma \in \mathbb{S}_k$.

Let $\tau \in \mathcal{T}_{b/w}^{pp,fl}(k)$ such that $|v_i| = n_i, i \in 1, \ldots k$ and set $m = \sum_{i=1}^k n_i - k - 1$ then there is an operation

$$\circ(\tau) : (O(n_1) \otimes \cdots \otimes O(n_k)) \to O(m) \tag{5.2}$$

which is obtained as follows. First label the vertex v_i by $op_{n_i} \in O(n_i)$. Let $\sigma \in \mathbb{S}_k$ be the permutation which is defined by the order of the tree, i.e. $v_{\sigma(1)} \prec^\tau \cdots \prec^\tau v_{\sigma(i)} \prec^\tau v_{\sigma(i+1)} \prec^\tau \cdots \prec^\tau v_{\sigma(k)}$.

Now starting at the bottom of the tree and going along the outside edge path, we read off the operation

$$\circ(\tau)(op_1 \otimes \cdots \otimes op_k) := (\cdots((op_{\sigma(1)} \circ_{i_1} op_{\sigma(2)}) \circ_{i_2} op_{\sigma(3)} \circ_{i_3}) \cdots) \circ_{i_{k-1}} op_{\sigma(k)} \tag{5.3}$$

where the i_k are given by $i_k = |\{v \in E_b(\tau) : v \prec^\tau v_{\sigma(k)}\}|$; recall that the root is considered a black vertex.

Vice versa, given a concatenation as above we can successively build up the corresponding tree.

Remark 5.1. Notice, we might have white leaves, which allows one to consider operads with a 0 component—such as CH^*. In general lifting the restriction on the n_i, we define the operations $\circ(\tau)$ to be zero if $|v_i| \neq n_i$.

5.1.1. The insertion partial meta-operad

Algebraically we can concatenate the operations $\circ(\tau)$ by substituting

$$op_i = \circ(\tau')(op'_1 \otimes \cdots \otimes op'_l) \quad \text{for some } \tau' \in \mathcal{T}_{b/w}^{pp,fl}(l)$$

into (5.1). In general, for $\tau \in \mathcal{T}_{b/w}^{pp,fl}(k)$, let $v_i := L_\tau(i)$ be the i-th white vertex of τ, and let $\tau' \in \mathcal{T}_{b/w}^{pp,fl}(l)$ with $|v_i| = |V_{\text{leaf}}(\tau')|$, we define the tree $\tau \circ_i \tau'$ to be the tree of the concatenated operation

$$\begin{aligned}&\circ(\tau \circ_i \tau')(op_1 \otimes \cdots \otimes op_{k+l-1})\\ &\quad := \circ(\tau)(op_1 \otimes \cdots \otimes op_{i-1} \otimes \circ(\tau')(op_i \otimes \cdots \otimes op_{i+l-1}) \otimes op_{i+l} \otimes \cdots \otimes op_{l+k-1}).\end{aligned} \tag{5.4}$$

Lemma 5.2. *The multiplication maps* $\circ_i$ (5.4) *together with the permutation action on the labels imbue* $\mathcal{T}^{pp,fl}_{b/w}$ *with the structure of a partial operad.*[2]

Proof. Straightforward. □

Remark 5.3. The partial concatenations $\circ_i$ insert a tree with k tails into the vertex v_i if $|v_i| = k$, by connecting the incoming edges of v_i to the tail vertices in the linear order at v_i and then contracting the images of the tail edges.

Definition 5.4. We will fix that for $\mathcal{O}$ in $\mathcal{S}et$ the direct sum which we again denote by $\mathcal{O}$ is given by the free Abelian group generated by $\mathcal{O}$ which we consider to be graded by the arity of the operations $op \in \mathcal{O}$ minus one. If the operad $\mathcal{O}$ is in $\mathcal{C}hain$ we can take the direct sum of the components as $\mathbb{Z}$-modules. In the case of an operad $\mathcal{O}$ in the category $\mathcal{V}ect_k$ we consider its direct sum to be the direct sum over k of its components. In all these cases, we say $\mathcal{O} = \{O(n)\}$ admits a direct sum and write $\mathcal{O} = \bigoplus_{n\in\mathbb{N}} O(n)$. We also consider $\mathcal{O}$ to be graded by $\mathbb{N}$ with the degree of $O(n)$ being $n-1$.

Remark 5.5. Consider an operad which admits a direct sum. Let $\mathcal{O}$ be its direct sum, then we obtain a map of partial operads.

$$\mathcal{T}^{pp,fl}_{b/w} \to \mathrm{Hom}(\mathcal{O}, \mathcal{O}).$$

In this sense one can say that $\mathcal{T}^{pp,fl}_{b/w}$ is the universal concatenation partial meta-operad.

5.2. The pre-Lie structure of an operad

In an operad which admits a direct sum, one can define the analog of the $\circ$ product of [11] and the iterated brace operation (cf. [13–15]).

Definition 5.6. Given any operad $\mathcal{O}$ in $\mathcal{S}et$, $\mathcal{C}hain$ or $\mathcal{V}ect_k$, we define the following map

$$O(m) \otimes O(n) \to O(m+n-1) \tag{5.5}$$

$$op_m \otimes op_n \mapsto \sum_{i=1}^{m} (-1)^{(i-1)(n+1)} op_m \circ_i op_n. \tag{5.6}$$

This extends to a map on $\mathcal{O} = \bigoplus_{n\in\mathbb{N}} O(n)$

$$\circ : \mathcal{O} \otimes \mathcal{O} \to \mathcal{O} \tag{5.7}$$

which we call the $\circ$ product.

We call the map which is defined in the same fashion as $\circ$, but with the omission of the signs $(-1)^{(i-1)(n+1)}$ the ungraded $\circ$ product.

Proposition 5.7. *The product* $\circ$ *defines the structure of a graded pre-Lie algebra on* $O := \bigoplus_{i\in\mathbb{N}} O(n)$. *Omitting the sign* $(-1)^{(i-1)(n+1)}$ *in the sum yields the structure of a non-graded pre-Lie algebra.*

[2] A partial operad is the notion obtained from an operad by requiring that the composition maps $\circ_i$ are only defined on a subobject of the $O(n)$. These compositions should be equivariant with respect to the $\mathbb{S}_n$ action and satisfy the associativity axioms if it is possible to compose them.

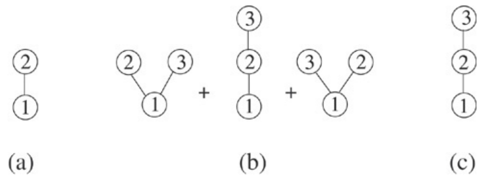

Fig. 5. (a) $op_1 \circ op_2$ (b) $(op_1 \circ op_2) \circ op_3$ and (c) $op_1 \circ (op_2 \circ op_3)$.

Proof. The proof is a straightforward calculation which is analogous to Gerstenhaber's original calculation [11]. We do not wish to rewrite the proof here, but in graphical notation the proof follows from Fig. 5. Without signs this notation is related to the one that can be found in [5], for the case with signs the interpretation of the resulting trees is discussed in Section 3.2 and Proposition 1.25. □

See also [31] for another variant of this fact.

Proposition 5.8. *If an operad admits a direct sum then its direct sum is an algebra over the symmetric top-dimensional chains of the little disc operad of the chain model provided by $CC_*(\mathcal{C}act^1)$ as well as over the shifted chains $(CC_n^{\text{top}})^{\mathbb{S}} \otimes (L^*)^{\otimes E_w}$.*

Proof. We have shown that the direct sum of an operad which admits such a sum has the structure of a pre-Lie algebra and a graded pre-Lie algebra so that the statement follows from Theorem 4.24. □

5.3. The insertion operad

The interesting property of the operation $\circ$ is that it effectively removes the dependence on the number of inputs of the factors. Using this logic systematically, we obtain a universal insertion operad.

5.3.1. Actions of $\mathcal{T}_{b/w}^{pp,nt,fl}$

Given an operad which admits a direct sum, we can also define other operations similar to $\circ_i$. The summands of these operations are brace operations, which are in natural correspondence with $\mathcal{T}_{b/w}^{pp,nt,fl}$. In fact these operations all appear in the iterations of $\circ$. They are given by inserting the operations into each other according to the scheme of the tree. In other words, we will show that every operad is a brace algebra.

Essentially, at this level of abstraction, we would not like to a priori specify the number of leaves, i.e. inputs and degrees of the operations, so we have to consider trees with all possible decorations by leaves. For this, we can use the foliage operators of Section 4.7.

Recall that there is an operation of $\mathcal{T}_{b/w}^{pp,fl}$ on homogeneous elements of $\mathcal{O}$ of the right degree. We extend this operation to all of $\mathcal{O}$ by extending linearly and setting to zero expressions which do not satisfy degree condition. Where the degree condition corresponding to a given τ applied to homogeneous elements op_k is that $op_k \in O(|v_k|)$.

Given τ in $\mathcal{T}_{b/w}^{pp,nt,fl}(n)$, we then define the operation

$$\circ(\tau)(op_1 \otimes \cdots \otimes op_n) := F(\tau)(op_1 \otimes \cdots \otimes op_n). \tag{5.8}$$

Notice that, although $F(\tau)$ is an infinite linear combination, for given $op_1, \ldots, op_n$ the expression on the right hand side is finite.

Examples of this are given in Fig. 5. Here the first tree yields the operation $op_1 \circ op_2$, i.e. the insertion (at every place) of op_2 into op_1. Iterating this insertion we obtain expression (b) which shows that inserting op_3 into $op_1 \circ op_2$ gives rise to three topological types: inserting op_3 in front of op_2, into op_2 and behind op_2. In the opposite iteration one just inserts $op_2 \circ op_3$ into op_1 which gives a linear insertion of op_2 into op_1 and op_3 into op_2. From the figure (up to signs) one can read off the symmetry in the entries 2 and 3 of the associator. The signs are fixed by the considerations of Section 3.2.

Remark 5.9. Again, inserting $op_i = \circ(\tau')(op'_1 \otimes \cdots \otimes op'_k)$ we obtain operad maps with respect to the symmetric group actions on the labels

$$\circ_i : \mathcal{T}^{pp,fl}_{b/w}(k) \otimes \mathcal{T}^{pp,nt,fl}_{b/w}(l) \to \mathcal{T}^{pp,nt,fl}_{b/w}(k+l-1)$$

by demanding that for all $op_1, \ldots, op_{k+l-1}$

$$\begin{aligned}&\circ(\tau \circ_i \tau')(op_1 \otimes \cdots \otimes op_{k+l-1})\\ &\quad = \circ(\tau)(op_1 \otimes \cdots \otimes op_{i-1} \otimes \circ(\tau')(op_i \otimes \cdots \otimes op_{i+l-1}) \otimes op_{i+l} \otimes \cdots \otimes op_{l+k-1}).\end{aligned}$$

Or in other words thinking about F as a formal power series, e.g. in $\mathcal{T}^{pp,fl}_{b/w}[[t]]$ where the powers of t keep track of the number of tails,

$$F(\tau \circ_i \tau') = F(\tau) \circ_i F(\tau').$$

Thus we have an insertion operad structure on $\mathcal{T}^{pp,nt,fl}_{b/w}$.

Remark 5.10. We can induce a pre-Lie operation on $\mathcal{T}^{pp,fl}_{b/w}$ via the $\circ$ operation.

$$F(\tau_1 * \tau_2) := F(\tau_1) \circ F(\tau_2). \tag{5.9}$$

Theorem 5.11. *Any chain operad which admits a direct sum is an algebra over the operad* $\mathcal{T}^{pp,nt,fl}_{b/w}$ *with the insertion products.*

Proof. Immediate from the preceding. □

5.4. Operad algebras and a generalized Deligne conjecture

Definition 5.12. We define an operad algebra to be an operad $\mathcal{O}$ which admits a direct sum together with an element $\cup \in O(2)$ which is associative, i.e. if $a \cup b := (-1)^{|a|}(\cup \circ_1 a) \circ_{|a+1|} b$ then $(a \cup b) \cup c = a \cup (b \cup c)$. Recall that $|a| = n - 1$ if $a \in O(n)$.

This definition is essentially equivalent to the definition of an operad with a multiplication of [15].

Definition 5.13. For $op \in O(m)$, $op'_i \in O(n_i)$, we define the generalized brace operations

$$op\{op'_1, \ldots, op'_n\} := \sum_{\substack{1 \le i_1 \le \cdots \le i_n \le m:\\ i_j + |op'_j + 1| \le i_{j+1}}} \pm(\cdots((op \circ_{i_1} op'_1) \circ_{i_2} op'_2) \circ_{i_3} \cdots) \circ_{i_n} op'_n \tag{5.10}$$

where the sign is defined to be the same one as in Eq. (4.6).

Lemma 5.14. *There is an operadic action of* $\mathcal{T}^{pp,nt}_{bp}$ *of any operad algebra.*

Proof. We can view the bipartite tree as a flow chart. For the white vertices, we use the brace operations (5.10) and for a black vertex with n incoming edges, we use the $n-1$ times iterated operation $\cup$. Notice that the order in which we perform these operations does not matter, since we took $\cup$ to be associative. □

Definition–Proposition 5.15. *Generalizing Gerstenhaber's [11] definition to an operad algebra, we define a differential on the direct sum by* $\partial f = f \circ \cup - (-1)^{|f|} \cup \circ f$.

Proof. The fact that this is a differential follows from the calculations of [11]. □

Remark 5.16. The analogous tree differential is given in Section 4.7.1. Replacing functions by elements of the operad the compatibility follows for the more general set-up of Definition–Proposition 5.15.

Theorem 5.17. *The generalized Deligne conjecture holds. That is, the direct sum of any operad algebra which admits a direct sum is an algebra over the chains of the little discs operad in the sense that it is an algebra over the dg-operad* $CC_*(\mathcal{C}act^1)$.

Proof. By the preceding Lemma 5.14, we have an operadic action of $\mathcal{T}_{bp}^{pp,nt}$ and thus an action of the chains $CC_*(\mathcal{C}act^1)$ which is a chain model for the little discs operad. The compatibility of the differentials follows directly from their definitions by a straightforward calculation as remarked above. □

6. The Hopf algebra of Connes and Kreimer and spineless cacti

6.1. The Hopf algebra of an operad

We have seen in Section 5.2 that any operad that admits a direct sum gives rise to a pre-Lie algebra. Now by the defining properties for a pre-Lie algebra the commutator of its product yields a Lie algebra or in the graded case an odd Lie algebra.

By the above considerations, we can thus naturally associate a pre-Lie algebra, a Lie algebra, and a Hopf algebra to each operad that admits a direct sum.

Definition 6.1. Given an operad $\{O(n)\}$ which admits a direct sum $\mathcal{O} = \bigoplus_n O(n)$, we define its pre-Lie algebra $\mathcal{PL}(\mathcal{O})$ to be the pre-Lie algebra $(\mathcal{O}, \circ)$ with $\circ$ as defined in Section 5.2, its Lie algebra $\mathcal{L}ie(\mathcal{O})$ to be the Lie algebra $(\mathcal{O}, [\, , \,])$ using the Lie bracket $[a, b] := a \circ b - b \circ a$ and its odd Lie algebra $\mathcal{L}ie^{\mathbb{Z}/2\mathbb{Z}}(\mathcal{O})$ to be the odd Lie algebra $(\mathcal{O}, \{\, , \,\})$ where $\{\, , \,\}$ is defined as usual via $\{a\,,\,b\} := a \circ b - (-1)^{(|a|+1)(|b|+1)} b \circ a$ for $a \in O(|a|)$ and $b \in O(|b|)$. Finally, the Hopf algebra of an operad $\mathcal{H}opf(\mathcal{O})$ is defined to be $U^*(\mathcal{L}ie(\mathcal{O}))$, i.e. the dual of the universal enveloping algebra of its Lie algebra.

As it turns out these objects or their $\mathbb{S}_n$-coinvariants are of interest. In particular, we obtain chain models in terms of chains of spineless cacti or even moduli space for some well known algebras.

6.2. Connes and Kreimer's Hopf algebra as the Hopf algebra of an operad

In [7] a Hopf algebra based on trees was defined to explain the procedure of renormalization in terms of the antipode of a Hopf algebra. This Hopf algebra was described directly, but also as the dual to the universal enveloping algebra of certain Lie algebra which was identified in [5] as the Lie algebra associated to the free pre-Lie algebra in one generator.

Definition 6.2. By the $\mathbb{S}_n$-coinvariants of an operad which admits a direct sum, we mean $\bigoplus_{n\in\mathbb{N}}(O(n))_{\mathbb{S}_n}$. We write $\mathcal{O}_{\mathbb{S}}$ for this sum. Here $\bigoplus_{n\in\mathbb{N}}(O(n))_{\mathbb{S}_n}$ is the shorthand notation explained in Definition 5.4.

In our notation we can rephrase the results of [5,7] as

Theorem 6.3. *The renormalization Hopf algebra of Connes and Kreimer H_{CK} is isomorphic to the Hopf algebra of $\mathbb{S}_n$-coinvariants of $\mathcal{H}opf(\mathcal{T}^{r,fl})$. This Hopf algebra is also isomorphic to the $\mathbb{S}_n$-coinvariants of $\mathcal{H}opf(\mathcal{P}l)$.*

6.3. A chain interpretation of H_{CK}

As shown in Theorem 4.24 there is a cell and thus a topological interpretation of the pre-Lie operad and the graded pre-Lie operad inside $\mathcal{C}act^1$ and thus inside the $\mathcal{A}rc$ operad. In this interpretation H_{CK} is also the Hopf algebra of the coinvariants of the shifted chain operad $CC_*^{\text{top}}(\mathcal{C}act)^{\mathbb{S}} \otimes (L^*)^{\otimes E_w}$. Recall that L^* is a free $\mathbb{Z}$-module generated by an element l of degree -1.

Proposition 6.4. *H_{CK} is isomorphic to the Hopf algebra of $\mathbb{S}_n$-coinvariants of the suboperad of top-dimensional symmetric combinations of shifted cells $((CC_*(\mathcal{C}act^1))^{\text{top}})^{\mathbb{S}} \otimes (L^*)^{\otimes E_w}$ of the shifted cellular chain operad of normalized spineless cacti CCcact $\otimes (L^*)^{\otimes E_w}$*

$$H_{CK} \simeq (\mathcal{H}opf((CC_*(\mathcal{C}act^1))^{\text{top}})^{\mathbb{S}} \otimes (L^*)^{\otimes E_w})_{\mathbb{S}}.$$

Proof. Immediate from Theorem 6.3 and Corollary 4.25. □

It is interesting to note that also the G structures and BV structures which are given by spineless cacti and cacti [17] are inside the symmetric (graded symmetric) combinations as well.

Definition 6.5. We define the planar Connes–Kreimer Hopf algebra H_{CK}^{pl} to be $(\mathcal{H}opf(CC_*^{\text{top}} \otimes (L^*)^{\otimes E_w}))_{\mathbb{S}}$. This is the straightforward generalization from rooted to planar planted trees.

Remark 6.6. We can also consider the graded Hopf algebra corresponding to the $\mathbb{S}_n$-coinvariants of graded top-dimensional cells $U^*(\mathcal{L}ie^{\mathbb{Z}/2\mathbb{Z}}(\mathcal{O}))_{\mathbb{S}}$.

Remark 6.7. The examples above should be relevant in the context of multiple polylogarithms. We expect to obtain other interesting examples of such Hopf algebras by applying the above constructions to other operads based on trees.

6.4. Comments on operads and H_{CK}

We have shown that any operad is an algebra over the operad $\mathcal{T}^{r,fl}$ in a natural way and thus the Hopf algebra H_{CK} naturally appears in any context involving operads, such as Deligne's conjecture. We have furthermore shown that there is a topological incarnation of the insertion product, which is based on surfaces. In this setting, we have constructed a chain representation of the algebra H_{CK}. This links the algebra H_{CK} and its underlying bracket for instance to string topology and in positive characteristic to Dyer–Lashof–Cohen operations [38].

7. Further developments and generalizations

7.1. Generalizations

7.1.1. The A_∞ case

All the statements made above in the context of algebras hold as well for A_∞-algebras. For this one has to change to an equivalent cell model for normalized spineless cacti. A sketch of the procedure is given below. A detailed account will be given in [25].

For $\tau \in \mathcal{T}^{pp,st,nt}_{b/w}$ let

$$C_\infty(\tau) := \prod_{v \in V_w} W_{|v|+1} \times \prod_{v \in V_b} K_{|v|}$$

where $W_{|v|+1}$ is the $|v|$-dimensional cyclohedron and $K_{|v|}$ is the $|v|-2$-dimensional Stasheff polytope a.k.a. associahedron.

Now one has to show a compatibility:

Lemma 7.1. *The tree differential ∂^∞_{tree} of* [24] *and the natural differential ∂_{poly} for the cyclohedra and associahedra are compatible in the following sense. The cells $C_\infty(\tau')$ appearing in the sum $C_\infty(\partial^\infty_{tree}\tau)$ are in* 1–1 *correspondence to the products appearing in $\partial_{poly}C_\infty(\tau)$ and furthermore the signs agree.*

Definition 7.2. Let $Cell(\mathcal{T}^{pp,st,nt}_{b/w})$ be the CW complex glued from the cells $C_\infty(\tau)$ using ∂^∞_{tree}.

Proposition 7.3. *$CC_*(Cell(\mathcal{T}^{pp,st,nt}_{b/w}))$ is equivalent to $CC_*(\mathcal{C}act^1)$ and hence a cell model for the little discs operad.*

Proof. Just contract the associahedra simultaneously blowing down the cyclohedra to simplices. The compatibility of this operation follows from the properties of the map st_∞. □

Using flow charts as in [24] where now a black vertex b corresponds to the $|b|$-th of the A_∞ multiplication $\mu_{|b|} : A^{\otimes|b|} \to A$ one obtains:

Theorem 7.4 (*[25]*). *Deligne's conjecture holds in the A_∞ case over $\mathbb{Z}$ for the chain model $CC_*(Cell(\mathcal{T}^{pp,st,nt}_{b/w}))$. Furthermore so does its generalization to an operad with an A_∞ multiplication.*

Here an operad with an A_∞ multiplication is an operad $\mathcal{O}$ with a collection of multiplications elements $m_n \in O(n)$ which satisfy the equations for A_∞ multiplications.

Remark 7.5. Using compatible realizations of cyclohedra and associahedra given as particular faces, one can also construct a topological quasi-operad [25]. An interesting open problem is the definition of a suboperad of $\mathcal{A}rc$ which is the natural topological operad model for these cells.

7.1.2. The cyclic case

The results and methods of this paper have been extended to the cyclic case in [19] in which it is proven that the Hochschild cochains of a Frobenius algebra carry an action of a chain model of the framed little discs operad. For this one needs other techniques than presented in this paper, notably operadic correlation functions.

7.2. Further developments

Building on the techniques of [19] one can extend the action to include all marked ribbon graphs and all $\mathbb{Z}/2\mathbb{Z}$ decorated ribbon graphs [20,21]. The latter generalizes all known graph actions on the Hochschild cochains such as the chains framed little discs and Sullivan Chord diagrams. These results yield an action of pseudo-chains of the decorated moduli space. There remains one caveat, since the ribbon graphs only give pseudo-cells, the dg-structure at the moment still needs some clarification.

In another new development, by taking up the philosophy of the present paper, we are able to realize all little k-cubes operads inside the arc formalism by using stabilization with respect to the genus operator of [26]. Using the formalism developed in this paper we then obtain a solution to Kontsevich's generalization of Deligne's conjecture to d-algebras. Moreover in the stable limit, we obtain an E_∞-suboperad [22]. This implies that the stabilization of the $\mathcal{A}rc$ operad yields an infinite loop space spectrum. We would like to point out, that this stabilization is different from the usual stabilization. For this one would use the gluing on of a fixed element of a particular type of the $\mathcal{A}rc$ operad which is not considered a stabilization in the above sense. Nevertheless, the proposed results of [22] could be regarded as a combinatorial/chain incarnation of the seminal results of Tillmann [36,37] and Madsen and Tillmann [33].

Acknowledgments

I would like to thank the Max-Planck Institute for Mathematics and the IHES for their hospitality. The visits to these Institutes were instrumental in the carrying out of the presented research; especially the discussions with Maxim Kontsevich. Furthermore, I would like to thank Ieke Moerdijk for interesting discussions. My thanks also goes to Wolfgang Lück and Peter Teichner for their hospitality and interest. In fact, the first steps for our solution to Deligne's conjecture were taken during a visit of the SFB in Münster and the final step occurred to me during a visit to San Diego in response to questions of Peter Teichner. Finally, I wish to thank Jim Stasheff for very enlightening discussions and encouragement as well as Murray Gerstenhaber for pointing out the similarities between his constructions and those of [26]. This observation was the initial spark for this work on Deligne's conjecture. We also wish to thank the referee for a careful reading and helpful suggestions.

The author also acknowledges partial financial support from the NSF under grant #DMS-0070681.

Appendix. Graph theoretic details

A.1. Dual graph constructions

A.1.1. The dual bipartite b/w graph

Fix a marked spineless treelike ribbon graph (Γ, f_0). We define a new bipartite b/w graph $\tau(\Gamma)$ as follows. Set

$$V_b(\tau(\Gamma)) := V(\Gamma), \qquad V_w(\tau(\Gamma)) := \{\text{cycles of } \Gamma\} \setminus \{c_0\}, \qquad F(\tau(\Gamma)) := F(\Gamma)$$

and specify f_0 as the distinguished flag. Now set

$$\delta_{\tau(\Gamma)}(f) := \begin{cases} \delta_\Gamma(f) & \text{if } f \in c_0 \\ c_i & \text{if } f \in c_i \neq c_0 \end{cases}, \qquad \imath_{\tau(\Gamma)}(f) := \begin{cases} N_\Gamma(\imath_\Gamma(f)) & \text{if } f \in c_0 \\ \imath_\Gamma(N_\Gamma^{-1}(f)) & \text{if } f \notin c_0. \end{cases}$$

We fix the ribbon graph structure by declaring the cyclic order at the white vertices to be the conjugate order of the cycle they represent. That is f is the predecessor of f' in the cyclic order at $v = c_i$ if f is the successor of f' in the cycle c_i. For the black vertices, we take the induced order from the identifications $V_b(\tau(\Gamma)) = V(\Gamma)$ and $F(\tau) = F(\Gamma)$. This means in particular that

$$N_{\tau(\Gamma)}(f) := \begin{cases} \imath_\Gamma(f) & \text{if } f \in c_0 \\ N_\Gamma(\imath_\Gamma(f)) & \text{if } f \notin c_0. \end{cases}$$

Notice that this graph only has one cycle and hence has genus 0, so it is a planar tree. This first statement can be seen directly or as follows, we have that the number of vertices of τ is equal to the number of vertices of Γ plus the number of cycles of Γ minus 1. The number of flags of τ is the number of flags of Γ which is 2 times the number of edges of Γ. So we get that $2 - 2g(\tau) = (|V(\Gamma)| + \#\text{cycles of } \Gamma - 1) - E(\Gamma) + \#\text{cycles of } \tau = 1 + \#\text{cycles of } \tau \leq 2$.

A.1.2. Ribbon graph defined by a b/w planar planted bipartite tree

The ribbon graph defined by a tree is given as follows: Given a b/w bipartite planar planted tree τ, we define a new graph $\Gamma(\tau)$ by setting

$$F(\Gamma(\tau)) := F(\tau), \qquad V(\Gamma(\tau)) := V_b(\tau)$$

and specifying f_0 as the distinguished flag. Furthermore, we define

$$\delta_{\Gamma(\tau)}(f) := \begin{cases} \delta_\tau(f) & \text{if } \delta_\tau(f) \in V_b(\tau) \\ \delta_\tau(\imath_\tau(f)) & \text{if } \delta_\tau(f) \in V_w(\tau) \end{cases}, \qquad \imath_{\Gamma(\tau)}(f) := \begin{cases} N_\tau & \text{if } \delta_\tau(f) \in V_b(\tau) \\ N_\tau^{-1} & \text{if } \delta_\tau(f) \in V_w(\tau). \end{cases}$$

We fix the ribbon graph structure by defining the cyclic order at a vertex as follows: if $\delta(f) \in V_b$ then the successor of f is the flag $\imath_\tau(f)$, the successor of this flag is defined to be the successor of f in the cyclic order at $v = \delta(f)$ of τ and so on. This means that if $\delta(f) \in V_w$ then the successor of f is $N_\tau(f)$. This implies that

$$N_{\Gamma(\tau)}(f) := \begin{cases} N_\tau^2(f) & \text{if } \delta(f) \in V_b \\ \imath_\tau(N_\tau^{-1}(f)) & \text{if } \delta(f) \in V_w. \end{cases}$$

Now Γ has a cycle which runs through all the edges and the other cycles of Γ are in 1–1 correspondence with the white vertices. The cycle that contains f_0 contains every second flag in the unique cycle of the tree, that is all the flags f with $\delta(f) \in V_b$. On the other hand if $\delta(f') = v \in V_w$ then all the flags in its cycle are incident to the same vertex and in effect $\imath_\tau(N_\tau^{-1}(f))$ is just the predecessor of f in the cyclic order of its vertex. Thus these cycles are in 1–1 correspondence with the sets $F(v)$ for $v \in V_w$. This means that $2 - 2g(\Gamma) = |V_b(\tau)| - |E(\tau)| + |V_w(\tau)| + 1 = 2$, so that the genus of Γ is zero. The cycle of f_0 contains all flags f with $\delta_\tau(f) \in V_b$ and for such a flag $\delta_\tau(\imath_\Gamma(f)) \in V_w$ and vice versa, so that either f or $\imath_\Gamma(f)$ lie in the N_Γ cycle of f_0 and indeed the graph Γ is a marked spineless treelike ribbon graph.

Lemma A.1. *The dual graph is a duality transformation that is $\Gamma(\tau(\Gamma)) = \Gamma$ and $\tau(\Gamma(\tau)) = \tau$.*

Proof. On the level of flags this is clear. For the vertices, this is also clear for the first order of iteration, in the second order the equality of the set of vertices follows from the observation explained above that the cycles of $\Gamma(\tau)$ which are not the distinguished cycle c_0 are in 1–1 correspondence to the white vertices. It remains to check the compatibility of the maps δ, $\imath$ and N which amounts to plugging in the definitions.

Here are some examples: $\delta_{\Gamma(\tau(\Gamma))}(f)$ with $f \in c_0$: $\delta_{\Gamma(\tau(\Gamma))}(f) = \delta_{\tau(\Gamma)}(f) = \delta_\Gamma(f)$. If $f \in c_i \neq c_0$ then $\delta_{\Gamma(\tau(\Gamma))}(f) = \delta_{\tau(\Gamma)}(\imath_{\tau(\Gamma)}(f)) = \delta_{\tau(\Gamma)}(N_\Gamma(\imath_\Gamma(f))) = \delta_\Gamma(N_\Gamma(\imath_G(f))) = \delta_\Gamma(f)$. As another example consider $N_{\tau(\Gamma(\tau))}(f)$ for f with $\delta_\tau(f) \in V_b$, we get $N_{\tau(\Gamma(\tau))}(f) = \imath_{\Gamma(\tau)}(f) = N_\Gamma(f)$ and if $\delta_\tau(f) \in V_w$ then $N_{\tau(\Gamma(\tau))}(f) = \imath_{\Gamma(\tau)}(f) = N_{\Gamma(\tau)}(\imath_{\Gamma(\tau)}(f)) = N_{\Gamma(\tau)}(N_\tau^{-1}(f)) = N_\tau^2(N^{-1}(f)) = N_\tau(f)$. Writing out the other calculations is tedious but straightforward. □

A.2. *Gluing spineless and normalized spineless cacti: The graph version*

A.2.1. S^1-*graphs*

To formulate the gluing in a purely graph theoretic way, we first need a new definition. An S^1-*graph* is a metric ribbon graph of genus 0 together with a distinguished flag f_0, such that

(i) if c_0 is the cycle of f_0 then for each flag either $f \in c_0$ or $\imath(f) \in c_0$
(ii) $\forall v : |F(v)| = 2$.

Such a graph has exactly two cycles that have the same length, which we call the radius of the S^1-graph.

An S^1-graph is equivalent to the data of a circle embedded into the plane with several marked points. To explain this, let $S^1_r := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = r\}$. We will use the natural coordinate θ on S^1: $x = r\cos(2\pi\theta)$, $y = r\sin(2\pi\theta)$. Let $0 = \theta_0 < \theta_1 < \cdots < \theta_n < 1$ be $n + 1$ points on S^1. Then $(S^1, (\theta_i))$ defines an S^1-graph by letting the points be the vertices, the arcs the edges and the flags are the half edges. To be very explicit each flag is an ordered pair (θ_i, θ_j) with $|i - j| = 1 \equiv (n - 1)$ and $\imath(v, w) = (w, v)$. We denote the edges by $\{\theta_i, \theta_j\}$ with the same restriction. Since $|F(v)| = 2$ there is only one choice of cyclic order at each vertex. We let $f_0 = (\theta_0, \theta_1)$ and let c_0 be the cycle f_0 that contains f_0. Finally, we set $\mu(\{\theta_0, \theta_n\}) := 1 - \theta_n$ and $\mu(\{\theta_i, \theta_{i+1}\}) = \theta_{i+1} - \theta_i$ for $i = 0, \ldots n - 1$. Conversely, any S^1-graph gives rise to the data above. For this we embed the realization of the CW complex which is an S^1 into the plane as a circle S^1_r of radius equal to the length of c_0, such that c_0 runs counterclockwise and that the vertex of f_0 is the point $\theta_0 = 0$. Each vertex then corresponds to a point θ_i.

A.2.2. *Gluing* S^1-*graphs*

We define a gluing for two S^1-graphs of the same radius. Given S and S', we scale S', so that the lengths of the radii of S and S' agree and then let $S \circ S'$ be the S^1-graph defined by taking the union of marked points. That is if $S = (S^1_r, \{\theta_i : i \in I\})$ and $S' = (S^1_r, \{\theta'_j : j \in J\})$ then $S \circ S' = (S^1_r, \{\theta_i : i \in I\} \cup \{\theta'_j : j \in J\})$.

We identify the set of vertices of the glued graphs with the union of the vertices of the two graphs: $V(S \circ S') = V(S) \cup V(S')$, we can also naturally identify $F(S \circ S')$ with $F(S) \cup F(S')$.

We extend the gluing of S^1-graphs to glueings of an S^1-subgraph with an S^1-graph.[3] Let (Γ, μ) be a metric ribbon graph and S be a subgraph which is an S^1-graph with the induced ribbon graph structure. Also fix an S^1-graph S', we define $(\Gamma, \mu) \circ_S S'$ by replacing the subgraph S by $S \circ S'$. This is well defined, since we can identify the vertices of S with vertices of $S \circ S'$. The result is again a metric ribbon graph.

This operation simply inserts new vertices into the subgraph, new vertices will have valence 2, so that it is naturally a ribbon graph.

[3] A subgraph is a subset of vertices and a subset of flags closed under $\imath_\Gamma$ and δ_Γ.

A.2.3. Lobes as S^1-graphs

Every lobe gives rise to an S^1-graph by considering the subgraph of vertices and edges of the cycle. We define the marked flag f_i of the S^1-graph to be the first flag f of the cycle c_0 such that $\imath(f) \in c_i$.

The fact that a cycle corresponding to a lobe is actually an S^1-subgraph, that is that each vertex v has $|F(v)| = 2$ when considering the subgraph, can most easily be seen from the dual tree. Here the statement corresponds to the fact that any two edges incident to a white vertex only have this vertex in common, which is true since the dual graph is a tree.

A.2.4. Chord diagrams

There is an S^1-graph covering each spineless cactus as follows. For a spineless cactus $c = (\Gamma, f_0, \mu)$ consider the S^1-graph obtained by going around the cycle c_0 and considering this an S^1-graph. The set of flags is defined to be the same as that of Γ and $\imath$ and μ are also defined identically in both graphs, but let the set of vertices of the S^1-graph be the set $F(\Gamma)$ and fix the map $\delta = id$. The marked flag is again f_0. We call this graph $S(c)$.

There is an equivalence relation on the vertices of $S(c)$ given by $v = \delta_{S(c)}(f) \sim v' = \delta_{S(c)}(f')$ if $\delta(f) = \delta(f')$ in c. We call $(S(c), \sim)$ the chord diagram of c. We wish to point out that as graphs $\Gamma = S(c)/\sim$ and that $(S(c)/\sim, f_0, \mu)$ is c if we give the $S(c)/\sim$ the ribbon structure of Γ that is the structure induced by first lifting N_Γ as a function on flags and then letting it descend to the quotient graph.

A.2.5. The glueings for spineless cacti

Fix two spineless cacti $c = (\Gamma, f_0, \mu)$ and $c' = (\Gamma', f_0', \mu')$, let S be the S^1-subgraph of c and represent c' as $(S(c'), \sim)$. Fix a cycle c_i of c and let r_i be the radius of c_i and R be the radius of c_0', the outside circle of c'.

We define the underlying metric graph of $c \circ_i c'$ to be the graph given by

$$\left[(\Gamma, \mu) \circ_{c_i} \frac{r_i}{R} S(c')\right] \Big/ \sim \tag{A.1}$$

where $\frac{r_i}{R}$ is the scaling action and $\sim$ is the equivalence relation given above extended to the graph $(\Gamma, \mu) \circ_{c_i} \frac{r_i}{R} S(c')$ by identifying the vertices of $\frac{r_i}{R} S(c')$ as a subset of the vertices of the glued graph. We mark this graph by the image of the flag f_0 of c. The ribbon structure is defined as follows. For all vertices of Γ which do not lie on c_i, we keep the order. For the other vertices, we fix the following linear order. Given v let $v_1, v_2, \dots, v_n$ be its pre-images in $(\Gamma, \mu) \circ_{c_i} \frac{r_i}{R} S(c')$ linearly ordered according to the distinguished cycle c_0' of $S(c')$ and its distinguished flag. Each of the sets $F(v_i)$ is also linearly ordered by using its cyclic order and declaring the first flag to be the first flag of c_0' which is incident to the vertex. Now $F(v) = \coprod_{i=1,\dots,n} F(v_i)$ and we enumerate the flags by the induced linear order starting to enumerate $F(v_1)$ in its linear order, continuing with $F(v_2)$ and finishing with $F(v_n)$ again in their linear orders. This gives a linear order on $F(v)$ and we fix the cyclic order of $F(v)$ to be the unique cyclic order which is compatible with that linear order.

It is now straightforward to check that the resulting data defines a spineless cactus. With the exception of c_i, all the cycles which were lobes descend unaltered as subsets of flags. The Euler-characteristic computation then shows that the genus is zero and that there is exactly one more cycle. This is the cycle of the flag f_0 and it passes through all the edges. It follows that the number of lobes is subadditive.

A.2.6. The glueings for normalized spineless cacti

Keeping the notations of the section above, now let c and c' be normalized spineless cacti and c_i a fixed cycle of Γ with radius r_i. Then set $\tilde{\mu}(e) = \mu(e)$ if e is not in c_i that is none of the two flags is in c_i. If one of the flags of e is in c_i then $\tilde{\mu}(e) = \frac{R}{r_i}(e)$. This causes the radius of c_i considered as a cycle of $\tilde{c}$ to be R, the radius of the outside circle of c'. We then define the gluing for normalized spineless cacti by setting the underlying metric graph to be

$$[(\Gamma, \tilde{\mu}) \circ_{c_i} S(c')]/\sim. \tag{A.2}$$

Again we mark the resulting graph by the image of the flag f_0 and induce a ribbon graph structure as above. As above it follows that the result is a normalized spineless cactus and the gluing is subadditive in the number of lobes.

References

[1] C. Berger, B. Fresse, Une décomposition prismatique de l'opérade de Barratt–Eccles, C. R. Math. Acad. Sci. Paris 335 (4) (2002) 365–370.

[2] F.R. Cohen, The homology of C_{n+1}-spaces, $n \geq 0$, in: The Homology of Iterated Loop Spaces, in: Lecture Notes in Mathematics, vol. 533, Springer, 1976.

[3] F.R. Cohen, Artin's braid groups, classical homotopy theory, and sundry other curiosities, in: Braids (Santa Cruz, CA, 1986), in: Contemp. Math., vol. 78, Amer. Math. Soc., Providence, RI, 1988, pp. 167–206.

[4] R.L. Cohen, J.D.S. Jones, A homotopy theoretic realization of string topology, Math. Ann. 324 (4) (2002) 773–798.

[5] F. Chapoton, M. Livernet, Pre-Lie algebras and the rooted trees operad, Int. Math. Res. Not. (8) (2001) 395–408.

[6] M. Chas, D. Sullivan, String topology, math.GT/9911159, Ann. of Math. (in press) Preprint.

[7] A. Connes, D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Comm. Math. Phys. 199 (1998) 203–242.

[8] P. Deligne, Catégories tannakiennes, in: The Grothendieck Festschrift, vol. II, in: Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 111–195.

[9] P. Deligne, Letter to Stasheff, Gerstenhaber, May, Schechtman and Drinfel'd, 1993 (Unpublished).

[10] A. Dzhumadil'daev, C. Löfwall, Trees, free right-symmetric algebras, free Novikov algebras and identities, Homology Homotopy Appl. 4 (2002).

[11] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. 78 (1963) 267–288.

[12] E. Getzler, Two-dimensional topological gravity and equivariant cohomology, Comm. Math. Phys. 163 (1994) 473–489.

[13] E. Getzler, Cartan homotopy formulas and the Gauss–Manin connection in cyclic homology, in: Quantum Deformations of Algebras and Their Representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), in: Israel Math. Conf. Proc., vol. 7, Bar-Ilan Univ., Ramat Gan, 1993 pp. 65–78.

[14] M. Gerstenhaber, A.A. Voronov, Higher-order operations on the Hochschild complex, Funktsional. Anal. i Prilozhen. 29 (1) (1995) 1–6, 96; translation in Funct. Anal. Appl. 29 (1) (1995) 1–5.

[15] M. Gerstenhaber, A.A. Voronov, Homotopy G-algebras and moduli space operad, Int. Math. Res. Not. (3) (1995) 141–153.

[16] T. Kadeishvili, The structure of the A(∞)-algebra, and the Hochschild and Harrison cohomologies, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 91 (1988).

[17] R.M. Kaufmann, On several varieties of cacti and their relations, Algebr. Geom. Topol. 5 (2005) 237–300.

[18] R.M. Kaufmann, Operads, moduli of surfaces and quantum algebras, in: N. Tongring, R.C. Penner (Eds.), Woods Hole Mathematics. Perspectives in Mathematics and Physics, in: Series on Knots and Everything, vol. 34, World Scientific, 2004.

[19] R.M. Kaufmann, A proof of a cyclic version of Deligne's conjecture via Cacti, math.QA/0403340, Preprint.

[20] R.M. Kaufmann, Moduli space actions on the Hochschild cochain complex I: Cell models, math.AT/0606064, MPIM2006-118, Preprint.

[21] R.M. Kaufmann, Moduli space actions on the Hochschild cochain complex II: Correlators, math.AT/0606065, MPIM2006-119, Preprint.

[22] R.M. Kaufmann, The arc spectrum: Detecting loop spaces (in preparation).
[23] M. Kontsevich, Operads and motives in deformation quantization, Lett. Math. Phys. 48 (1999) 35–72.
[24] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne's conjecture, in: Conférence Moshé Flato 1999, vol. I, (Dijon), in: Math. Phys. Stud., vol. 21, Kluwer Acad. Publ., Dordrecht, 2000, pp. 255–307.
[25] R.M. Kaufmann, R. Schwell, Associahedra, Cyclohedra and the A_∞-Deligne conjecture (in preparation).
[26] R.M. Kaufmann, M. Livernet, R.B. Penner, Arc operads and arc algebras, Geom. Topol. 7 (2003) 511–568.
[27] P. van der Laan, I. Moerdijk, The renormalisation bialgebra and operads, hep-th/0210226, Preprint.
[28] P. van der Laan, Operads—Hopf algebras and coloured Koszul duality, Ph.D. Thesis, Utrecht University, 2004.
[29] J.E. McClure, J.H. Smith, H. Jeffrey, A solution of Deligne's Hochschild cohomology conjecture, in: Recent Progress in Homotopy Theory (Baltimore, MD, 2000), in: Contemp. Math., vol. 293, Amer. Math. Soc., Providence, RI, 2002, pp. 153–193.
[30] J.E. McClure, J.H. Smith, H. Jeffrey, Multivariable cochain operations and little n-cubes, J. Amer. Math. Soc. 16 (3) (2003) 681–704.
[31] M. Markl, S. Shnider, Drinfel'd algebra deformations, homotopy comodules, and the associahedra, Trans. Amer. Math. Soc. 348 (1996) 3505–3547.
[32] M. Markl, S. Shnider, J. Stasheff, Operads in algebra, topology and physics, in: Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002, p. x+349.
[33] I. Madsen, U. Tillmann, The stable mapping class group and $Q(\mathbb{C}\mathrm{P}^\infty_+)$, Invent. Math. 145 (3) (2001) 509–544.
[34] R.C. Penner, Decorated Teichmüller theory of bordered surfaces, Comm. Anal. Geom. 12 (4) (2004) 793–820.
[35] D. Tamarkin, Another proof of M. Kontsevich formality theorem, math/9803025, Preprint; Formality of chain operad of small squares, Lett. Math. Phys. 66 (1–2) (2003) 65–72.
[36] U. Tillmann, Higher genus surface operad detects infinite loop spaces, Math. Ann. 317 (3) (2000) 613–628.
[37] U. Tillmann, Strings and the stable cohomology of mapping class groups, in: Proceedings of the International Congress of Mathematicians, Beijing, 2002, vol. II, Higher Ed. Press, Beijing, 2002, pp. 447–456.
[38] V. Tourtchine, Dyer–Lashof–Cohen operations in Hochschild cohomology, math.RA/0504017, Preprint.
[39] A.A. Voronov, Homotopy Gerstenhaber algebras, in: Conférence Moshé Flato 1999, vol. II (Dijon), in: Math. Phys. Stud., vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 307–331.
[40] A.A. Voronov, Notes on universal algebra, in: M. Lyubich, L. Takhtajan (Eds.), Graphs and Patterns in Mathematics and Theoretical Physics, in: Proc. Sympos. Pure Math., vol. 73, AMS, Providence, RI, 2005, pp. 81–103.