PATH SPACE DECOMPOSITIONS FOR THE VIRASORO ALGEBRA AND ITS VERMA MODULES

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Starting from a detailed analysis of the structure of path spaces of the $\mathcal{A}$ fusion graphs and the corresponding irreducible Virasoro algebra quotients $V(c, h)$ for the $(2, q$ odd) models, we introduce the notion of an admissible path space representation. The path spaces $\mathcal{P}_A$ over the $\mathcal{A}$ graphs are isomorphic to the path spaces over Coxeter A graphs that appear in FB models. We give explicit construction algorithms for admissible representations. From the finite-dimensional results of these algorithms we derive a decomposition of $V(c, h)$ into its positive and negative definite subspaces w.r.t. the Shapovalov form and the corresponding signature characters. Finally, we treat the Virasoro operation on the lattice induced by admissible representations, adopting a particle point of view. We use this analysis to decompose the Virasoro algebra generators themselves. This decomposition also takes into account the nonunitarity of the $(2, q)$ models.

1. Introduction

Ever since Ref. 1, the close connection between CFT and statistical mechanics has been known. Subsequent to that book, Belavin, Polyakov and Zamolodchikov[2] gave an in-depth analysis of the implications of conformal invariance in two dimensions.[3-5] The aforementioned correspondence can effectively be analyzed on the level of graphs and their path spaces. These made their first appearance in statistical mechanics as the configuration spaces of integrable models defined by Andrews, Forrester and Baxter.[6] Since the subsequent identification of critical parameters by Huse[7] as belonging to the unitary discrete FQS series of minimal models,[8] there has been a desire to construct a Virasoro representation on these path spaces in order to better understand the appearance of this algebra in this context. This would be especially intriguing, since the algebra appears even in off-critical cases in the calculation of local height probabilities.[9] One interesting aspect would be a correspondence between the Temperley–Lieb–Jones algebra predominant in these

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models[10] and the Virasoro algebra itself providing a link between statistical systems and the corresponding conformal field theories.[11,12] There has been a lot of research in developing new models along the lines of Ref. 6 and in the study of their properties.[10,13–19] In Ref. 20 Forrester and Baxter introduced new models whose critical behavior is nonunitary.[9] In this context one can find the $(2, q$ odd) series of minimal models in the critical behavior of statistical models based on Coxeter A_n graphs, which will be studied in this paper. The class of models based on Coxeter graphs can be extended to include all A–D–E graphs[21] parallel to the classification of modular invariant partition functions.[22–25]

There are two ways in which to proceed so as to arrive at the correspondence mentioned above. One can start from the statistical model side or from the CFT side. There has been some development on the statistical side (see Ref. 26 and references therein). In Ref. 26, the authors propose a double limit in order to obtain Virasoro generators from the Temperley–Lieb–Jones algebra. For a different approach see Refs. 27–29.

We shall tackle the above problem (construction of a Virasoro representation) from the CFT side, where little is known. On the CFT side, path spaces appear in sum formulas for the conformal characters.[30] These path spaces, however, are not over simple Coxeter graphs, but are more involved in the sense that the structure of the graphs which are considered is a bit more complicated. However, the $\mathcal{A}$ graphs which appear in the $(2, q)$ models (q odd) can be linked via path space isomorphisms to the Coxeter A_n graphs[30] (see also Sec. 2). In Ref. 31, another way of introducing graphs into CFT was found by rewriting character formulas.

In this paper, we will proceed from the path spaces over the $\mathcal{A}$ graphs. In Sec. 2 we will give the relevant definitions and notations which we will need in the following. Then, in Sec. 3, we will start the analysis of the path space and the Verma module quotient structure in order to establish a connection between them. As a result, we will introduce the notion of an admissible path space representation. In Sec. 4, we will give and discuss possible constructions for these representations. The calculations involving these constructions will then lead us to the main conjecture, which states signature character formulas for the $(2, q)$ models as well as the corresponding decomposition of the irreducible Virasoro quotient $V(c, h)$ into positive and negative definite subspaces $V(c, h) = V(c, h)^+ \oplus V(c, h)^-$. Finally, in Sec. 5, we turn to the structure of the Virasoro action induced by admissible representations on the path space. To this end, a particle interpretation is given. The action of the Virasoro generators L_n can then be described by a shift $(\widehat{L}_n)$ and a one-particle creation (resp. annihilation) (C_n) operator. The main conjecture can be restated in formulas for these operators and their adjoints. This reformulation explains the degree of nonunitarity of the $(2, q)$ models or, in other words, the degree of non-self-adjointness of the Virasoro algebra in the path space metric. We conclude the paper with a summary of the results and an outlook for further fields of study.

2. Preliminaries and Notations

Since we are interested in representations of the Virasoro algebra, it is useful to first fix notations. By the Virasoro algebra Vir we mean the complex Lie algebra generated by L_n, $n \in \mathbb{Z}$ and C with commutation relations

$$[L_n, L_m] = (m - n)L_{m+n} + \delta_{m+n,0}\frac{1}{12}C(m^3 - m), \tag{2.1}$$

$$[L_n, C] = 0. \tag{2.2}$$

Note the choice of signs in (2.1). As usual, we denote by $M(c, h)$ the Verma module to the eigenvalue c of C and lowest (choice of signs) weight h for L_0. Furthermore, let $J(c, h)$ be the unique maximal submodule and $V(c, h)$ the unique irreducible quotient:

$$V(c, h) = M(c, h)/J(c, h). \tag{2.3}$$

In this paper, we will concentrate on certain values of c, namely the ones belonging to the $(2, q)$ series. Here (p, q) denotes the c value:

$$c(p, q) = 1 - 6\frac{(p - q)^2}{pq}. \tag{2.4}$$

For these specific models, an explicit basis for the quotients $V(c, h)$ was given by Feigin, Nakanishi and Ooguri (FNO) in Ref. 32, the lowest weights being

$$h_j = \frac{-j(q - 2 - j)}{2q}, \quad j = 0, \ldots, N, \tag{2.5}$$

with $q = 2N + 3$. The basis is given by elements

$$L_{n_1} \cdots L_{n_m}|h_j\rangle, \tag{2.6}$$

with $n_1 \geq \cdots \geq n_m \geq 1$, which satisfy the following two conditions:

(1) $n_i - n_{i+N} \geq 2$ (difference 2 condition),
(2) $\#\{n_i = 1\} \leq j$ (initial condition).

Reference 32 also introduced an order which will be quite helpful later on. One defines

$$L_{m_1} \cdots L_{m_r} \succ L_{n_1} \cdots L_{n_s} \tag{2.7}$$

if

(i) $r > s$,
(ii) $r = s$ and $\sum_i m_i > \sum_i n_i$, or
(iii) $r = s$, $\sum_i m_i = \sum_i n_i$ and $m_t > n_t$, $m_i = n_i$ for $1 \leq i \leq t - 1$.

For the above basis a graphical enumeration method was given in Ref. 30. The graphs used for this are the fusion graphs $\mathcal{A}_{N+1}$ of the corresponding theory with $c(2N+3,2)$ (for a precise definition see Ref. 30). Here we just define the graph $\mathcal{A}_N$ by its incidence matrix. For $0 \leq i,j \leq N-1$,

$$(\mathcal{A}_N)_{i,j} = \begin{cases} 1 & \text{if } i+j < N, \\ 0 & \text{otherwise}. \end{cases}$$

The basis itself is then given by paths over the corresponding graphs. By this we mean the following: A path over a graph $\mathcal{G}$ with vertices numbered by a set I is a map : $\mathbb{N}_0 \mapsto I$, i.e. a sequence of vertices $(l_i)_i$, with the restriction that two successive vertices are linked.

Furthermore, let $\mathcal{P}_{\{\mathcal{G},l_0,l_\infty\}}$ denote the Hilbert space with basis given by the paths on the graph $\mathcal{G}$ which start at l_0 and end in l_∞. This means that for $i \gg 1$, $l_i = l_\infty$. The scalar product on the space is chosen to be the one in which all paths are mutually orthonormal.

We then have a bijection[30] between $\mathcal{P}_{\{\mathcal{A}_{N+1}, N-j, 0\}}$ and $V(c(2,2N+3), h_j)$ which is given by the simple map:

$$((l_i)_i) \mapsto \cdots L_2^{l_2} L_1^{l_1} = \prod_{i=\infty}^{1} L_i^{l_i} \quad \text{for } i \in \mathbb{N}. \tag{2.8}$$

The restrictions for the paths given by the graph are directly translated into the difference 2 condition of the FNO basis. The index $N+1$ fixes c to $c(2,2N+3)$ and the first term of the sequence l_0 dictates the h value of the theory and guarantees compliance with the initial condition. In the path spaces we have an L_0 which provides the usual grading:

$$L_0(l_i)_i := \left(h_{N-l_0} + \sum_{i \in \mathbb{N}} i l_i \right) \cdot ((l_i)_i). \tag{2.9}$$

By K-theoretic arguments one can show that the path space over the $\mathcal{A}_{N+1}$ graph is isomorphic to the Coxeter $A_{2(N+1)} = A_{q-1}$ graph considered in Refs. 9 and 20.

Example. In the case of the Lee–Yang edge singularity [the (2,5) model], an isomorphism can easily be given. We just relabel the A_4 graph:
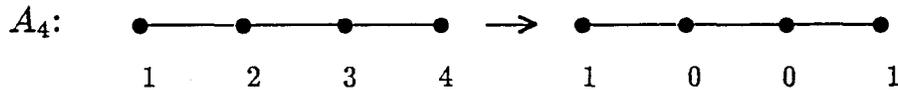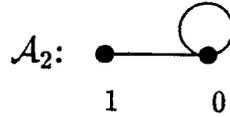

Fig. 1. Relabeling of the A_4 graph.

and see that in this labeling the path space over A_4 is just the same as over the $\mathcal{A}_2$ graph.

Fig. 2. $\mathcal{A}_2$ fusion graph.

One can also check that L_0 [Eq. (2.9)] gives the same spectrum over the ground state as in the FB models.

This example shows the obvious advantages of the $\mathcal{A}$ graphs. The ground state $(\cdots 32323 \cdots)$ of the FB model reads simply $(000 \cdots)$ and the operator L_0 just depends on the site i, not on three neighboring sites. We also see that the energy of the ground state is 0 and thus it is much easier to find the energy of the excited state, since no ground state contribution has to be subtracted.

In general, we can see that in the A_{2n} models there are just n distinct labels which can appear in the even (resp. odd) lattices, and thus a graph with N vertices should suffice. Furthermore, one feature of the $\mathcal{A}$ graphs is that the vertex 0 is always connected to itself. So we always have a ground state $(000 \cdots)$ which replaces a ground state $(\cdots l, l-1, l, l-1 \cdots)$, and thus we also have a simpler structure for the excitations. In fact, part of this has already been realized in Ref. 6, which introduced the following relabeling of the Coxeter A_{r-1} graph, if r is odd $(r = 2N + 3)$:

$$
l_i \mapsto \begin{cases} \frac{1}{2}(2N + 1 - l_i) & \text{if } l_i \text{ odd}, \\ \frac{1}{2}(l_i - 2) & \text{if } l_i \text{ even}. \end{cases}
\tag{2.10}
$$

These labels correspond to those of the fusion graphs. Furthermore they map the initial conditions of the fusion and FB path spaces for the h values onto each other. Thus one can reassociate the fields of the given CFT with the edges of the graph in a procedure reverse to the one presented in Ref. 30.

In particular, for the (2,5) model we see that the fusion $\mathcal{A}_2$ and the relabeled Coxeter A_4 graph are identical. In this way, all results for the path spaces of the former can be translated back to the path space of the Coxeter A_4 graph. For the translation one must, however, first make a choice for the odd and even sublattices (see Ref. 6).

The above relabeling was interpreted in Ref. 6 as a lattice gas picture. For the ABF (resp. FB) models, the restriction given by the graph amounts to the restriction that the sum of particles on neighboring lattice sites must be N or $N - 1$. In this line of thinking, the fusion graphs describe a lattice gas with the perhaps more natural restriction that there are at most N particles on two neighboring sites. The described particle interpretation as a lattice gas is adopted in this paper.

In the following we want to extend the operator L_0 [Eq. (2.9)] to a full representation of Vir.

3. Admissible Representations for $\mathcal{P}_\mathcal{A}$

3.1. *General situation*

We are looking for an irreducible representation of the Virasoro algebra on the path space $\mathcal{P}_\mathcal{A}$. One other way to state this problem is given by the following:

Remark. If such a representation exists, then

$$\mathcal{P}_\mathcal{A} \cong V(c, h) \quad \text{as Vir modules}. \tag{3.1}$$

Because of the universal property of the Verma module there exists a surjective homomorphism $\pi : M(c, h) \to \mathcal{P}_\mathcal{A}$.[33] Then $K := \ker(\pi)$ is a submodule. Therefore K is a submodule of the unique maximal submodule $J(c, h)$. Hence $K = J(c, h)$, since $\dim_n K := \dim_n J(c, h)$, using the isomorphism (2.8).

So the problem can be reformulated in the following way: The object we are looking for is an isomorphism

$$\Phi : \mathcal{P}_\mathcal{A} \longrightarrow V(c, h). \tag{3.2}$$

One such isomorphism can be easily given in the following way:

Example 1

$$\Phi((l_i)_i) := \prod_{i=\infty}^{1} L_i^{l_i}. \tag{3.3}$$

This is possible since the $\mathcal{A}$ graphs encode exactly the restrictions which appear for the FNO basis.

Example 1, however, does not respect all of the various natural structures of $\mathcal{P}_\mathcal{A}$ and $V(c, h)$, which will be introduced in the following subsection.

3.2. *Natural structures on $\mathcal{P}_\mathcal{A}$ and $V(c, h)$*

If we regard $\mathcal{P}_\mathcal{A}$ (denoted in the following by $\mathcal{P}$) simply as a configuration space of path space origin, we can associate the following structures: first we have two operators L_0 and K on $\mathcal{P}$, where L_0 is the usual path space L_0 [Eq. (2.9)]:

$$L_0((l_i)_i) := \left(h_{N-l_0} + \sum_{i \in \mathbb{N}} i l_i \right) \cdot ((l_i)_i), \tag{3.4}$$

and K is defined by

$$K((l_i)_i) := \left(\sum_{i \in \mathbb{N}} l_i \right) \cdot ((l_i)_i). \tag{3.5}$$

Together they provide a double grading:

$$\mathcal{P} = \bigoplus_{n,k} \mathcal{P}_{n,k} \tag{3.6}$$

where

$$L_0|_{\mathcal{P}_{n,k}} = (n + h_{N-l_0}) \cdot id \,,$$

$$K|_{\mathcal{P}_{n,k}} = k \cdot id \,.$$

Furthermore, we have a natural scalar product $(\ ,\ )_{\mathcal{P}}$ of the Hilbert space which takes into account the configuration space nature. It is defined by the fact that all paths are mutually orthonormal:

$$(((l_i)_i), ((g_i)_i))_{\mathcal{P}} := \prod_i \delta_{l_i, g_i} \,. \tag{3.7}$$

This metric turns (3.6) into an orthogonal decomposition,

$$\mathcal{P} = \perp_{n,k} \mathcal{P}_{n,k} \,. \tag{3.8}$$

So the path space basis should yield an orthogonal basis for the Verma module quotient under a reasonable representation (resp. isomorphism Φ). And we should find a structure corresponding to the double grading (3.6).

If, on the other hand, we look at $V(c, h)$, we have the usual grading given by L_0 and a canonical Hermitian form, the Shapovalov form (see e.g. Ref. 33), which we denote by $\langle\ ,\ \rangle_S$.

Together, these provide the usual orthogonal decomposition of $V(c, h)$:

$$V(c, h) = \perp_n V_n(c, h) \,. \tag{3.9}$$

Although there is no obvious second grading which would provide another orthogonal splitting, we do have a filtration.

Consider the spaces

$$\widetilde{V}_{n,k} := \mathrm{span}\left\langle L_{n_l} \cdots L_{n_1} | n_l \geq \cdots \geq n_1, l \leq k \text{ and } \sum_i n_i = n \right\rangle$$

(i.e. linear combinations of products of at most k L_i's of energy n).

These spaces give the above-mentioned filtration of each $V_n(c, h)$:

$$\emptyset = \widetilde{V}_{n,-1} \subset \widetilde{V}_{n,0} \subset \cdots \subset \widetilde{V}_{n,k-1} \subset \widetilde{V}_{n,k} \subset \widetilde{V}_{n,k+1} \subset \cdots \subset \widetilde{V}_{n,\infty}$$

$$= V_n(c, h) \,. \tag{3.10}$$

The successive quotients of this filtration are especially interesting, since they have the same dimensions as the respective path space counterparts. To this end let $\widehat{V}_{n,k} := \widetilde{V}_{n,k}/\widetilde{V}_{n,k-1}$ denote these quotients; then we again have, using (2.8),

$$\dim \widehat{V}_{n,k} = \dim \mathcal{P}_{n,k} \,. \tag{3.11}$$

We denote by $\pi_{n,k}$ the canonical projection $\pi_{n,k} : \widetilde{V}_{n,k} \to \widehat{V}_{n,k}$.

After these considerations it is clear which special features an isomorphism (3.2) should have in order to preserve the natural structures of $\mathcal{P}$. The following definition is therefore central.

We call an isomorphism $\Phi : \mathcal{P}_{\mathcal{A}} \to V(c, h)$ *admissible* if the following two conditions hold:

(a) $\Phi(\mathcal{P}_{n,k}) \subset \widetilde{V}_{n,k}$ and thus $\pi_{n,k} \circ \Phi(\mathcal{P}_{n,k}) = V_{n,k}$;

(b) $\Phi^*\langle \, , \, \rangle_S$ is orthogonal in the path basis, i.e. $\langle \Phi((l_i)_i), \Phi((g_i)_i) \rangle = \epsilon((l_i)_i, (g_i)_i) \cdot ((l_i)_i, (g_i)_i)_{\mathcal{P}}$, with $\epsilon((l_i)_i, (g_i)_i) \in \mathbb{C}$.

Remarks

(1) We can always normalize so that the ϵ of (b) is ± 1.

(2) The remark in condition (a) encodes the information (3.11) about the dimensions of the corresponding spaces. Condition (a) itself coarsely identifies the number of L's with the sum of the nonzero entries in the path sequence. In particular, an entry k signifies the presence of k L's (except for l_0, of course, which just fixes h).

(3) Condition (b) guarantees the existence of an orthogonal basis complementing the previous one in each step of the filtration. Reformulated, condition (b) is equivalent to the existence of orthogonal sections $s_{n,k}$ (i.e. $\pi_{n,k} \circ s_{n,k} = id$).

$$0 \longrightarrow \widetilde{V}_{n,k-1} \longrightarrow \widetilde{V}_{n,k} \underset{s_{n,k}}{\overset{\pi_{n,k}}{\underset{\longleftarrow}{\longrightarrow}}} \widehat{V}_{n,k} \longrightarrow 0 \,, \tag{3.12}$$

which are orthogonal in the sense that

$$s_{n,k}(\widehat{V}_{n,k}) \perp \widetilde{V}_{n,k-1}. \tag{3.13}$$

Together, these sections give rise to an orthogonal decomposition: let $V_{n,k} := s_{n,k}(\widehat{V}_{n,k})$; then

$$V(c, h) = \perp_{n,k} V_{n,k} \,, \tag{3.14}$$

corresponding to the double grading (3.6) of the path space. The existence of such a basis, however, is nontrivial; it is, rather, the crucial point. If we look at Example 1 [Eq. (3.3)], for instance, we see that it satisfies condition (a) and fails to satisfy condition (b). In other words, the configuration space nature of the path space leads to the prediction of a specific orthogonal basis of $V(c, h)$.

(4) An isomorphism Φ' can be reconstructed from the sections $s_{n,k}$ by

$$\Phi'((l_i)_i) = s_{n,k} \circ \pi_{n,k} \left(\prod_{i=\infty}^{1} L_i^{l_i} \right) \,, \quad \text{for } (l_i)_i \in \mathcal{P}_{n,k} \,. \tag{3.15}$$

Although this isomorphism is not necessarily admissible, it satisfies the following two conditions:

(a) $\Phi'(\mathcal{P}_{n,k}) \subset \widetilde{V}_{n,k}$,

(b') $\Phi'(\mathcal{P}_{n,k}) \perp \Phi'(\mathcal{P}_{n',k'})$ w.r.t. $\langle\ ,\ \rangle_S$, if $n \neq n'$ or $k \neq k'$.

An isomorphism satisfying conditions (a) and (b') will be called *weakly admissible*. Given a weakly admissible isomorphism Φ', we can always construct an admissible isomorphism Φ by orthogonalizing inside the spaces $\Phi'(\mathcal{P}_{n,k})$. This is always possible, since the Shapovalov form is nondegenerate and by condition (b') we already have an orthogonal decomposition $V(c,h) = \perp_{n,k} \Phi'(\mathcal{P}_{n,k})$, so that the form is nondegenerate on the different parts of this decomposition, and hence diagonalizable. Independently of the chosen orthogonalization procedure we have

$$\Phi'(\mathcal{P}_{n,k}) = \Phi(\mathcal{P}_{n,k}) := V_{n,k}\,. \tag{3.16}$$

So the spaces $V_{n,k}$ as well as the decomposition corresponding to the double grading of the path space $V(c,h) = \perp_{n,k} V_{n,k}$ depend only on the weakly admissible structure.

Additional remark. In some cases (if $h = 0$), it is useful to restrict oneself to the quasiprimary objects. Let

$$V_n^{\text{q.p.}} := \ker(L_{-1}|_{V(c,h)_n})\,,$$

$$\widetilde{V}_{n,k}^{\text{q.p.}} := \ker(L_{-1}|_{\widetilde{V}_{n,k}})\,,$$

$$\text{resp. } \widehat{V}_{n,k}^{\text{q.p.}} := \ker(L_{-1}|_{\widehat{V}_{n,k}})\,.$$

Analogously, we define orthogonal sections $s_{n,k}^{\text{q.p.}}$ by

$$0 \longrightarrow \widetilde{V}_{n,k-1}^{\text{q.p.}} \longrightarrow \widetilde{V}_{n,k}^{\text{q.p.}} \underset{s_{n,k}^{\text{q.p.}}}{\overset{\pi_{n,k}}{\underset{\longleftarrow}{\longrightarrow}}} \widehat{V}_{n,k}^{\text{q.p.}} \longrightarrow 0\,. \tag{3.17}$$

The whole information can be retrieved due to the fact that the $SL(2,\mathbb{C})$ sub-Verma modules based on $V_n^{\text{q.p.}}$ provide a basis for $V(c,h)$:

$$\bigoplus_n U(L_1)V_n^{\text{q.p.}} = V(c,h)\,. \tag{3.18}$$

The contravariance of the Shapovalov form and the fact that $L_1|_{\mathcal{P}_{n,k}} \in \mathcal{P}_{n,k}$ if $h = 0$ (see Subsec. 4.1) guarantee that the orthogonal decomposition of the quasiprimary part

$$V(c,h)^{\text{q.p.}} = \perp_{n,k} \widehat{V}_{n,k} \tag{3.19}$$

$[\widehat{V}_{n,k} := s_{n,k}^{\text{q.p.}}(V_{n,k}^{\text{q.p.}})]$ will induce the decomposition (3.14) on the whole of $V(c,h)$. This fact will be used for a construction procedure in the next section.

4. Admissible Representations and Signature Characters

4.1. *Constructions for admissible representations*

One way to find an orthogonal basis of $V(c, h)$ (resp. the sections $s_{n,k}$) with the required restrictions is the following:

Construction 1. We simply orthogonalize the basis vectors of FNO with respect to the FNO order (2.7).

As mentioned before, we do not know *a priori* that this algorithm will work, since the Shapovalov form is not definite and isotropic vectors may occur. But, on the other hand, the existence (resp. the success of the algorithm) will be exactly the new information gained. The corresponding isomorphism would be

$$\Phi_{\text{FNO}}((l_i)_i) := \left(\prod_{i=\infty}^{1} L_i^{l_i} \right)^{\perp_{\text{FNO}}}, \tag{4.1}$$

where the superscript $\perp_{\text{FNO}}$ refers to the above-mentioned orthogonalization. In doing the calculations, we proceeded as follows: we just generated the Shapovalov form in the FNO basis and used it to do the orthogonalization. This was carried out up to $V_{32,5}$ (i.e. for a 93-dimensional subspace) for the (2,5) model and up to $V_{15,5}$ (i.e. for a 37-dimensional subspace) for the $(2, q$ odd) models for $q < 35$.

We could also use any other order in which

$$L_{m_1} \cdots L_{m_r} \succ L_{n_1} \cdots L_{n_s} \tag{4.2}$$

if $r > s$. The FNO order is, however, best suited (see Subsec. 4.2).

Construction 2. As mentioned before, if $h = 0$ we can restrict ourselves to the quasi-primary objects. Hence, we first enlarge a basis of $\widetilde{V}_{n,k-1}^{\text{q.p.}}$ which we already know by induction to a basis of $\widetilde{V}_{n,k}^{\text{q.p.}}$ and orthogonalize the new vectors in an arbitrary fashion.

This procedure can be refined, in the sense that the FNO ordering provides an even finer filtration than (3.10) [as above, we could choose any other ordering with the condition (4.2)], so we can order the new basis and orthogonalize w.r.t. this ordering as in Construction 1.

Although this construction seems more tedious, especially if one wants to recover the full basis, it can be quite useful for the calculation of the signature, since all L_1 descendent vectors of quasi-primary ones have the same sign of the metric. The calculation itself was performed in the following manner: if $h = 0$ we have the nice feature that each singular vector can be taken to be of the form

$$L_m L_m L_{n_r} \cdots L_{n_1} + \text{lower order terms w.r.t. the ordering (2.7)} . \tag{4.3}$$

If $k = 2$ this is especially nice, since $V_{n,2}^{\text{q.p.}}$ is one-dimensional. The construction was carried out for $k = 2$ up to $n = 200$ for all $(2, q)$ models.

We have used both these constructions explicitly up to different grades for n, k. The results of the algorithms and the terms of the resulting signature characters are contained in the next subsection.

4.2. *Signature characters for the* $(2, q)$ *models*

Since any admissible representation results in an orthogonal decomposition and, in addition, provides an orthogonal basis of $V(c, h)$ (corresponding to the path basis), it can be used to calculate signatures of the different parts of the decomposition (3.14). As mentioned above, this has been carried out for various values of n and k.

For all calculated examples we find that

$$\text{sign}(\langle\,,\,\rangle_S)|_{V_{n,k}} = (-1)^k \dim V_{n,k}\,. \tag{4.4}$$

In other words, the Shapovalov form is $(-1)^k$ definite on $V_{n,k}$. This is truly a remarkable result: from the path space nature we not only get an orthogonal basis, but also find — and this is very important — a basis which decomposes the path space of $V(c, h)$ into positive and negative definite subspaces:

$$V(c, h) = V(c, h)^+ \oplus V(c, h)^-\,. \tag{4.5}$$

For a discussion on the alternating sign of the definiteness, see the Remark in Subsec. 5.3.

The data presented are far out of the region of mere coincidence and thus lead us to the following conjecture:

Main conjecture. In the $(2, q)$ minimal models admissible representations exist, and they provide a splitting (4.5) of $V(c, h)$ into positive and negative definite subspaces by

$$V(c, h)^+ := \bigoplus_{\substack{n \\ k \text{ even}}} V_{n,k}\,, \quad V(c, h)^- := \bigoplus_{\substack{n \\ k \text{ odd}}} V_{n,k}\,. \tag{4.6}$$

The resulting signature character would be

$$\sigma(c(2, 2N+3), h_j)(q) = \sum_{n_1,\ldots,n_N \geq 0} (-1)^{\left(\sum_{i=1}^{N} i n_i\right)} \frac{q^{N_1^2 + \cdots + N_N^2 + N_j + \cdots + N_N}}{(q)_{n_1} \cdots (q)_{n_N}} \tag{4.7}$$

with $N_i = \sum_{j=i}^{N} n_j$ and $(q)_i = \prod_{j=1}^{i} (1 - q^j)$.

Remarks

(1) The signature character formulas (4.7) have been previously conjectured by Nahm.[34] They have been compared to the ones given by Kent[35,36] up to $O(q^{100})$ for the $(2, 5)$ and the $(2, 7)$ model, and the two coincide.[37]

(2) The exact structure of the sections or of the orthogonalization is of no importance for the induced metric on $\mathcal{P}$, since it is positive (resp. negative)

definite on each of the $\mathcal{P}_{n,k}$. So any other choice of Φ would yield the same results. Only the weak admissibility is of importance.

(3) The characters corresponding to (4.7) are the same except for the term $(-1)^{\sum_{i=1}^{N} i n_i}$ and have been given in Ref. 32. They correspond to a series of sum rules for characters which can be interpreted in terms of quasi-particles.[31] In this setting, the exponent $\sum_{i=1}^{N} i n_i$ is just the sum over the number of quasi-particles of type i weighted with their energy weights (see also Subsec. 5.1).

5. Virasoro on the Lattice

5.1. *General remarks*

Any isomorphism Φ [see (3.2)] is, of course, nothing but a realization of the Virasoro algebra on the lattice in the sense of Sec. 1. If we now look at admissible representations only, we can describe the action on the paths themselves more explicitly. The total structure is, however, dependent on the specific choice of basis induced by the choice for Φ. In general, when considering only the weakly admissible structure of an admissible representation, we can find the following:

$$L_m(\widetilde{V}_{n,k}) \subset \widetilde{V}_{n+m,k+1}, \tag{5.1}$$

$$L_{-m}(\widetilde{V}_{n,k}) \subset \widetilde{V}_{n-m,k}. \tag{5.2}$$

Furthermore, we see from the restrictions for the annihilating ideals[32] that

$$L_1^l(\widetilde{V}_{n,k}) \subset \widetilde{V}_{n+l,k+j}, \tag{5.3}$$

$$L_2^r(\widetilde{V}_{n,k}) \subset \widetilde{V}_{n+2r,k+N+j}, \tag{5.4}$$

for $c = c(2, 2N + 3)$ and $h = h_j$.

The first inclusion is given by the initial condition and the second by the difference 2 condition.

The inclusions (5.1) and (5.2) are most easily seen in the FNO basis:

$$L_m(L_{n_k} \cdots L_{n_1}) = \sum_{i=k}^{r} [(n_i - m) L_{n_k} \cdots L_{n_i+m} \cdots L_{n_1}]$$
$$+ L_{n_k} \cdots L_{n_{r+1}} L_m L_{n_r} \cdots L_{n_1}, \tag{5.5}$$

where $n_{r+1} > m \geq n_r$.

If any of the above vectors violates the conditions for the FNO basis, its expression in terms of the basis vectors contains not more L_i's than before. This is seen directly from the annihilating ideals and the ordering as in the proof in Ref. 32.

If we now pull the Virasoro action back with an admissible Φ onto the path space, the above inclusions translate directly into restrictions on the $\mathcal{P}_{n,k}$:

$$L_m(\mathcal{P}_{n,k}) \subset \mathcal{P}_{n+m,k} \oplus \mathcal{P}_{n+m,k+1} \,, \tag{5.6}$$

$$L_{-m}(\mathcal{P}_{n,k}) \subset \mathcal{P}_{n-m,k-1} \oplus \mathcal{P}_{n-m,k} \,. \tag{5.7}$$

This can be seen in the pulled-back Shapovalov form. First of all, we have the inclusions (5.1) and (5.2) for the path spaces as well:

$$L_m(\mathcal{P}_{n,k}) \subset \bigoplus_{j=1}^{k+1} \mathcal{P}_{n+m,j} \,, \tag{5.8}$$

$$L_{-m}(\mathcal{P}_{n,k}) \subset \bigoplus_{j=1}^{k} \mathcal{P}_{n-m,j} \,. \tag{5.9}$$

Furthermore we see from the orthogonality (3.14) that

$$\text{for } (g_i)_i \in \mathcal{P}_{n,k} \quad \text{and} \quad (l_i)_i \in \bigoplus_{j=1}^{k-1} \mathcal{P}_{n+m,j}$$

$$\Phi^* \langle (l_i)_i, L_m((g_i)_i) \rangle_S = \langle \Phi((l_i)_i), L_m\Phi((g_i)_i) \rangle_S$$

$$= \langle L_{-m}\Phi((l_i)_i), \Phi((g_i)_i) \rangle_S$$

$$= 0 \tag{5.10}$$

by (3.13), since $L_{-m}\Phi((l_i)_i) \in \widetilde{V}_{n,k-1}$ and $\Phi((g_i)_i) \in s_{n,k}(\widehat{V}_{n,k})$.

So (5.6) follows. (5.7) follows in the same manner. From the initial condition (5.3) and from the difference 2 condition (5.4), we find by similar arguments that

$$L_1^l(\mathcal{P}_{n,k}) \subset \mathcal{P}_{n+l,k} \oplus \cdots \oplus \mathcal{P}_{n+l,k+j} \,, \tag{5.11}$$

$$L_2^r(\mathcal{P}_{n,k}) \subset \mathcal{P}_{n+2r,k} \oplus \cdots \oplus \mathcal{P}_{n+2r,k+N+j} \,, \tag{5.12}$$

if $c = c(2, 2N+3)$ and $h = h_j$.

These restrictions concerning the action on the paths lead us to the following additive splitting of the usual Virasoro generators:

$$\text{Let } L_m = \widehat{L}_m + C_m, \text{ for } m \in \mathbb{Z} \,, \tag{5.13}$$

$$\text{where } \widehat{L}_m|_{\mathcal{P}_{n,k}} = P_{n,k} \circ L_m \,,$$

$$C_m|_{\mathcal{P}_{n,k}} = P_{n,k+1} \circ L_m, \text{ for } m > 0 \,,$$

$$C_m|_{\mathcal{P}_{n,k}} = P_{n,k-1} \circ L_m, \text{ for } m < 0 \,,$$

where $P_{n,k}$ denotes the orthogonal projection onto $\mathcal{P}_{n,k}$. The relations (5.3) and (5.4) can now be written as

$$C_1^{j+1} = 0 \,, \tag{5.14}$$

$$C_2^{N+j+1} = 0, \text{ if } c = c(2, 2N+3) \text{ and } h = h_j \,. \tag{5.15}$$

This means that the information about h (resp. c) is encoded in the operators C_1 (resp. C_2). These characteristic quantities can now be simply read off from the nilpotency index of the respective operator.

We can also find commutation relations for these operators by substituting (5.13) into the basic Virasoro relation (2.1) and then comparing the degrees of the various operators w.r.t. the K grading.

To simplify things, consider from now on $\widehat{L}_n$ and C_n for $n \in \mathbb{N}$ only and $\widehat{L}_n^\dagger = \widehat{L}_{-n}$ (resp. $C_n{}^\dagger = C_{-n}$), where the dagger is the adjoint w.r.t. the Shapovalov form.

We then have (as well as the resp. daggered equations)

$$[\widehat{L}_n, \widehat{L}_m] = (m - n)\widehat{L}_{n+m}, \tag{5.16}$$

$$[\widehat{L}_n, C_m] + [C_n, \widehat{L}_m] = (m - n)C_{n+m}, \tag{5.17}$$

$$[C_n, C_m] = 0, \tag{5.18}$$

and as mixed commutators if $m \geq n$,

$$[\widehat{L}_n^\dagger, \widehat{L}_m] + [C_n{}^\dagger, C_m] = (m + n)\widehat{L}_{m-n} + \delta_{m,n}\frac{c}{12}(m^3 - m), \tag{5.19}$$

$$[\widehat{L}_n^\dagger, C_m] = (m + n)C_{m-n}, \tag{5.20}$$

$$[C_n{}^\dagger, \widehat{L}_m] = 0. \tag{5.21}$$

If $m \leq n$ we obtain the daggered equations:

$$[\widehat{L}_n^\dagger, \widehat{L}_m] + [C_n{}^\dagger, C_m] = (m + n)\widehat{L}_{n-m}^\dagger + \delta_{m,n}\frac{c}{12}(m^3 - m), \tag{5.22}$$

$$[C_n{}^\dagger, \widehat{L}_m] = (m + n)C_{n-m}^\dagger, \tag{5.23}$$

$$[\widehat{L}_n^\dagger, C_m] = 0. \tag{5.24}$$

Although these relations seem a bit more complicated than the original Virasoro relations, there are several reasons for the introduction of the operators $\widehat{L}_n$ and C_n. One is that in a particle interpretation of the path space they can be viewed as shift (resp. creation–annihilation) operators. There are several ways in which to give a particle interpretation for the sequences $(l_i)_i$. In Ref. 31, for instance, a quasi-particle interpretation was developed. According to this any $(2, q)$ model contains N different types of quasi-particles.

Here we shall adopt a simpler point of view. A nonzero l_i is just taken to signify the presence of l_i particles on the site i. The connection to the quasi-particle picture is just looking at a quasi-particle of type i as being made up of i single particles. In this context, the spaces $\mathcal{P}_{n,k}$ are the configuration spaces of k particles of total energy n. In these terms C_n are one-particle creation operators and $C_n{}^\dagger$ one-particle annihilation operators, while $\widehat{L}_n$, $\widehat{L}_n^\dagger$ conserve the total number of particles and

thus just produce a net number of n shifts to the right (resp. left). In particular, we have

$$[\widehat{L}_1, \widehat{L}_n] = (n - 1)\widehat{L}_{n+1} \quad \text{for } i \in \mathbb{N}, \tag{5.25}$$

$$[\widehat{L}_1, C_n] + [C_1, \widehat{L}_n] = (n - 1)C_{n+1}. \tag{5.26}$$

This expresses the fact that higher order shifts are obtained by nested $\widehat{L}_1$ commutators of $\widehat{L}_2$. Furthermore, the commutators simplify for $h = 0$. Then we have, from the initial condition,

$$C_1 = C_{-1} = 0, \quad \text{and thus} \quad \widehat{L}_1 = L_1, \tag{5.27}$$

and the above relations simplify to

$$[L_{\pm 1}, \widehat{L}_n] = (n \mp 1)\widehat{L}_{n\pm 1}, \tag{5.28}$$

$$[L_{\pm 1}, C_n] = (n \mp 1)C_{n\pm 1}. \tag{5.29}$$

Now higher order shifts and creation (resp. annihilation) of particles by Virasoro action are obtained by nested L_1 commutators of $\widehat{L}_2$ and C_2 (resp. C_{-2}). The above remarks characterize the net action of the Virasoro on a k-particle state. The finer structure — how the shifts look on the single particles or where along the chain a new particle is created — depends on the specific choice of basis for the isomorphism Φ. In the next section we will discuss this question for Constructions 1 [(4.1), i.e. the orthogonalized FNO basis] and 2 (quasi-primary construction).

5.2. *Shifts and creation*

The simplest action of L_n one could imagine would be that $\widehat{L}_n$ shifts each particle n sites and C_n creates a particle at site n. This is almost the case in Example 1. An action like this would be something like the Sugawara construction.[33] Problems arise, however, if the new configuration is not allowed. Here the action becomes very complicated. It can happen, for instance, that all particles are shifted or more than one particle is annihilated, etc. This is an effect induced by the nonadmissibility of this example. The admissibility guarantees an overall control over the action, as discussed in the previous subsection. The price paid is the "loss" of a simple shift and creation structure. In Construction 1, however, we have further control over the Virasoro action. To this end we associate with a sequence $(l_i)_i$ its finite number of nonzero members:

$$(l_i)_i \longrightarrow (l_1^{i_1}, \ldots, l_m^{i_m}), \tag{5.30}$$

i.e. the configuration with l_k particles at site i_k.

We now examine the specific action of the Virasoro for Construction 1:

(1) $\widehat{L}_n$ results in a sum of configurations where the highest particle is shifted at most n steps. In the ordering (2.7) we have

$$\widehat{L}_n(n_1^{i_1}, \ldots, n_m^{i_m}) = \sum_{\text{config.} \in \mathcal{P}_{..k}} \lambda_{\text{config.}} \text{config.} \tag{5.31}$$

with $\sum_i n_i = k$ and config. $\succeq (n_1^{i_1}, \ldots, (n_m - 1)^{i_m}, 1^{i_m + n})$.

(2) C_n creates a sum of configurations, in which the new particle is at most at site n:

$$C_n(n_1^{i_1}, \ldots, n_m^{i_m}) = \sum_{\text{config.} \in \mathcal{P}_{..k+1}} \lambda_{\text{config.}} \text{config.} \tag{5.32}$$

with $\sum_i n_i = k$ and

$$\text{config.} \preceq \begin{cases} (n_1^{i_1}, \ldots, n_r^{i_r}, 1^n, n_{r+1}^{i_{r+1}}, \ldots, n_m^{i_m}) & \text{if } r < n < r+1, \\ (n_1^{i_1}, \ldots, (n_r + 1)^{i_r}, \ldots, n_m^{i_m}) & \text{if } r = n. \end{cases}$$

If any of the configurations above is not allowed, the respective coefficient is 0. These relations are proven analogously to the inclusions (5.1) and (5.2). Of course, there are analogous restrictions for the daggered operators.

Example. In particular, in the vacuum sector of the (2,5) model C_2 creates only particles at site 2. So, for instance,

$$C_2(1^2, n_2^{i_2}, \ldots, n_m^{i_m}) = 0,$$

$$C_2(1^3, n_2^{i_2}, \ldots, n_m^{i_m}) = 0,$$

$$C_2(1^4, n_2^{i_2}, \ldots, n_m^{i_m}) = \sum_{\text{config.} \in \mathcal{P}_{..k+1}} \lambda_{\text{config.}} \text{config.}$$

with   config. $\preceq (1^2, 1^4, \ldots, n_m^{i_m})$,

resp.

$$C_2^\dagger(1^2, n_2^{i_2}, \ldots, n_m^{i_m}) = \sum_{\text{config.} \in \mathcal{P}_{..k-1}} \lambda_{\text{config.}} \text{config.}$$

with   config. $\succeq (n_2^{i_2}, \ldots, n_m^{i_m})$,

$$C_2^\dagger(1^3, n_2^{i_2}, \ldots, n_m^{i_m}) = 0,$$

$$C_2^\dagger(1^4, n_2^{i_2}, \ldots, n_m^{i_m}) = 0.$$

The relevant coefficients of this example have been calculated up to $n=16$ and comply with the above restrictions.

If we now turn to Construction 2, we see that the action of L_1 is particularly simple. In fact one defines the whole isomorphism by the action

$$L_1(n_1^{i_1}, \ldots, n_m^{i_m}) = (n_1^{i_1}, \ldots, (n_m - 1)^{i_m}, 1^{i_m + 1}). \tag{5.33}$$

The rest of the Virasoro action is, however, very complicated, since there are no further restrictions apart from (5.6) and (5.7), so that in the generic case all coefficients $\lambda_{\text{config.}} \neq 0$. In fact, we have not found any other admissible representations

with harder restrictions on the overall action of Vir than Construction 1. For constructions resulting from a different ordering this is fairly easily seen.

5.3. *Nonunitarity and the path space metric*

If we regard the adjoint ‡ with respect to $(\, , \,)_\mathcal{P}$, we have no nice formulas for L_n. This is a general fact which results from the negative c values. It has been remarked several times[9,28,29,33] that one can define a positive definite scalar product on the configuration space, but that in this product the Virasoro algebra is not self-adjoint. In fact, it cannot be, due to the negative c. The complex structure of the shifts can also be explained in this way. For the treated c values one cannot find a Sugawara construction in which the Heisenberg algebra and the Virasoro algebra are both self-adjoint.[33] From the main conjecture, we have, however,

$$\widehat{L}_n^{\ddagger} = \widehat{L}_{-n} \, , \tag{5.34}$$

$$C_n^{\ddagger} = -C_{-n} \, . \tag{5.35}$$

These equations establish the connection between the positive definite scalar product on the configuration space and the way the Virasoro algebra behaves under adjunction. In this way, the above relations can also be taken to be a reformulation of the main conjecture. In the particle language, this reads in the following way: The nonunitarity of the $(2, q)$ models manifests itself only in the different sign between the creation and annihilation operators under adjunction.

Remark. In a way this is the simplest possible nonunitarity. If we want the pulled-back Shapovalov form to be definite (positive or negative) on the configuration space of k particles $\oplus_n \mathcal{P}_{n,k}$, then the definiteness changes sign from the space of k particles to the space of $k+1$ particles. Let $(l_i)_i \in \mathcal{P}_{n,k}$ and $m \gg n$. From our assumption of definiteness we have (5.34) and $C_n^{\ddagger} = \pm C_{-n}$, so

$$\begin{aligned}
((l_i)_i, [L_{-m}, L_m](l_i)_i)_\mathcal{P} &= 2mn + \frac{c}{12}(m^3 - m) < 0 \\
&= ((l_i)_i, L_{-m}L_m(l_i)_i) \\
&= ((l_i)_i, \widehat{L}_{-m}\widehat{L}_m(l_i)_i)_\mathcal{P} + ((l_i)_i, C_{-m}C_m(l_i)_i)_\mathcal{P} \\
&= (\widehat{L}_m(l_i)_i, \widehat{L}_m(l_i)_i)_\mathcal{P} \pm (C_m(l_i)_i, C_m(l_i)_i)_\mathcal{P} \, ; \tag{5.36}
\end{aligned}$$

thus the second term in Eq. (5.36) must be negative, and so we must have $C_n^{\ddagger} = -C_{-n}$.

6. Conclusion and Outlook

We have investigated the action of the Virasoro algebra on the path space given by the $\mathcal{A}_{N+1}$ graphs. These spaces are related by a path space isomorphism to the configuration space of the FB models on coxeter A_{q-1} graphs whose critical indices

are those of the $(2, q)$ models $(q = 2N + 3)$. A detailed study of the spaces under consideration led us to the central notion of an admissible representation. Explicit constructions for such representations were given. The finite-dimensional results concerning these representations led us to conjecture signature character formulas for the above-stated models as well as the respective decomposition of $V(c, h)$ into positive and negative definite subspaces. Furthermore, the explicit Virasoro action corresponding to an admissible representation was studied. As a result, we found that the action of L_n can be described by a shift $(\widehat{L}_n)$ and a creation (C_n) operator. The parameters N and j which specify the central charge c and the lowest weight h of the theory appear as nilpotency indices for these operators. Finally, it was shown that the degree of nonunitarity (resp. of the non-self-adjointness) of the Virasoro algebra of the $(2, q)$ models can be understood in terms of the formulas for the adjoint operator of $\widehat{L}_n$ and C_n.

These results establish a close connection between integrable models and the CFT's considered here. On the level of the graphs the path space isomorphism between the fusion graph $\mathcal{A}$ explains the appearance of the $(2, q)$ series in the A_{q-1} models. From the interplay between the two perceptions of the objects as belonging to CFT or statistical mechanics, we gain insight into their structure. From the CFT side we learn how the Virasoro algebra operates as a sum of shifts and creation. The integrable model side provides us with signature characters and their corresponding decompositions into positive and negative definite subspaces.

We hope that in this spirit we can learn more about the connection between CFT and integrable models. There are also other series of CFT's in which graphs appear in sum rules for characters.[31] A similar treatment would be of interest. Perhaps there is also a connection with the quasi-particle interpretation of sum rules.[38-40] In this way, one can perhaps also understand the implications of the simple structure of the signature characters and their role in statistical mechanics.

Acknowledgments

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