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The algebra of discrete torsion

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Abstract

We analyze the algebraic structures of G-Frobenius algebras which are the algebras associated to global group quotient objects a.k.a. global orbifolds. Here G is any finite group. First, we show that these algebras are modules over the Drinfel'd double of the group ring $k[G]$ and are moreover $k[G]$-module algebras and $k[G]$-comodule algebras.

We furthermore consider G-Frobenius algebras up to projective equivalence and define universal shifts of the multiplication and the G-action preserving the projective equivalence class. We show that these shifts are parameterized by $Z^2(G, k^*)$ and by $H^2(G, k^*)$ when considering of G-Frobenius algebras up to isomorphism. We go on to show that these shifts can be realized by the forming of tensor products with twisted group rings, thus providing a group action of $Z^2(G, k^*)$ on G-Frobenius algebras acting transitively on the classes G-Frobenius related by universal shifts. The multiplication is changed according to the cocycle, while the G-action transforms with a different cocycle derived from the cocycle defining the multiplication. The values of this second cocycle also appear as a factor in front of the trace which is considered in the trace axiom of G-Frobenius algebras. This allows us to identify the effect of our action by $Z^2(G, k^*)$ as so-called discrete torsion, effectively unifying all known approaches to discrete torsion for global orbifolds in one algebraic theory. The new group action of discrete torsion is essentially derived from the multiplicative structure of G-Frobenius algebras. This yields an algebraic realization of discrete torsion defined via the perturbation of the multiplication resulting from a tensor product with a twisted group ring.

Additionally we show that this algebraic formulation of discrete torsion allows for a treatment of G-Frobenius algebras analogous to the theory of projective representations of groups, group extensions and twisted group ring modules.

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Lastly, we identify another set of discrete universal transformations among G-Frobenius algebras pertaining to their super-structure and classified by $\mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ which are essential for the application to mirror symmetry for singularities with symmetries.

Introduction

G-Frobenius algebras were introduced in [26] to explain the algebraic structure replacing Frobenius algebras when one is dealing with global group quotients by the action of a finite group G, a.k.a. global orbifolds, in theories such as (quantum) cohomology of global quotients [18,22], K-theory [23], local rings of singularities [26,29], etc. G-Frobenius algebras have also provided exactly the right structure to describe the cohomology of symmetric products [27,28] whose structure is closely related to that of Hilbert schemes [21]. Physically they can be thought of as topological field theories with a finite gauge group. The precise sense of this interpretation in terms of functors from a cobordism category to a linear category is contained in [26].

The characteristic features of G-Frobenius algebras are that they are group graded non-commutative algebras with a group action and a controlled non-commutativity. These algebras take into account the geometry of all the various fixed point sets under the action of the various group elements and their interrelations. The group degree g part of the algebra encodes the properties of the fixed point set of the action by $g \in G$. For a non-trivial element g the group degree g parts are called the twisted sectors, while the part of group degree of the identity is called the untwisted sector. The data of the G-grading together with the required type of G-action and several compatibilities can be concisely restated by saying G-Frobenius algebras are modules over the Drinfel'd double of the group ring $D(k[G])$. This gives an important link of G-Frobenius algebras to other orbifold theories. The role of the Drinfel'd double $D(k[G])$ in the description of orbifold conformal field theories as described by [9] was first observed in [2,3,11], for a discussion see below.

As we prove below, the algebraic structures of a G-Frobenius algebra are as follows: it is naturally a left $k[G]$-module algebra as well as a right $k[G]$-comodule algebra. Moreover, it satisfies the Yetter–Drinfel'd (YD) condition for bimodules and is thus also a module over $D(k[G])$, the Drinfel'd double of $k[G]$. Thus one could call it a $D(k[G])$-*module algebra*.

The invariants of a G-Frobenius algebra under the group action yield a commutative algebra which is naturally graded by the conjugacy classes of the group which have been previously regarded for instance in the context of cohomology [6]. The larger non-commutative algebras are more adapted, however, to reflect the geometry and general properties. The most pertinent example being the existence of a G-graded multiplication in lieu of a complicated grading by conjugacy classes.

Another feature which is best described in the non-commutative setup is the phenomenon of discrete torsion. This is the main topic of the present paper. Discrete torsion is a phenomenon which is expected to appear in the setting of global quotients by finite groups. In different settings discrete torsion takes on different meanings, but in all of them one thinks about a new "degree of freedom" which one should take to mean that there is not a unique description of an orbifold, but that there is a whole set of descriptions indexed by cohomological data of the group in question. In our setting of G-Frobenius algebras an ambiguity in the G-action dubbed discrete torsion was first observed in [26] for so-called Jacobian G-Frobenius algebras.

In the present paper, we approach the "ambiguity" by considering transformations on the multiplication and the G-action which map one G-Frobenius algebra into a projectively equivalent one. We show that those transformations, also called twists, which can be defined universally are exactly parameterized by $Z^2(G, k^*)$. Here universal means that the twists are defined for any G-Frobenius algebra.

We then go on to show that these universal twists for all G-Frobenius algebras can be realized by the operation of tensoring with twisted group rings. This operation preserves the dimensions of the twisted sectors and generates G-Frobenius algebras which are projectively equivalent by changing the multiplication and the G-action, where the change in the G-action is completely determined by the change in the multiplication. Moreover, this leads to a group action of $Z^2(G, k^*)$ on the set of G-Frobenius algebras, thus realizing discrete torsion for the first time as a group action.

The effect the twisting of the multiplication has on the G-action is via a cocycle ε, which, as we show, also appears as factor in front of the trace considered in the trace axiom for G-Frobenius algebras. In the case the elements g, h considered in the trace axiom commute, these factors are what has classically been called discrete torsion. This observation justifies the name and furthermore allows us to relate to various other definitions of discrete torsion in different settings, see below.

We would like to note that due to the consideration of the multiplicative structure of G-Frobenius algebras, not only have we identified the possible choices of discrete torsion, but we have found a group of transformations acting by taking tensor products with twisted group rings, which transforms from one choice of discrete torsion to any other. Also, for these considerations it is essential that we deal not only with the invariants, but with the whole G-Frobenius algebra, since the dimensions of the invariants of the untwisted sectors can change. The operation of twisting by discrete torsion by tensoring with the appropriate twisted group ring changes the metric, the multiplication and the group action in a projective fashion. We would also like to stress that it is actually the group of cocycles which plays a major role and not only the cohomology classes. Two cohomologous cycles in $Z^2(G, k^*)$ give rise to isomorphic, but different G-Frobenius algebras.

As demonstrated in [28], the presented "algebra of discrete torsion" allows for a simple explanation of a sign for the metric which appears in passing from the symmetric product to the Hilbert scheme that had been much discussed in the literature (see, e.g., [18,21]). Moreover, the consideration of cocycles versus cohomology classes also explains the existence of a whole family of multiplications [33] associated to the change of sign, as shown in [28]. This application exhibits the importance of dealing with the cocycles themselves rather than their cohomology classes, since the twists actually come from non-trivial cocycles whose cohomology class is, however, trivial when considering $\mathbb{C}^*$ coefficients.

Another application of the current presentation of discrete torsion and super-twists (see below) is given by the results of [27] which characterize the G-Frobenius algebra related to symmetric products as unique up to the above mentioned twists, which are explicitly computed [27].

At this point it might be instructive to compare and contrast the discrete torsion appearing in G-Frobenius algebras with other incarnations of discrete torsion that have appeared in the literature.

In order to make the operation of discrete torsion transparent, we have phrased the present article purely in terms of algebra. To connect to the literature it will be, however, necessary to make some digression into physics and vertex operator algebras (VOAs). The reader not enthusiastic about these subjects can safely skip ahead, since none of the material is necessary for the presentation of the algebraic facts.

As mentioned previously in terms of physics one can think of G-Frobenius algebras as a non-commutative incarnation of topological field theories with a finite gauge group *before projection*.

In physics, orbifolds were first considered for string theory [9,10,38], Landau–Ginzburg theories (see, e.g., [15,37]), conformal field theory [16], and gauge theory [17]. The relationship between G-Frobenius algebra, orbifold string theory and orbifold conformal field theory is, as is customary in considerations related to mirror symmetry, given by regarding chiral rings and suitably twisted models such as Landau–Ginzburg B-models or A- and B-twisted Sigma-models [39].

In all these theories discrete torsion makes its appearance as an ambiguity in assigning a fixed "model" to a given orbifold geometry. These treatments also all include the consideration of twisted sectors and the operation of the orbifold group G by "conjugation" on the twisted sectors. Here "conjugation" means that the action of an element g takes the h-twisted sector to the ghg^{-1} twisted sector. Another common property of all orbifold descriptions is that the twisted sectors are modules over the untwisted sector. The common approach is then to consider the invariants under the group action and calculate for instance the dimensions, which yield partition functions. A more mathematical approach is to classify the possible twisted the modules over the invariants untwisted sector in examined.

As observed in [2], the particular structure of the G-action on the twisted sectors provides a link to the Drinfel'd double. By considering operators for the group action and operators for the projection to the twisted sectors and the algebra they form one arrives at $D(k[G])$. Strictly speaking in the physical setup (e.g., [16]), the twisted sectors are only considered after the projection to invariants, so that the projections to the g-twisted sectors are only possible in the case of an Abelian group. Otherwise the twisted sectors are indexed only by conjugacy classes. So, to be precise after the projection there is not a $D(k[G])$-module structure, but a module structure of the twisted sectors over the various centralizers. These representations then can be induced to representations of $D(k[G])$ [11]. A striking fact is that all the irreducible representation theory can be obtained by inducing from representations of the centralizers as noticed in [3,11] (see [25] for a generalization). The classical treatment of $D(k[G])$ in the orbifold setup [3,11] actually only uses $D(k[G])$ as a separate structure, whose representation theory miraculously reproduces the fusion rules of the conformal field theory in the special case of so-called holomorphic RCFTs, which were considered in [16]. For a nice explanation of the underlying philosophy in terms of an equivalence of categories of modules and specific calculations in this direction see [31].

Classically, that is to say in physics, the ambiguity of discrete torsion manifests itself as a consistent choice of factors in front of the partition functions on tori and more mathematically in the possibility of having twisted modules over the untwisted sector. More generally given a CFT one can consider the partition function over any closed surface and this will yield factors depending on the surface. All these factors can however be expressed essentially via the ones from the torus (see, e.g., [4,37]).

In the considerations of open string theory and D-branes discrete torsion has also made its appearance in the form of projective representation of the orbifold group [7,8], see also [35]. In this form the "phase factors" are associated to surfaces with boundary instead of closed surfaces. The fact that the phase factors defined in this fashion [7] agree with those defined for the closed surface [36] was observed in [1]. Here the equality is a formal statement, since the surfaces are decidedly different, one is closed and the other has a boundary.

The mathematical formulation of conformal field theory is mostly done in terms of vertex operator algebras. These algebraic objects then inherit the structures discussed above. This leads to the classification of all twisted modules over the invariants of the untwisted sector as in [12–14,24].

Our setup is slightly different, but can be related to the ones described above. The first major difference is that in the construction involving G-Frobenius algebras we do not only consider the invariants, but define the multiplication and all other structures *before taking invariants*.

Thus in our setup, since we consider all twisted sectors before projection to the invariants, a G-Frobenius algebra is indeed a $D(k[G])$-module. We actually derive this property using the fact that the G-grading gives a $k[G]$-comodule structure, and the G-action gives a $k[G]$-module structure which satisfy the Yetter–Drinfel'd (YD) condition of [32]. Moreover, we show that the multiplicative structure is also compatible, that is that a G-Frobenius algebra is not only $k[G]$-module and a $k[G]$-comodule, but actually a $k[G]$-module algebra and a $k[G]$-comodule algebra.

The second major difference is that we are most interested in the possible *algebra* structures given underlying linear data. The consideration of the $D(k[G])$ module and the twisted sectors as modules over the untwisted sector are only intermediate steps as explained for instance in [26,27,29]. In the setting of VOAs a multiplication of the twisted sectors is not usually considered and only known for the special example of the $\mathbb{Z}/2\mathbb{Z}$ orbifold [12] used to construct the monster and in the cases recently constructed in [19].

As we explained above the main tool in the description of the group action of discrete torsion are the possible G-Frobenius *algebra* structures on a G-graded linear space all of whose graded components are one-dimensional which precisely the twisted group algebras. Moreover, once the multiplication is fixed all other structures of the G-Frobenius algebra can be derived from it. Thus it is the *multiplicative structure* which characterizes the twisted group algebras and hence discrete torsion. This gives a different interpretation of the group of cocycles $Z^2(G,k^*)$ in the setting of orbifolds, which is related to the usual one via the operation of forming tensor products. Namely, using our results on universal twists presented below, it follows that tensoring with the twisted group ring realizes all discrete torsions by a *group action* of $Z^2(G,k^*)$. Classically one discrete torsion is only *indexed* by $H^2(G,k^*)$. Our group action descends to $H^2(G,k^*)$ if we consider G-Frobenius algebras up to isomorphism. As noted above this entails that two isomorphic G-Frobenius algebras might have different metrics. The universality condition for the twists can be compared to the genericity conditions used in [2,3,11,16] to derive the fusion rules.

Thus one can say the *multiplicative* structure of G-Frobenius algebras and twisted group rings in particular explain, classify and realize what has to be called "discrete torsion" for G-Frobenius algebras as a group action.

It turns out that while the cocycles themselves describe the twists in the multiplication a derived cocycle ε describes the twist of the group action. It is this cocycle ε evaluated on commuting elements which makes it possible to link our results to other considerations in the literature on various quantities which all seem to carry the same name of discrete torsion. The most important observation here is that the cocycle ε appears, as we show, as a factor in front of the trace in the trace axiom which by [26] corresponds to the partition function on the torus with one boundary. Here we do not need commuting monodromies. Moreover, using the cobordism construction for commuting monodromies this trace corresponds at the same time to the closed torus and to the torus with one boundary component with monodromy around that boundary being identity, see [26]. Via the cobordism description of [26] one can also calculate the discrete torsion for other surfaces. These considerations yield an alternative description of the results of [1], with the additional benefit that we do not have to switch between open and closed string theory and compare two actually different surfaces on a formal level. Also, we do not have to restrict ourselves to commuting elements, but can use a unifying framework of cobordisms.

We stress that since G-Frobenius algebras already encode the whole cobordism theory [26], only the torus makes a direct appearance through the trace axiom. This is enough for all surfaces though. These correspond to different traces, whose calculation is then more or less straightforward using [26].

The action of the modular group is built in to the theory by cutting and gluing operations on the torus [26], whose algebraic manifestation is the trace axiom. One astonishing upshot about the treatment of discrete torsion is that the modular transformation properties, viz. the trace axiom, hold automatically for twisted group rings, so we do not have to deal with this separately. For a discussion of the action of the modular group in the representation theory of $D(k[G])$ see [3].

Another new point is that the phase factor ε is essentially derived from the multiplicative structure as it results from the twist for the group operation which is defined via the multiplication in the twisted group ring. As an additional benefit, we do not need any recourse to one of the different geometrical schemes or an ad hoc introduction of phase factors. Also the relationship of the cocycles in $Z^2(G, k^*)$ and cocycles ε is transparent and has a simple algebraic reason. And as mentioned previously, ε is naturally defined on the whole of $G \times G$ and not just only on the commuting elements.

In terms of the mathematical theory there are twists by discrete torsion in orbifold cohomology as explained in [34] which agree with the ones we define in the setting of global orbifolds [18,22]. The other cases of discrete torsion which were previously found for Jacobian G-Frobenius algebras [26] and the algebraic discrete torsion of [27] are all subsumed in the present formulation.

Thus one can say that our present treatment unifies and extends all the different approaches to discrete torsion [1,7,8,26,27,34–36,38] in the setting of G-Frobenius algebras, that is topological field theories with a finite gauge group in the sense of [26].

We go on to show that our treatment of discrete torsion allows an interpretation which analogous to the theory of projective representations of groups, group extensions and twisted group ring modules. Loosely said given a cocycle $\alpha \in Z^2(G, k^*)$ as above on can find an Abelian extension G^α whose representations correspond to lifts of projective representations of G with the cocycle α, see [30] and Section 4 for details. This classical fact has for instance been successfully exploited in [20] to obtain character tables and quiver diagrams for projective representations necessary to discuss the fusion rules.

Our motivation is different and comes from the fact that the group G plays a central role in the structure of a G-Frobenius algebra, so it ought to be possible to find a G^α-Frobenius algebra which encodes the situation after the twist by discrete torsion. We show that there is indeed a canonical way to construct a G^α-Frobenius algebra which is a lift of the G-Frobenius algebra twisted by α. Notice that this entails a new bigger grading group and thus new twisted sectors and an extension of the action to the bigger group.

In a geometric interpretation this amounts to taking a twisted Cartesian product with *point*$/H$ where H is the Abelian group used in the extension. In the case of trivial α this corresponds to extending the action of the group G to the group $G \times H$ with H acting trivially. Informally, one can either compare this to a fattening of a point by a trivial action or more generally to different stacks having the same coarse moduli space. We plan to elaborate on these geometrical aspects in [23].

We would like to point out that even after twisting the G-Frobenius algebra remains a G-Frobenius algebra, that is there is still has a true action of G and not only a projective one. This seems to be the reason, parallel to the discussion in [17], that our discrete torsion is in $Z^2(G, k^*)$—respectively $H^2(G, k^*)$ up to isomorphism—rather than in $H^3(G, k^*)$ as discussed in [2,3,11,16]. Another observation in the same spirit is that since the condition of [16,17], as explained in [3], is met one can expect to get the same representation theory from the quantum double as from the twisted quantum double [3]—although in our setting, we never have change from $D(k[G])$ to its twisted version.

We furthermore examine on the generic super-structures one can impose on a given G-Frobenius algebra and show that these are again given by tensor product, but now with "superized" versions of $k[G]$. These are a second type of discrete deformation, which is actually different from the one of discrete torsion.

This freedom of choice is essential for applications to orbifolding and mirror symmetry for singularities with symmetries [26,29].

In summary, we have obtained a new algebraic way of describing discrete torsion through a *group operation* of $Z^2(G, k^*)$ via forming of tensor products with twisted group rings. Here both the description of discrete torsion as a group action and the relevance of the use of cocycles rather than just the cohomology classes are novel points. We also use a new approach, since we essentially use the *multiplicative* structure of G-Frobenius algebras which was not previously discussed. In our formulation discrete torsion is primarily a twist of the multiplication, which has as a secondary consequence a twist in the G-action. This also clarifies the appearance of $Z^2(G, k^*)$ which naturally classifies the multiplications in the various twisted group rings.

It is this algebraic description which via the cobordism considerations of [26] reproduces, extends, and unifies all the previously known incarnations of discrete torsion for global quotients—and arbitrary surfaces.

We would like to emphasize that the elegant and lucid picture of discrete torsion as given by the operation of tensoring with twisted group rings is only possible when considering the whole non-commutative G-Frobenius *algebra* rather than its invariants or only its $D(k[G])$-module structure—which are the focus of all previous work on the subject—where such a description cannot exist.

The paper is organized as follows.

In the first section, we recall the notion of G-Frobenius algebras, and show that group rings and twisted group rings are G-Frobenius algebras.

In the second section, we give the algebraic properties of G-Frobenius algebras. The main theorem is that a G-Frobenius algebra has the natural algebraic structure of a $k[G]$-module algebra and a $k[G]$-comodule algebra which satisfies the YD condition.

We also characterize G-Frobenius algebras which are Galois over their identity sector as $k[G]$-comodule algebras.

The third section contains the realization of discrete torsion as an action of $Z^2(G,k^*)$ on G-Frobenius algebras and of $H^2(G,k^*)$ on the isomorphism classes of G-Frobenius algebras. This is done by analyzing universal twists of G-Frobenius algebras which are twists of the multiplication and the group action preserving the projective class of the G-Frobenius algebra. The main statements then are, that

(1) the universal twists are entirely governed by the multiplicative structure,
(2) the universal twists are in 1-1 correspondence with $Z^2(G,k^*)$,
(3) these twists can be realized by tensoring with the respective twisted group ring, which
(4) renders a *group action* of $Z^2(G,k^*)$ on the set of G-Frobenius algebra. Furthermore,
(5) the induced change in the G-action is given by a cocycle ε, which
(6) appears in front of the trace of the trace axiom.

The result (6) legitimizes the name of discrete torsion and gives the link to the other quantities of the same name as discussed above.

Additionally, we study the generic super-structures ($\mathbb{Z}/2\mathbb{Z}$-gradings) which one can impose on a given G-Frobenius algebra and show that they are classified by $\mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z})$ and can be implemented by tensoring with super group algebras. We prove that both twists "commute" in the sense that one can twist by twisted super group algebras or first by twisted group algebras and then by super group algebras.

In the fourth section we introduce a theory for twists of G-Frobenius algebras in analogy with projective representations of a group and their relation to modules over the twisted group algebra and extensions of the group. Here the final result is that given any Abelian group H and a cocycle $[\alpha'] \in H^2(G,H)$ then for any central extension G^α of G by H with class $[\alpha']$ and any G-Frobenius algebra A there is a natural $G^{\alpha'}$-Frobenius algebra $A^{\alpha'}$ to which the Frobenius algebra A_α (A twisted by α) can be lifted. Here $[\alpha] \in H^2(G,k^*)$ is the image under the transgression map associated to $[\alpha']$ of a $\chi \in \mathrm{Hom}(H,k^*)$. Vice versa, the above χ gives a push down map, which maps A^α onto A_α. Lastly, we show that there is a universal setup of this kind if there is a representation group for G.

1. G-Frobenius algebras

We fix a finite group G and denote its unit element by e. We furthermore fix a ground field k of characteristic zero for simplicity. With the usual precautions the characteristic of the field does not play an important role and furthermore the group really only needs to be completely disconnected.

1.1. Definition. A *G-twisted Frobenius algebra*—or G-Frobenius algebra for short—over a field k of characteristic 0 is $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$, where

G	finite group;
A	finite dim G-graded k-vector space, $A = \bigoplus_{g\in G} A_g$, A_e is called the untwisted sector and the A_g for $g \neq e$ are called the twisted sectors;
$\circ$	a multiplication on A which respects the grading: $\circ : A_g \otimes A_h \to A_{gh}$;
1	a fixed element in A_e—the unit;
η	non-degenerate bilinear form which respects grading, i.e., $g\|_{A_g\otimes A_h} = 0$ unless $gh = e$;
φ	an action of G on A (which will be by algebra automorphisms), $\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, s.t. $\varphi_g(A_h) \subset A_{ghg^{-1}}$;
χ	a character $\chi \in \mathrm{Hom}(G, k^*)$,

satisfying the following axioms:

Notation. We use a subscript on an element of A to signify that it has homogeneous group degree—e.g., a_g means $a_g \in A_g$—and we write $\varphi_g := \varphi(g)$ and $\chi_g := \chi(g)$.

(a) *Associativity*:

$$(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k).$$

(b) *Twisted commutativity*:

$$a_g \circ a_h = \varphi_g(a_h) \circ a_g.$$

(c) *G invariant unit*:

$$1 \circ a_g = a_g \circ 1 = a_g \quad \text{and} \quad \varphi_g(1) = 1.$$

(d) *Invariance of the metric*:

$$\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k).$$

(i) *Projective self-invariance of the twisted sectors*:

$$\varphi_g|A_g = \chi_g^{-1}\mathrm{id}.$$

(ii) *G-invariance of the multiplication*:

$$\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h).$$

(iii) *Projective G-invariance of the metric*:

$$\varphi_g^*(\eta) = \chi_g^{-2}\eta.$$

(iv) *Projective trace axiom*:

$$\forall c \in A_{[g,h]} \text{ and } l_c \text{ left multiplication by } c\text{:}$$
$$\chi_h \operatorname{Tr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \operatorname{Tr}(\varphi_{g^{-1}} l_c|_{A_h}).$$

1.1.1. Special G-Frobenius algebras. We briefly review special G-Frobenius algebras. For details see [26,27].

We call a G-Frobenius algebra special if all A_g are cyclic A_e modules via the multiplication $A_e \otimes A_g \to A_g$. Fixing a cyclic generator $1_g \in A_g$ the algebra is completely characterized by two compatible cocycles, namely $\gamma \in \bar{Z}^2(G, A_e)$ and $\varphi \in Z^1(G, k^*[G])$ where $\bar{Z}$ are graded cocycles (see [26]) and $k^*[G]$ is the group ring restricted to invertible coefficients with G-module structure induced by the adjoint action:

$$\phi(g) \cdot \left(\sum \mu_h h\right) = \sum_h \mu_h g h g^{-1}.$$

We set $\varphi(g) = \sum_h \varphi_{g,h} g h g^{-1}$ and $\gamma_{g,h} = \gamma(g,h)$.

The multiplication and G-action are determined by

$$1_g 1_h = \gamma_{g,h} 1_{gh}, \qquad \varphi_g(1_h) = \varphi_{g,h} 1_{ghg^{-1}}.$$

There are two compatibility equations:

$$\varphi_{g,h}\gamma_{ghg^{-1},g} = \gamma_{g,h} \quad \text{and} \tag{1.1}$$

$$\varphi_{k,g}\varphi_{k,h}\gamma_{kgk^{-1},khk^{-1}} = \varphi_k(\gamma_{g,h})\varphi_{k,gh}. \tag{1.2}$$

Notice that if $\gamma_{g,h}$ is non-zero, i.e., $A_g A_h \neq 0$, then (1.1) determines $\varphi_{g,h}$. We also would like to remark that (1.2) is automatically satisfied if $A_g A_h A_k \neq 0$ (cf. [26]).

1.2. The group ring $k[G]$. Let $k[G]$ denote the group ring of G.

1.2.1. The Hopf structure of $k[G]$. Recall that $k[G]$ is a Hopf algebra with the natural multiplication, the comultiplication induced by $\Delta(g) = g \otimes g$, counit $\varepsilon(g) = 1$ and antipode $S(g) = g^{-1}$.

1.2.2. The G-Frobenius structure of $k[G]$. When considering $k[G]$ as a G-Frobenius algebra we will consider $k[G]$ as a left $k[G]$-module with respect to conjugation, i.e., the map $k[G] \otimes k[G] \to k[G]$ given by

$$\sum_g \nu_g g \otimes \sum_h \mu_h h \mapsto \sum_{g,h} \nu_h \mu_g g h g^{-1}.$$

The other structures are the naturally G graded natural multiplication on $k[G]$ with the unit e, the metric $\eta(g,h) = \delta_{gh,e}$ and $\chi_g \equiv 1$. It is trivial to check all axioms.

If we were to choose a grading $\tilde{}\in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$, then $\chi_g = (-1)^{\tilde{g}}$ and $\varphi_g(h) = (-1)^{\tilde{g}\tilde{h}}$.

1.3. The twisted group ring $k^\alpha[G]$. Recall that given an element $\alpha \in Z^2(G, k^*)$ one defines the twisted group ring $k^\alpha[G]$ to be given by the same linear structure with multiplication given by the linear extension of

$$g \otimes h \mapsto \alpha(g,h) gh \tag{1.3}$$

with 1 remaining the unit element. To avoid confusion, we will denote elements of $k^\alpha[G]$ by $\hat{g}$ and the multiplication with $\cdot$. Thus

$$\hat{g} \cdot \hat{h} = \alpha(g,h) \widehat{gh}.$$

For α the following equations hold:

$$\alpha(g,e) = \alpha(e,g), \qquad \alpha\big(g, g^{-1}\big) = \alpha\big(g^{-1}, g\big). \tag{1.4}$$

Furthermore,

$$\hat{g}^{-1} = \frac{1}{\alpha(g, g^{-1})} \hat{g}^{-1}.$$

1.3.1. Remark. Given a two cocycle α and possibly extending the field by square roots we can find a cocycle $\tilde{\alpha}$ in the same cohomology class which also satisfies

$$\tilde{\alpha}\big(g, g^{-1}\big) = 1. \tag{1.5}$$

If one wishes to consider $\mathbb{C}$ as a ground field, one can work with such cocycles.

1.3.2. Lemma. *Set*

$$\varepsilon(g,h) = \frac{\alpha(g,h)}{\alpha(ghg^{-1}, g)},$$

then the left adjoint action of $k^\alpha[G]$ on $k^\alpha[G]$ is given by

$$g \otimes h \xmapsto{\mathrm{ad}} \varepsilon(g,h) \widehat{ghg^{-1}}. \tag{1.6}$$

Proof. By the definition of multiplication in $k^\alpha[G]$,

$$\hat{g}\cdot\hat{h}\cdot\hat{g}^{-1}=\frac{\alpha(g,h)\alpha(gh,g^{-1})}{\alpha(g,g^{-1})}\widehat{ghg^{-1}}.$$

Now by associativity,

$$\alpha\bigl(gh,g^{-1}\bigr)\alpha\bigl(ghg^{-1},g\bigr)=\alpha(gh,e)\alpha\bigl(g^{-1},g\bigr)=\alpha\bigl(g,g^{-1}\bigr).$$

So

$$\frac{\alpha(g,h)\alpha(gh,g^{-1})}{\alpha(g,g^{-1})}=\frac{\alpha(g,h)}{\alpha(ghg^{-1},g)}. \qquad \square$$

1.3.3. The G-Frobenius algebra structure of $k^\alpha[G]$. Recall from [26,27] the following structures which turn $k^\alpha[G]$ into a special G-Frobenius algebra:

$$\gamma_{g,h}=\alpha(g,h),\qquad \varphi_{g,h}=(-1)^{\tilde{g}\tilde{h}}\frac{\alpha(g,h)}{\alpha(ghg^{-1},g)}=:\varepsilon(g,h),$$
$$\eta\bigl(\hat{g},\hat{g}^{-1}\bigr)=\alpha\bigl(g,g^{-1}\bigr),\qquad \chi_g=(-1)^{\tilde{g}}. \tag{1.7}$$

Here the second line induces the third via

$$\hat{g}\cdot\hat{h}=\alpha(g,h)\widehat{gh},\qquad \widehat{ghg^{-1}}\cdot\hat{g}=\alpha\bigl(ghg^{-1},g\bigr)\widehat{gh}.$$

We recall that if k^* is two divisible, we could scale s.t. $\eta(g,g^{-1})=1$ and ε would indeed yield the adjoint action. The last equation follows from the special case of the trace axiom since the dimension of all sectors is one.

It is an exercise to check all axioms. All compatibility equations follow automatically, since $\alpha(g,h)\neq 0$. The only axiom which is not straightforward is the trace axiom, but see [27] for a proof.

1.3.4. Remark. By the general theory (see above), $\varepsilon\in H^1(G,k^*[G])$ where $k^*[G]$ is the group ring restricted to invertible coefficients with G-module structure induced by the adjoint action:

$$\phi(g)\cdot\Bigl(\sum\mu_h h\Bigr)=\sum_h\mu_h ghg^{-1}.$$

1.3.5. Relations. The $\varepsilon(g,h)$ satisfy the equations:

$$\varepsilon(g,e)=\varepsilon(g,g)=1, \tag{1.8}$$
$$\varepsilon(g_1g_2,h)=\varepsilon\bigl(g_1,g_2hg_2^{-1}\bigr)\varepsilon(g_2,h), \tag{1.9}$$
$$\varepsilon(g,h)^{-1}=\varepsilon\bigl(g^{-1},ghg^{-1}\bigr), \tag{1.10}$$

$$\varepsilon(k, gh) = \varepsilon(k, g)\varepsilon(k, h)\frac{\alpha(kgk^{-1}, khk^{-1})}{\alpha(g, h)}, \tag{1.11}$$

where (1.9) is the statement that $\varphi \in \operatorname{Hom}(G, \operatorname{Aut}(A))$, (1.10) is a consequence of (1.9), and (1.11) is the compatibility equation which also ensures the invariance of the metric.

Furthermore, the trace axiom holds [27] which is equivalent to the equation

$$\alpha\big([g, h], hgh^{-1}\big)\varepsilon(h, g) = \alpha\big([g, h], h\big)\varepsilon\big(g^{-1}, ghg^{-1}\big) \tag{1.12}$$

or

$$\varepsilon(h, g) = \varepsilon\big(g^{-1}, ghg^{-1}\big)\frac{\alpha([g, h], h)}{\alpha([g, h], hgh^{-1})}. \tag{1.13}$$

In the case that the group elements in the equations commute we obtain the famous conditions of discrete torsion which make ε into a bicharacter on commuting elements.

For *commuting elements*:

$$\begin{gathered}\varepsilon(g, e) = \varepsilon(g, g) = 1, \qquad \varepsilon(g_1g_2, h) = \varepsilon(g_1, h)\varepsilon(g_2, h),\\ \varepsilon(g, h)^{-1} = \varepsilon\big(g^{-1}, h\big), \qquad \varepsilon(g, h) = \varepsilon\big(h^{-1}, g\big) = \varepsilon(h, g)^{-1},\\ \varepsilon(h, g_1g_2) = \varepsilon(h, g_1)\varepsilon(h, g_2),\end{gathered} \tag{1.14}$$

where the last equation is now a consequence of the second and the fourth and the third equation follows from the second.

1.3.6. Fact. One can show [27] that the twisted group algebras $k^\alpha[G]$ are the only G-Frobenius algebras with the property that all A_g are one-dimensional. To be completely precise there is an additional freedom of choosing a super (i.e., $\mathbb{Z}/2\mathbb{Z}$) structure determined by a homomorphism $\sigma \in \operatorname{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ (see [26] and 3.4 below).

1.3.7. Geometry of $k^\alpha[G]$. From the point of view of Jacobian Frobenius algebras [26] it is natural to say that $k[G]$ is the Frobenius algebra naturally associated to *point*$/G$. The existence of the twisted algebras suggests that there are several equivalent ways of taking the group quotient. This is made precise by Theorem 3.3.2 below.

2. Algebraic structures of a G-Frobenius algebra

We fix a G-Frobenius algebra $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$.

2.1. Theorem. *A G-Frobenius algebra is naturally a left $k[G]$-module algebra as well as a right $k[G]$-comodule algebra. Moreover, it satisfies the Yetter–Drinfel'd* (*YD*) *condition for bimodules and is thus a module over $D(k[G])$, the Drinfel'd double of $k[G]$. Where the YD condition reads*

$$\sum h_1 \cdot m_0 \otimes h_2m_1 = \sum (h_2 \cdot m)_0 \otimes (h_2 \cdot m)_1h_1. \tag{2.1}$$

Here we used the usual notation for coalgebras and right comodules. I.e., if $\Delta : H \to H \otimes H$ is the comultiplication of H then for $h \in H$ we write $\Delta(h) = \sum h_1 \otimes h_2$ and if $\check{\rho} : M \to H$ is a right comodule map, then for $m \in M$ we write $\check{\rho}(m) = \sum m_0 \otimes m_1$.

Proof. The theorem follows from the collection of facts below and the general statement that any H bimodule satisfying the YD-condition is a module over $D(H)$ (see, e.g., [32]).

2.1.1. Remark. In this particular case the YD condition states that the comodule structure is $k[G]$-equivariant with respect to the adjoint action of $k[G]$ on itself, viz. as a tensor product of G-Frobenius algebras of left $k[G]$ modules. See below.

2.1.2. The $k[G]$-module structure. Since A is a k algebra, the G-action φ turns A into a right $k[G]$ module. More precisely, for $a \in A \sum_g \nu_g g \in k[G]$,

$$\Bigl(\sum \nu_g g\Bigr) \otimes a \mapsto \sum_g \nu_g \varphi_g(a). \tag{2.2}$$

Since $\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(A))$ this is a module structure.

2.1.3. The $k[G]$-comodule structure. Since A is a G graded algebra it is naturally a $k[G]$-comodule.

More precisely, for $a \in A$, $a = \bigoplus_g a_g$ the $k[G]$-comodule structure $\rho : A \to A \oplus k[G]$ is given by

$$a \mapsto \sum_g (a_g \otimes g), \tag{2.3}$$

which obviously yields a comodule.

2.1.4. Lemma. *A G-Frobenius algebra is a $k[G]$-module algebra and a $k[G]$-comodule algebra or equivalently a $k[G]^*$-module algebra.*

Proof. For the module algebra structure notice that:

(1) A is a left $k[G]$ module as noticed before.
(2) The $k[G]$-action induced by φ is by definition by algebra automorphisms, and $\Delta(g) = g \otimes g$ thus

$$\varphi_g(ab) = \varphi_g(a)\varphi_g(b).$$

(3) Since the unit is invariant:

$$\varphi_g(1) = 1 = \varepsilon(g)1.$$

The structure of comodule algebra follows from the fact that

$$\varphi(a_g b_h) = a_g b_h \otimes gh,$$

which, as is well known, is nothing but the condition of A being a G graded algebra

$$A_g A_h \subset A_{gh}. \qquad \square \tag{2.4}$$

2.1.5. Remark. Notice that the condition (2.4) is usually given by a strict inclusion, so that it is usually not $k[G]$-Galois—which is equivalent to $A_g A_h = A_{gh}$ (cf. [32]). In case it is, the structure of the algebra is particularly transparent. We will come back to this later.

2.1.6. The compatibility. We will view $k[G]$ as a left $k[G]$-module using the adjoint action. Then $A \otimes k[G]$ turns into a left $k[G]$-module by using the diagonal action. This is the natural left $k[G]$-module structure on the tensor product of left Hopf modules:

$$\left(\sum_h \mu_h h\right)\left(\sum_g a_g \otimes \nu g\right) = \sum_{h,g} \mu_h \nu \varphi(h)(a_g) \otimes ghg^{-1}.$$

2.1.7. Lemma. *The comodule structure is $k[G]$-equivariant and thus the comodule map is a map of left $k[G]$-modules where we use the left adjoint action of $k[G]$ on itself as the left $k[G]$-action*:

$$\begin{aligned}\rho\left(\left(\sum_h \mu_h h\right)(a)\right) &= \rho\left(\sum_h \mu_h \varphi_h(a)\right) = \sum_{h,g} \mu_h \varphi_h(a_g) \otimes hgh^{-1}\\ &= \left(\sum_h \mu_h h\right) \cdot \left(\sum_g a_g \otimes g\right) = \left(\sum_h \mu_h h\right) \cdot \rho(a).\end{aligned} \tag{2.5}$$

2.1.8. The YD condition. Plugging in the coproduct and action yields

$$\varphi_g(a_h) \otimes gh = \varphi_g(a_h) \otimes \left(ghg^{-1}\right)g, \tag{2.6}$$

which verifies the YD condition for A.

2.1.9. Proposition. *If A is a G-Frobenius algebra that is $k[G]$-Galois over A_e as a $k[G]$-comodule algebra, then A is special and $\gamma \in Z^2(G, A^*)$ where A^* are the units of A. So in particular γ determines φ uniquely.*

Moreover, if A_e is one-dimensional, then $A = k^\alpha[G]$, for some $\alpha \in H^2(G, k^*)$ with a choice of parity $\tilde{} \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$.

Proof. Since $A_{g^{-1}} A_g = A_e$, there are elements $a_g \in A_g$, $b_{g^{-1}} \in A_{g^{-1}}$ s.t. $b_{g^{-1}} a_g = 1$ then a_g is a cyclic generator since $\forall c_g \in A_G$ $c_g = c_g(b_{g^{-1}} a_g) = (c_g b_{g^{-1}}) a_g$ and $c_g b_{g^{-1}} \in A_e$. Choosing generators 1_g in this way it is easy to check that the cocycles need to be invertible and thus the φ are fixed by (1.1). Furthermore, notice that the multiplication map induces an isomorphism of A_e modules between A_e and A_g via $a \mapsto a1_g$ where A_e is a cyclic A_e module over itself via left multiplication. This follows by associativity from $a = a(1_g 1_{g^{-1}}) = (a1_g)1_{g^{-1}}$ and thus $a1_g \neq 0$ and the map $A_e \to A_g$ is also injective. Thus the restriction maps are all isomorphisms and graded cocycles coincide with the usual ones. □

3. The action of discrete torsion

3.1. Twisting G-Frobenius algebras. Given a G-Frobenius algebra A we can re-scale the multiplication and G-action by a scalar. More precisely, let $\lambda : G \times G \to k^*$ be a function. For $a = \bigoplus_g a_g \in A$ we define

$$\varphi^\lambda(g)(a) = \bigoplus_h \lambda(g,h)\varphi(g)(a_h).$$

Given another function $\mu : G \times G \to k^*$ we can also define a new multiplication $\circ^\mu$,

$$a_g \circ^\mu b_h = \mu(g,h) a_g \circ b_h.$$

3.1.1. Remark. These twists arise from a projectivization of the G-structures induced on a module over A as for instance the associated Ramond-space (cf. [26]). In physics terms this means that each twisted sector will have a projective vacuum, so that fixing their lifts in different ways induces the twist. Mathematically this means that g twisted sector is considered to be a Verma module over A_g based on this vacuum.

3.1.2. Induced shift on the metric. Due to the invariance of the metric, the twist in the multiplication results in a twisted metric

$$\eta^\mu\bigl(a_g, b_g^{-1}\bigr) := \mu\bigl(g, g^{-1}\bigr)\eta\bigl(a_g, b_g^{-1}\bigr).$$

3.1.3. Definition. We define $s(\mu,\lambda)(A)$ to be the collection

$$\bigl\langle G, A, \circ^\mu, 1, \eta^\lambda, \varphi^\lambda, \chi\bigr\rangle.$$

3.1.4. Proposition. *$s(\mu,\lambda)(A)$ is G-Frobenius algebra if and only if the following equations hold for μ,λ:*

$$\mu(e,g) = \mu(g,e) = 1. \tag{3.1}$$

Furthermore, $\forall g,h,k \in G$ *s.t.* $A_g A_h A_k \neq 0$:

$$\mu(g,h)\mu(gh,k) = \mu(h,k)\mu(g,hk) \tag{3.2}$$

and if $A_g A_h \neq 0$*, then*

$$\lambda(g,h) = \frac{\mu(g,h)}{\mu(ghg^{-1},g)}. \tag{3.3}$$

If $A_g A_h \neq 0$ *as well as* $A_g A_h A_k = 0$*,*

$$\lambda(g,hk)\mu(h,k) = \lambda(g,h)\lambda(g,k)\mu\bigl(ghg^{-1},gkg^{-1}\bigr). \tag{3.4}$$

Furthermore,

$$\begin{gathered}\lambda(gh,k) = \lambda(h,k)\lambda\bigl(g,hkh^{-1}\bigr),\\ \lambda(e,g) = \lambda(g,g) = 1,\\ \mu\bigl([g,h],hgh^{-1}\bigr)\lambda(h,g) = \mu\bigl([g,h],h\bigr)\lambda\bigl(g^{-1},ghg^{-1}\bigr),\end{gathered} \tag{3.5}$$

where the third equation has to hold for all pairs g,h *s.t.* $\exists c \in A_{[g,h]}$ *s.t.* $\chi_h \operatorname{Tr}(l_c\varphi_h|_{A_g}) \neq 0$*, where* l_c *is the left multiplication by* c*. In particular, it must hold for all pairs* g,h *with* $[g,h]=e$*.*

Proof. The first equation (3.1) expresses that 1 is still the unit for the algebra. The statement (3.2) for μ is the obvious form of associativity. The statement (3.3) comes from the compatibility equation of the group action with the multiplication.

Equation (3.4) ensures the equivariance of the multiplication. It is automatic if $A_g A_h A_k \neq 0$ and also if $A_g A_h = 0$.

The first equation (3.5) for λ is equivalent to the fact that φ^λ is still a G-action.

Notice that $\lambda(e,g) = 1$ since $A_e A_g = A_g$ and thus

$$\lambda(e,g) = \frac{\mu(e,g)}{\mu(g,e)} = 1$$

and so the identity remains invariant.

Also notice that there is no twist to the character!

$$\chi_g^\lambda = (-1)^{\tilde{g}} \dim A_g \operatorname{Str}^{-1}\left(\varphi_g^{\lambda)}\big|_{A_e}\right) = \chi_g \lambda(g,e) = \chi_g.$$

This in turn implies the second statement in the second line by projective self-invariance:

$$\begin{gathered}\chi_g^{-1}\mathrm{id}|A_g = \varphi_g^\lambda|A_g = \lambda(g,g)\varphi_g|A_g = \lambda(g,g)\chi_g^{-1}\mathrm{id}|A_g,\\ 1 = \lambda(e,k) = \lambda\bigl(g^{-1}g,k\bigr) = \lambda(g,k)\lambda\bigl(g^{-1},gkg^{-1}\bigr),\end{gathered}$$

so

$$\lambda(g,h) = \lambda\bigl(g^{-1},ghg^{-1}\bigr)^{-1}.$$

The third equation follows from the projective trace axiom.

$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:

$$\chi_h \operatorname{Tr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \operatorname{Tr}(\varphi_{g^{-1}} l_c|_{A_h}). \tag{3.6}$$

Thus we must have $\chi_h \operatorname{Tr}(l_c \varphi_h^\lambda|_{A_g}) = \chi_{g^{-1}} \operatorname{Tr}(\varphi_{g^{-1}}^\lambda l_c|_{A_h})$ but this is equivalent to the third equation in view of (3.6).

Now we check the other axioms.

The invariance of the metric follows from associativity:

$$\begin{aligned}
\eta^\mu(a_g, b_h \circ^\mu c_{h^{-1}g^{-1}}) &= \mu\big(g, g^{-1}\big)\mu\big(h, h^{-1}g^{-1}\big)\eta(a_g, b_h \circ c_{h^{-1}g^{-1}}) \\
&= \mu(g,h)\mu\big(gh, h^{-1}g^{-1}\big)\eta(a_g \circ b_h, c_{h^{-1}g^{-1}}) \\
&= \eta^\mu(a_g \circ^\mu b_h, c_{h^{-1}g^{-1}}).
\end{aligned}$$

The projective invariance of the metric reads as

$$\lambda(g,k)\lambda\big(g, k^{-1}\big)\mu\big(gkg^{-1}, gk^{-1}g^{-1}\big) = \mu\big(k, k^{-1}\big),$$

which is automatic since $A_g A_k A_{k^{-1}} \neq 0$. □

3.1.5. Definition. We call a twist $s(\lambda, \mu)$ *universal* if it transforms *any* G-Frobenius algebra into a G-Frobenius algebra. We call two twists $s(\lambda, \mu)$ and $s(\lambda', \mu')$ isomorphic if for any G-Frobenius algebra A the algebras $s(\lambda, \mu)(A)$ and $s(\lambda', \mu')(A)$ are isomorphic.

3.1.6. Theorem. *The universal twists are in* 1-1 *correspondence with elements* $\alpha \in Z^2(G, k^*)$ *and the isomorphism classes of universal twists are given by* $H^2(G, k^*)$.

Proof. If the twist is universal then there are no restrictions on the equations. In particular $\mu \in Z^2(G, k^*)$ and λ is completely determined by μ via (3.3). All the other properties are then automatic. The claim about isomorphism classes is obvious by noticing that if α and α' are cohomologous and $\alpha/\alpha' = d\beta$ for some $\beta \in Z^1(G, k^*)$ then a diagonal rescaling of the generators of $k^\alpha[G]$ by β yields $k^{\alpha'}[G]$ so the result follows from the characterization of universal twists as taking tensor produce with twisted group rings below. □

3.2. Discrete torsion. In this subsection we prove that universal twists are exactly given by twisting with discrete torsion.

3.2.1. Reminder. Given two G-Frobenius algebras $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$ and $\langle G, A', \circ', 1', \eta', \varphi', \chi'\rangle$ we defined [26] their tensor product as G-Frobenius algebras to be the G-Frobenius algebra

$$\left\langle G, \bigoplus_{g \in G} (A_g \otimes A'_g), \circ \otimes \circ', 1 \otimes 1', \eta \otimes \eta', \varphi \otimes \varphi', \chi \otimes \chi' \right\rangle.$$

We will use the short hand notation $A \mathbin{\widehat{\otimes}} B$ for this product.

3.2.2. Definition. Given a G-Frobenius algebra A and an element $\alpha \in H^2(G,k)$ we define the α-twist of A to be the G-Frobenius algebra $A^\alpha := A \mathbin{\widehat{\otimes}} k^\alpha[G]$.

Notice that

$$A^\alpha_g = A_g \otimes k \simeq A_g. \tag{3.7}$$

Using this identification the G-Frobenius structures are given by:

3.3. Lemma. *The induced structures under the isomorphism* (3.7) *are*

$$\begin{aligned} \circ^\alpha|_{A^\alpha_g \otimes A^\alpha_h} &= \alpha(g,h)\circ, \qquad \varphi^\alpha_g|_{A^\alpha_h} = \varepsilon(g,h)\varphi_g, \\ \eta^\alpha|_{A^\alpha_g \otimes A^\alpha_{g^{-1}}} &= \alpha\bigl(g,g^{-1}\bigr)\eta, \qquad \chi_g = \chi_g. \end{aligned} \tag{3.8}$$

Proof. We notice that the two algebras have the same linear structure $A_{\alpha,g} = A_g \otimes kg \simeq A_g$ with the isomorphism given by $a_g \otimes g \mapsto a_g$. Now the multiplication is given by

$$(a_g \otimes g) \otimes (a_h \otimes h) \mapsto a_g a_h \otimes \alpha(g,h)gh = \alpha(g,h)a_g a_h \otimes gh,$$

which yields the twisted multiplication.

The twist for the G-action is computed to be

$$\varphi_{\alpha,h}(a_g \otimes g) = \varepsilon(g,h)\varphi_h(g) \otimes hgh^{-1}.$$

This leads us to the following proposition.

3.3.1. Proposition. $A_\alpha \simeq s(\alpha,\varepsilon)A$.

3.3.2. Theorem. *The set of universal twists are described by tensoring with twisted group algebras which identifies this operation with twisting by discrete torsion.*

In other words given a generic G-Frobenius algebra A there are exactly $H^2(G,k)$ twists of it by discrete torsion.

3.3.3. Discrete torsion as phases for the partition sum. Notice that for any $c \in A^\alpha_{[g,h]} \simeq A_{[g,h]}$,

$$\chi_h \operatorname{Str}(l_c\varphi_h|_{A^\alpha_g}) = \varepsilon(h,g)\chi_h \operatorname{Str}(l_c\varphi_h|_{A_g}). \tag{3.9}$$

This is the original freedom of choice of a phase for the summands of the partition function postulated by physicists. In this context, we should regard g,h: $[g,h]=e$ and set $c=e$. More precisely, set

$$Z(A) = \sum_{g,h\in G:\ [g,h]=e} \operatorname{Str}(\chi_g\varphi_g|_{A^\alpha_h}) := \sum_{g,h\in G:\ [g,h]=e} Z_{g,h}, \tag{3.10}$$

$$Z(A_\alpha) = \sum_{g,h\in G:\ [g,h]=e} \varepsilon(g,h)Z_{g,h}. \tag{3.11}$$

We could omit the factors χ_g, but from the point of view of physics we should take the trace in the Ramond space (cf. [26]) where the $k[G]$ module structure is twisted by χ.

3.4. Supergrading. In this subsection we wish to address questions of supergrading. There is a general theory of supergraded G-Frobenius algebras and special G-Frobenius algebras. We will expose the structures here for the group ring.

3.4.1. Super *G*-Frobenius algebras. If the underlying algebra of a G-Frobenius algebra has a supergrading $\tilde{}$, then the axioms of a G-Frobenius algebra have to be changed to

(b^σ) *Twisted super-commutativity*:

$$a_g \circ a_h = (-1)^{\tilde{a}_g \tilde{a}_h} \varphi_g(a_h) \circ a_g.$$

(iv^σ) *Projective super-trace axiom*:

$$\forall c \in A_{[g,h]} \text{ and } l_c \text{ left multiplication by } c\text{:}$$
$$\chi_h \operatorname{STr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \operatorname{STr}(\varphi_{g^{-1}} l_c|_{A_h}),$$

where STr is the super-trace.

For details on the super-structure as well as the role of the super structure for special G-Frobenius algebras we refer to [26].

3.4.2. Supergraded twisted group rings. Fix $\alpha \in H^2(G, k^*)$, $\sigma \in \operatorname{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ then there is a twisted super-version of the group ring where now the relations read

$$\hat{g}\hat{h} = \alpha(g,h)\widehat{gh}, \tag{3.12}$$

and the twisted commutativity is

$$\hat{g}\hat{h} = (-1)^{\sigma(g)\sigma(h)} \varphi_g(\hat{h})\hat{g}, \tag{3.13}$$

and thus

$$\varphi_g(\hat{h}) = (-1)^{\sigma(g)\sigma(h)} \alpha(g,h)\alpha\bigl(gh, g^{-1}\bigr)\widehat{ghg^{-1}} =: \varphi_{g,h}\widehat{ghg^{-1}}, \tag{3.14}$$

and thus

$$\varepsilon(g,h) := \varphi_{g,h} = (-1)^{\sigma(g)\sigma(h)} \frac{\alpha(g,h)}{\alpha(ghg^{-1}, g)}. \tag{3.15}$$

We would just like to remark that the axiom (iv^σ) shows the difference between super twists and discrete torsion.

3.4.3. Definition. We denote the α-twisted group ring with super-structure σ by $k^{\alpha,\sigma}[G]$. We still denote $k^{\alpha,0}[G]$ by $k^{\alpha}[G]$ where 0 is the zero map and we denote $k^{0,\sigma}[G]$ just by $k^{\sigma}[G]$ where 0 is the unit of the group $H^2(G,k^*)$.

A straightforward calculation shows

3.4.4. Lemma. *$k^{\alpha,\sigma}[G] = k^{\alpha}[G] \otimes k^{\sigma}[G]$ and more generally.*

3.4.5. Lemma. *Let A be the G-Frobenius algebra or more generally super Frobenius algebra with supergrading˜ $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$, then $A \otimes k^{\sigma}[G]$ is isomorphic to the super G-Frobenius algebra $\langle G, A, \circ, 1, \eta, \varphi^{\sigma}, \chi^{\sigma}\rangle$ with supergrading $\tilde{\sigma}$, where*

$$\varphi^{\sigma}_{g,h} = (-1)^{\sigma(g)\sigma(h)}\varphi_{g,h}, \qquad \chi^{\sigma} = (-1)^{\sigma(g)}\chi_g, \qquad \tilde{a}^{\sigma}_g = \tilde{a}_g + \sigma(g). \tag{3.16}$$

Using arguments and definitions for universal twists as for discrete torsion, we can obtain the following proposition. Here universal means that there is no assumption on the particular structure of the G-Frobenius algebra, in other words it pertains to generic G-Frobenius algebras.

3.4.6. Proposition. *Given a (super) G-Frobenius algebra A the universal super G-Frobenius algebra re-gradings are in 1-1 correspondence with $\mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ and these structures can be realized by tensoring with $k^{\sigma}[G]$ for $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$.*

4. Projective representations, extensions, and twisted group algebras

In this section we first assemble classical facts about groups which will be extended to G-Frobenius algebras. As an intermediate step we analyze twisted group algebras, which belong to both worlds.

4.1. Part I: groups.

4.1.1. Projective representations. A projective representation ρ of a group is a map $\rho: G \to GL(V)$, V being a k-vector space, which satisfies

$$\rho(g)\rho(h) = \alpha(g,h)\rho(gh), \qquad \rho(e) = \mathrm{id}. \tag{4.1}$$

It is easy to check that $\alpha(g,h) \in Z^2(G,k^*)$. Moreover, with a natural notion of projective isomorphy two projective representations are isomorphic if their classes are cohomologous (cf., e.g., [5,30]).

4.1.2. Extensions. Given a central extension

$$1 \mapsto A \to G^* \xrightarrow{\pi} G \to 1 \tag{4.2}$$

fix a section s of π and define $\alpha : G \times G \to A$ by $s(g)s(h) = \alpha(g,h)s(gh)$. It is easy to see that indeed $\alpha \in Z^2(G,A)$ and furthermore changing the section or changing the extension by an isomorphism preserves the cohomology class of α.

Vice versa a cycle in $\alpha \in Z^2(G,A)$ were A is an Abelian group gives rise to a central group extension of G:

$$1 \mapsto A \to G^\alpha \stackrel{\pi}{\to} G \to 1, \tag{4.3}$$

where $G^\alpha = A \rtimes G$. The maps are given by $a \mapsto (A, e_G)$, $(a,g) \mapsto g$ and the multiplication is given by $(a,g)(a',g') = (aa'\alpha(g,g'), gg')$.

4.1.3. The transgression map. Given a cycle $\alpha \in H^2(G,A)$ there is a natural map

$$\mathrm{Tra}_\alpha : \mathrm{Hom}\big(A, k^*\big) \to H^2\big(G, k^*\big), \tag{4.4}$$

which sends $\chi \in \mathrm{Hom}(A,k^*)$ to the cocycles defined by $(g,h) \mapsto \chi\alpha(g,h)$. Actually this map maps into the cohomology group with values in the torsion subgroup of k^* which we call $\mathrm{tors}(k^*)$:

$$\mathrm{Tra}_\alpha : \mathrm{Hom}\big(A, k^*\big) \to H^2\big(G, \mathrm{tors}\big(k^*\big)\big). \tag{4.5}$$

4.1.4. Facts. We briefly give the facts linking group cohomology, projective representations and twisted group algebras. For a detailed account see [30].

(1) The classes of central extensions of a group G by an Abelian group A are in 1-1 correspondence with $H^2(G,A)$.
(2) Any projective α-representation is a module over the α-twisted group algebra $k^\alpha[G]$. (This is in fact an equivalence of categories.)
(3) Every projective representation with cycle α is projectively equivalent to one that can be lifted to linear representation on $G^{\hat{\alpha}}$ if $[\alpha]$ is in the image of the transgression map associated to $[\hat{\alpha}]$.
(4) If $H^2(G,k^*) = H^2(G, (\mathrm{tors}(k^*))$ then:
    (a) any projective representation can be lifted to a suitable group, and
    (b) there is a universal extension

$$1 \to A \to G^* \to G \to 1$$

such that *any* projective representation lifts to G^* and moreover the group $A \simeq H^2(G,k^*)$.

Assumption. For the remainder of the section we will assume that k has the property that $H^2(G,k^*) = H^2(G, (\mathrm{tors}(k^*))$. This is the case, e.g., if is algebraically closed or $k = \mathbb{R}$, see, e.g., [30].

4.2. Part II: the twisted group algebra revisited. Fix $[\alpha'] \in H^2(G, A)$, an element $[\alpha] \in \mathrm{Im}(\mathrm{Tra}_{[\alpha']})$ and a pre-image character $\chi \in \mathrm{Hom}(A, k^*)$.

This yields a central extension:

$$1 \mapsto A \to G^{\alpha'} \stackrel{\pi}{\to} G \to 1 \tag{4.6}$$

with a section s of π s.t. the cocycle corresponding to s is α'. The map χ induces map

$$\chi : k\bigl[G^\alpha\bigr] \to k[G] : ag \mapsto \chi(a)g \tag{4.7}$$

while the section s induces a map

$$s : k[G] \to k\bigl[G^\alpha\bigr] : g \mapsto 1_A g. \tag{4.8}$$

4.2.1. Projective algebra. Using the maps s, χ, we can also characterize the multiplication μ^α in $k^\alpha[G]$ as follows: it is the map which makes the following diagram commutative:

$$\begin{tikzcd} k\bigl[G^\alpha\bigr] \otimes k\bigl[G^\alpha\bigr] \arrow[r, "\mu"] & k\bigl[G^\alpha\bigr] \arrow[d, "\chi"] \\ k[G] \otimes k[G] \arrow[u, "s\otimes s"] \arrow[r, "\mu^\alpha"] & k[G]. \end{tikzcd}$$

We already know that μ^α induces the structure of an algebra. This diagram captures the statement about lifts of projective representations of G to linear representations of G^α.

This is essentially 4.1.4(1).

4.2.2. Projective coalgebra. Using the diagram as above, we define a comultiplication by commutativity of:

$$\begin{tikzcd} k\bigl[G^\alpha\bigr] \arrow[r, "\Delta"] & k\bigl[G^\alpha\bigr] \otimes k\bigl[G^\alpha\bigr] \arrow[d, "\chi\otimes\chi"] \\ k[G] \arrow[u, "s"] \arrow[r, "\Delta^\alpha"] & k[G] \otimes k[G]. \end{tikzcd}$$

The coalgebra structure we induce in this way on $k[G]$ is actually the old coalgebra structure, but $k^\alpha[G]$ ceases to be a bialgebra.

4.2.3. Remark: braiding. If one would like a bialgebra structure on the group ring $k^\alpha[G]$ then one has to consider braided objects, where the braiding is inverse to the twist. It should be possible to find analogous statements to the ones presented in this article by considering structures over $k^\alpha[G]$ in braided categories.

4.2.4. Adjoint action. Let ad denote the adjoint action $k[G^\alpha]$. Then there is an induced action on $k[G]$,

$$\begin{tikzcd} k\bigl[G^\alpha\bigr] \otimes k\bigl[G^\alpha\bigr] \arrow[r, "\mathrm{ad}"] & k\bigl[G^\alpha\bigr] \arrow[d, "\chi"] \\ k[G] \otimes k[G] \arrow[u, "s\otimes s"] \arrow[r, "\mathrm{ad}^\varepsilon"] & k[G]. \end{tikzcd}$$

According to 1.3.2 this action is given by

$$\mathrm{ad}^\varepsilon(g)(h) := \varepsilon(g,h)ghg^{-1}.$$

4.3. Part III: *G*-Frobenius algebras. We now apply the logic of part II to general G-Frobenius algebras.

Let H be an Abelian group. Fix $[\alpha'] \in H^2(G,H)$, an element $[\alpha] \in \mathrm{Im}(\mathrm{Tra}_{[\alpha']})$ and a pre-image character $\chi \in \mathrm{Hom}(H, k^*)$ and a central extension:

$$1 \mapsto H \to G^{\alpha'} \xrightarrow{\pi} G \to 1 \tag{4.9}$$

with a section s of π s.t. the cocycle corresponding to s is α'.

4.3.1. Definition. Let A^α be a G^α-Frobenius algebra. We say that a G-Frobenius algebra F can be lifted to A^α if there are maps $i: A \to A^\alpha$ and $\mathrm{res}: A^\alpha \to A$ such that the structural maps fit into the commutative diagrams

$$\begin{tikzcd} A^\alpha \arrow[r, "\rho^\alpha"] & A^\alpha \otimes k\bigl[G^\alpha\bigr] \arrow[d, "\mathrm{res}\otimes\chi"] \\ A \arrow[u, "i"] \arrow[r, "\rho"] & A \otimes k[G] \end{tikzcd}$$

and

$$\begin{tikzcd} A^\alpha \otimes A^\alpha \arrow[r, "\mu^\alpha"] & A^\alpha \arrow[d, "\mathrm{res}"] \\ A \otimes A \arrow[u, "i\otimes i"] \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} k\bigl[G^\alpha\bigr] \otimes A^\alpha \arrow[r, "\varphi^\alpha"] & A^\alpha \arrow[d, "\mathrm{res}"] \\ k[G] \otimes A \arrow[u, "s\otimes s"] \arrow[r, "\varphi"] & A \end{tikzcd}$$

and all algebraic structures are compatible.

4.3.2. Definition. We say that an H-Frobenius algebra B is H homogeneous if it is endowed with an additional left H-action τ which shifts group degree and is equivariant w.r.t. multiplication. More precisely the following two equations hold:

$$\tau(h)(A_{h'}) \subset A_{h,h'}, \qquad \tau(h)(ab) = a\tau(h)(b). \tag{4.10}$$

It is standard to see that

4.3.3. Remark. With the notation as above, the left action τ of H on B is necessarily by isomorphisms and thus B is a special H-Frobenius algebra whose components are all isomorphic. Moreover, B is Galois as a $k[H]$-comodule over B_e.

4.3.4. Definition. Given a G-Frobenius algebra A, an H-homogeneous H-Frobenius algebra B and a cocycle $\alpha \in Z^2(G, H)$ we define the *crossed product* of A and B to be the G^α-Frobenius algebra

$$A \#_\alpha B := \left\langle G^\alpha, A \otimes B, \circ \#_\alpha \circ', 1 \otimes 1, \eta \otimes \eta', \varphi \#_\varepsilon \varphi', \chi \otimes \chi' \right\rangle, \tag{4.11}$$

where

$$(a_g \otimes b_h) \circ \#_\alpha \circ' (c_{g'} \otimes d_{h'}) = a_g c_{g'} \otimes \tau\bigl(\alpha(g, g')\bigr) b_h d_{h'} \tag{4.12}$$

and

$$\varphi \#_\varepsilon \varphi'(g, h)(a_{g'} \otimes b_{h'}) = \varphi_g(a_{g'}) \otimes \tau\bigl(\alpha(g, g')\alpha\bigl(gg', g^{-1}\bigr)\bigr)\varphi_h(b_{h'}). \tag{4.13}$$

We leave it to the reader to verify all axioms, since it is analogous to previous calculations.

4.3.5. Quantum symmetry group. The postulated second left action by translation τ can be viewed as the quantum symmetry group postulated by physicists. Notice that it acts freely. The invariants are linearly isomorphic to $\bigoplus A_g \otimes B_{e_H}$ where e_H is the unit element of H.

4.3.6. Lemma. *The linear map above induces an isomorphism*

$$^{H_\tau}(A \#_\alpha B) \simeq \bigoplus_{g \in G} (A_g \otimes B_{e_H})$$

as G-Frobenius algebras with trivial action on the second factor.

Here we denoted the invariants under the action of H by τ by $^{H_\tau}$.

4.3.7. Definition. Fix $\chi \in \mathrm{Hom}(H, k^*)$ then there is a natural map from B to B_e given by $\tau(h)b_e \mapsto \chi(h)b_e$. This map induces a map

$$A \#_\alpha B \to \bigoplus_{g \in G} (A_g \otimes B_{e_H}),$$

which induces a structure of G-Frobenius algebra on $\bigoplus_{g\in G}(A_g \otimes B_{e_H})$, the G-action on the second factor being trivial. We define

$$(A \#_\alpha B)^\chi$$

to be this G-Frobenius algebra.

It is easy to check that the following holds:

4.3.8. Lemma. *Keeping the notation above, let* $[\alpha'] = \mathrm{Tra}_{[\alpha]}(\chi)$ *and more precisely on the level of cocycles let* $\alpha'(g, g') = \chi\alpha(g, g')$.

Then

$$(A \#_\alpha B)^\chi \simeq \Big(\bigoplus (A_g \otimes B_{e_H})\Big)_{\alpha'}.$$

4.3.9. Definition. Given a cocycle $\alpha \in Z^2(G, H)$, a central extension G^α of G by H and a G-Frobenius algebra A we define A^α to be the G^α-Frobenius algebra

$$A^\alpha := A \#_\alpha k[H].$$

4.3.10. Theorem. *Given a G-Frobenius algebra A and cocycles $\alpha \in Z^2(G, k^*)$, $\alpha' \in Z^2(G, H)$ which are related by $\chi \in \mathrm{Hom}(G, k^*)$ via $\alpha(g, g') = \chi(\alpha'(g, g'))$.*

Then the twist A_α of A lifts to the $G^{\alpha'}$-Frobenius algebra $A^{\alpha'}$ and moreover,

$$\big(A^\alpha\big)^\chi \simeq A_\alpha.$$

Finally, if G^* is the universal extension of G whose cocycle is $\beta \in H^2(G, H^2(G, k^*))$, then any twist A_α of a G-Frobenius algebra A lifts to A^β.

Proof. Choose a section s of the extension yielding α. We denote the unit element of H by e_H and denote $s(g)$ by $e_H g$. We let $i : A_g \to A^{\alpha'}_{e_H g}$ be the map given by $A_g \to A_g \otimes k e_H : a_g \mapsto a_g \otimes e_H$ and define $\mathrm{res}: A^{\alpha'}_{hg} \simeq A_g \otimes k e_H \mapsto A_g$ to be the map $a_g \otimes e_H \mapsto \chi(h) a_g$.

Then

$$\begin{aligned}(\mathrm{res} \otimes \chi)\big(\rho^{\alpha'}\big(i(a_g)\big)\big) &= (\mathrm{res} \otimes \chi)\big(\rho^{\alpha'}(a_g \otimes e_H)\big)\\ &= (\mathrm{res} \otimes \chi)(a_g \otimes e_H) \otimes (e_H g) = a_g \otimes g,\end{aligned}$$

which assures the comodule algebra structure

$$\chi\big(\mu^\alpha\big((i \otimes i)(a_g \otimes b_{g'})\big)\big) = \chi\big(a_g b_{g'} \otimes \alpha(g, g') g g'\big) = \alpha(g, h) a_g b_{g'},$$

since $A_{e_H g} A_{e_H g'} \subset A_{\alpha(g,g') g g'}$ which assures the algebra structure

$$\chi \circ \varphi \circ (s \otimes s)(g \otimes a_h) = \chi \circ \varphi\big((e_H g \otimes A_{e_H g'})\big) = \varepsilon(g, g') \varphi_g(a_{g'}),$$

which assures the module algebra structure, since $\varphi_{e_H g}(A_{e_H h}) \subset A_{\varepsilon'(g,h)gh}$, where we set

$$\varepsilon'(g, g') = \frac{\alpha'(g, h)}{\alpha'(ghg^{-1}, g)}$$

to be the cocycle of the adjoint action. Then by 1.3.2,

$$\chi\big(\varepsilon'(g, g')\big) = \varepsilon(g, g').$$

For the last statement notice that (cf., e.g., [30])

$$k\big[G^*\big] = \prod_{\alpha \in T} k\big[G^\alpha\big],$$

where T is a transversal for $B^2(G, k^*)$ in $Z^2(G, k^*)$. So $A^\beta \simeq \bigoplus_{\alpha \in T} A^\alpha$ and we can lift to the appropriate component. $\square$

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References

[1] P.A. Aspinwall, A note on the equivalence of Vafa's and Douglas's picture of discrete torsion, J. High Energy Phys. 12 (2000) 029.
[2] P. Bántay, Orbifolds and Hopf algebras, Phys. Lett. B 245 (1990) 477–479.
[3] P. Bántay, Orbifolds, Hopf algebras, and the Moonshine, Lett. Math. Phys. 22 (3) (1991) 187–194.
[4] P. Bántay, Symmetric products, permutation orbifolds and discrete torsion, Lett. Math. Phys. 63 (3) (2003) 209–218.
[5] C.W. Curtis, I. Reiner, Methods of Representation Theory, vol. I. With Applications to Finite Groups and Orders, Pure Appl. Math., Wiley, New York, 1988.
[6] W. Chen, Y. Ruan, A new cohomology theory of orbifold, Comm. Math. Phys., in press.
[7] M.R. Douglas, D-branes and discrete torsion, preprint, hep-th/9807235.
[8] M. Douglas, Two lectures on D-geometry and noncommutative geometry, in: Nonperturbative Aspects of Strings, Branes and Supersymmetry, Trieste, 1998, World Scientific, River Edge, NJ, 1999, pp. 131–156.
[9] L. Dixon, J. Harvey, C. Vafa, E. Witten, Strings on orbifolds, Nuclear Phys. B 261 (1985) 678–686.
[10] L. Dixon, J. Harvey, C. Vafa, E. Witten, Strings on orbifolds. II, Nuclear Phys. B 274 (1986) 285–314.
[11] R. Dijkgraaf, V. Pasquier, P. Roche, Quasi Hopf algebras, group cohomology and orbifold models, Nuclear Phys. B. Proc. Suppl. 18 (60) (1990) 60–72.

[12] C. Dong, H. Li, G. Mason, Modular-invariance of trace functions in orbifold theory and generalized Moonshine, Comm. Math. Phys. 214 (2000) 1–56.
[13] C. Dong, H. Li, G. Mason, Twisted representations of vertex operator algebras, Math. Ann. 310 (1998) 571–600.
[14] C. Dong, G. Mason, On quantum Galois theory, Duke Math. J. 86 (1997) 305–321.
[15] R. Dijkgraaf, E. Verlinde, H. Verlinde, Topological strings in $d < 1$, Nuclear Phys. B 352 (1) (1991) 59–86.
[16] R. Dijkgraaf, C. Vafa, E. Verlinde, H. Verlinde, The operator algebra of orbifold models, Comm. Math. Phys. 123 (1989) 485–526.
[17] R. Dijkgraaf, E. Witten, Topological gauge theories and group cohomology, Comm. Math. Phys. 129 (1990) 393–429.
[18] B. Fantechi, L. Goettsche, Orbifold cohomology for global quotients, Duke Math. J. 117 (2003) 197–227.
[19] V. Gorbounov, F. Malikov, Vertex algebras and the Landau–Ginzburg/Calabi–Yau correspondence, preprint, math.AG/0308114.
[20] Feng, A. Hanany, Y.-H. He, N. Prezas, Discrete torsion, covering groups and quiver diagrams, J. High Energy Phys. 4 (2001) 27, Paper 37.
[21] M. Lehn, C. Sorger, The cup product of Hilbert schemes for $K3$ surfaces, Invent. Math. 152 (2003) 305–329.
[22] T. Jarvis, R. Kaufmann, T. Kimura, Pointed admissible g-covers and g-cohomological field theories, Compositio Math., in press.
[23] T. Jarvis, R. Kaufmann, T. Kimura, Orbifold K-theory of global quotients, in preparation.
[24] V.G. Kac, I.T. Todorov, T. Ivan, Affine orbifolds and rational conformal field theory extensions of $W_{1+\infty}$, Comm. Math. Phys. 190 (1997) 57–111.
[25] Y. Kashina, G. Mason, S. Montgomery, Computing the Frobenius–Schur indicator for abelian extensions of Hopf algebras, J. Algebra 251 (2) (2002) 888–913.
[26] R.M. Kaufmann, Orbifolding Frobenius algebras, Internat. J. Math. 14 (2003) 573–619.
[27] R.M. Kaufmann, Second quantized Frobenius algebras, Comm. Math. Phys. 248 (2004) 33–83.
[28] R.M. Kaufmann, Discrete torsion, symmetric products and the Hilbert scheme, in: C. Hertling, M. Marcolli (Eds.), Frobenius Manifolds, Quantum Cohomology, and Singularities, Vieweg, Wiesbaden, 2004, in press.
[29] R.M. Kaufmann, Singularities with symmetries, orbifold Frobenius algebras and mirror symmetry, preprint, math.AG/0312417; Contemp. Math., in press.
[30] G. Karpilovsky, Projective Representations of Finite Groups, Dekker, 1985.
[31] G. Mason, S.-H. Ng, Group cohomology and gauge equivalence of some twisted quantum doubles, Trans. Amer. Math. Soc. 353 (2001) 3465–3509.
[32] S. Montgomery, Hopf algebras and their actions on rings, CBMS Regional Conf. Ser. in Math., vol. 82, Amer. Math. Soc., Providence, RI, 1993.
[33] Z. Qin, W. Wang, Hilbert schemes and symmetric products: a dictionary, in: Orbifolds in Mathematics and Physics, in: Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 233–257.
[34] Y. Ruan, Discrete torsion and twisted orbifold cohomology, preprint, math.AG/0005299.
[35] E. Sharpe, Discrete torsion and gerbes $\mathrm{I} + \mathrm{II}$, preprints, hep-th/9909120, hep-th/9909108.
[36] C. Vafa, Modular invariance and discrete torsion on orbifolds, Nuclear Phys. B 273 (1986) 592–606.
[37] C. Vafa, Topological Landau–Ginzburg models, Modern Phys. Lett. A 6 (1991) 337–346.
[38] C. Vafa, E. Witten, On orbifolds with discrete torsion, J. Geom. Phys. 15 (1995) 189–214.
[39] E. Witten, Mirror manifolds and topological field theory, in: Essays on Mirror Manifolds, Internat. Press, Hong Kong, 1992, pp. 120–158.