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The Intersection Form in $H^*(\overline{M}_{0n})$ and the Explicit Künneth Formula in Quantum Cohomology

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0 Introduction

Let $\overline{M}_{0n}$ be the moduli space of genus-0 curves with n marked points. Its cohomology ring was determined by Keel [Ke], who gave a presentation in terms of boundary divisors, their intersections, and their relations. A boundary divisor is specified by a 2-partition $S_1 \amalg S_2$ of $\overline{n} := \{1, \ldots, n\}$. The additive structure of this ring was studied and presented in [KM] and [KMK]. Although much about the structure of this ring is known, there are still several open questions. The complete study of the intersection theory of this space, however, is of importance for the theory of quantum cohomology. In particular, it is necessary in order to understand the Künneth formula for quantum cohomology, which is given by the tensor product of cohomological field theories; cf. [KM] and [KMK].

In this paper, we prove a general formula for the intersection form of two arbitrary monomials in boundary divisors. Furthermore, we present a tree basis of the cohomology of $\overline{M}_{0n}$. With the help of the intersection form we determine the Gram matrix for this basis and give a formula for its inverse. This enables us to calculate the tensor product of the higher order multiplications arising in quantum cohomology and formal Frobenius manifolds. In the context of quantum cohomology this establishes the explicit Künneth formula.

In §2 of this paper, we prove a formula for the intersection form for any two polynomials in the boundary divisors of complementary degree. More precisely, after the introduction of the notion of trees with multiplicities and good multiplicity orientations, we can formulate the following.

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Theorem. Let $\mathrm{mon}(\sigma_1, m_1)$ and $\mathrm{mon}(\sigma_2, m_2)$ be two monomials of complementary degree in $H^*(\overline{M}_{0n})$. If there is no good multiplicity orientation of $(\tau, m) := \tau(\sigma_1 \cup \sigma_2, m_1 + m_2)$, then $\langle \mathrm{mon}(\sigma_1, m_1)\, \mathrm{mon}(\sigma_1, m_1)\rangle = 0$. If there does exist one, then

$$\langle \mathrm{mon}(\sigma_1, m_1)\, \mathrm{mon}(\sigma_1, m_1)\rangle = \prod_{v \in V_\tau} (-1)^{|v|-3} \frac{(|v|-3)!}{\prod_{f \in F(v)} (\mathrm{mult}(f))!^2} \prod_{e \in E_\tau} (m(e) - 1)!,$$

where mult is the unique multiplicity orientation for (τ, m), provided by the Lemma 2.3, whose value is given in the formula (2.3). □

The notation $\mathrm{mon}((\tau, m)) := \prod_{e \in E_\tau} D_{\sigma(e)}^{m(e)}$ used in this theorem, along with an exposition of the different combinatorics of trees involved in the intersection theory of $\overline{M}_{0n}$, is explained in the introductory §1 where also Keel's presentation is briefly reviewed.

In §3 we will give a monomial basis $\mathcal{B}_n$ of $H^*(\overline{M}_{0n})$ together with a tree representation for it. The basis that is presented here and the proof of linear independence are inspired by the work of Yuzvinsky [Yu], who worked out a basis in another presentation of the cohomology ring developed by De Concini and Procesi [CP] via hyperplane arrangements. Using the results of §2 we can write down the Gram matrix for this basis and give a formula for its inverse.

The realization of this basis in terms of boundary divisors is necessary for applications to quantum cohomology and operads [KM], [G], since these structures make explicit use of the presentation of $H^*(\overline{M}_{0n})$ in terms of tree strata.

As an application to this field we use the results of §3 to calculate the tensor product of the higher-order products and correlation functions stemming from a tree-level cohomological field theory, which appear in the tensor product of formal Frobenius manifolds and yield the explicit Künneth formula for quantum cohomology.

Corollary. For two projective algebraic manifolds V and W, the potential $\Phi^{V \times W}$ yielding the quantum cohomology of $V \times W$ in terms of Φ^V and Φ^W is given by the formula

$$\Phi^{V \times W}(\gamma' \otimes \gamma'') = \sum_{n \geq 3} \frac{1}{n!} \sum_{\mu, \nu \in \mathcal{B}_n} Y'(\check{\mu})(\gamma'^{\otimes n}) m_{\mu\nu} Y''(\check{\nu})(\gamma''^{\otimes n}),$$

where $\mathcal{B}_n$ is the basis of §3, $(m_{\mu\nu})_{\mu\nu \in \mathcal{B}_n}$ is its Gram matrix (3.14), $\{\check{\mu} | \mu \in \mathcal{B}_n\}$ is the dual basis obtained via the inverse Gram matrix (3.15) and $\{Y'(\tau)\}$ (resp. $\{Y''(\tau)\}$) are the operadic abstract correlation functions obtained from Φ^V (resp. Φ^W) via (4.3) and (4.5). □

As an example, the first higher-order products are written out explicitly.

0.1 Notation

Throughout this paper we will denote by $\subset$ the strict inclusion, and use $\subseteq$ for the not necessarily strict relation. Furthermore we denote by $\mathbb{N}$ the positive integers, and we will use the notation $\overline{n}$ to denote the set $\{1, \ldots, n\}$.

1 Partitions and trees

1.1 Notation

We will consider a tree τ to be a collection of sets of vertices, edges, and tails (V_τ, E_τ, T_τ) with given incidence relations. A flag will be a pair (vertex, incident edge) or (vertex, incident tail). The set of all flags will be denoted by F_τ, those incident to one vertex v by $F_\tau(v)$.

1.2 Keel's presentation

Usually the cohomology ring of $\overline{M}_{0S}$ is presented in terms of classes of boundary divisors as generators and quadratic relations as introduced by [Ke]. The additive structure of this ring and the respective relations can then naturally be described in terms of stable trees (see [KM] and [KMK]). The boundary divisors of $\overline{M}_{0S}$ are in one-to-one correspondence with unordered 2-partitions $\{S_1, S_2\}$ of S, satisfying $|S_1| \geq 2$ and $|S_2| \geq 2$ (stability). Let $\{D_\sigma | \sigma = \{S_1, S_2\}$ a stable S-partition$\}$ be a set of commuting independent variables. Consider the ideal $I_n \subset F_n$ in the graded polynomial ring $F_S := K[D_{\{S_1,S_2\}}]$ generated by the following relations:

(i) $D_{\{S_1,S_2\}} D_{\{S_1',S_2'\}}$, if the number of nonempty pairwise intersections of these sets equals 4.

(ii) For all distinct $i, j, k, l \in S : \sum_{ij\sigma kl} D_\sigma - \sum_{kj\tau il} D_\tau$
where the notation of the type $ij\sigma kl$ is used to imply that $\{i, j\}$ and $\{k, l\}$ are subsets of different parts of σ.

Set $H_S^* := F_S / I_S$. Keel's theorem states that the map

$$D_\sigma \longmapsto \text{dual cohomology class of the boundary divisor in } \overline{M}_{0n} \text{ corresponding to the partition } \sigma$$

induces the isomorphism of rings (doubling the degrees)

$$H_S^* \xrightarrow{\sim} H^*(\overline{M}_{0n}, K). \tag{1.1}$$

1.3 Additive structure of H_n^*

The additive structure of the cohomology can be nicely presented in terms of trees (see [KMK]). There (Proposition 1.3) it is proved that the set of trees with r edges is in bijection with the set of good monomials of degree r. We will briefly quote some of the notions and results from that paper. A monomial $D_{\sigma_1} \cdots D_{\sigma_a} \in F_S$ is called good if the family of 2-partitions $\{\sigma_1, \dots, \sigma_a\}$ is good, i.e., $a(\sigma_i, \sigma_j) = 3$, where for two unordered stable partitions $\sigma = \{S_1, S_2\}$ and $\tau = \{T_1, T_2\}$ of S,

$$a(\sigma, \tau) := \text{the number of non-empty pairwise} \\ \text{distinct sets among} \quad S_i \cap T_j,\ i, j = 1, 2.$$

1.3.1 Lemma (1.2 in [KMK]). Let τ be a stable S-tree with $|E_\tau| \geq 1$. For each $e \in E_\tau$, denote by $\sigma(e)$ the 2-partition of S corresponding to the one edge S-tree obtained by contracting all edges except for e. Then

$$\text{mon}(\tau) := \prod_{e \in E_\tau} D_{\sigma(e)} \tag{1.2}$$

is a good monomial. □

1.3.2 Proposition (1.3 in [KMK]). For any $1 \leq r \leq |S| - 3$, the map $\tau \longmapsto \text{mon}(\tau)$ establishes a bijection between the set of good monomials of degree r in F_S and stable S-trees τ with $|E_\tau| = r$ modulo S-isomorphism. There are no good monomials of degree greater than $|S| - 3$. □

1.3.3 Additive relations.

In [KMK] it is shown that the good monomials span the cohomology space and furthermore that all linear relations between them are generated by the relative versions of (ii),

$$\sum_{ij\tau' kl} \text{mon}(\tau') = \sum_{ik\tau'' jl} \text{mon}(\tau''), \tag{1.3}$$

where $\{ij\tau' kl\}$ and $\{ik\tau'' lj\}$ are the preimages of the contraction onto a given τ contracting exactly one edge onto a fixed vertex v seperating the flags marked by i, j and k, l (resp. i, k and j, l) in such a way that they lie on different components after severing e, where the markings i, j, k, l refer to flags that are part of the edges of the unique paths from v to the tails i, j, k, l in τ, and it is required that the paths start along different edges.

1.4 Trees with multiplicity

Since we will have to deal with monomials, which are not necessarily good, we will extend the notion of trees to that of trees with multiplicity.

1.4.1 Definition. An S-tree with multiplicity is a pair (τ, m) consisting of an S-tree and a function $m : E_\tau \to \mathbb{N}$. If no multiplicity function is given, we will assume that it is identically 1.

Call a monomial $D_{\sigma_1}^{m_1} \cdots D_{\sigma_k}^{m_k}$ nice if $a(\sigma_i, \sigma_j) = 2$ or 3. Set

$$\mathrm{mon}((\tau, m)) := \prod_{e \in E_\tau} D_{\sigma(e)}^{m(e)}. \tag{1.4}$$

1.4.2 Proposition. For any $1 \leq r \leq |S| - 3$, the map: $(\tau, m) \longmapsto \mathrm{mon}((\tau, m))$ establishes a bijection between the set of nice monomials of degree r in F_S and stable S-trees with multiplicity (τ, m) with $deg(\tau, m) := \sum_{e \in E_\tau} m(e) = r$. □

Proof. Immediate from 1.3.2. ■

1.4.3 Remark. Notice that unlike the case of good monomials, a nice monomial can represent a zero class even if the degree is less than or equal to $|S| - 3$.

1.5 Rooted trees and ordered partitions

1.5.1 Remark. If we choose a distinguished element $s \in S$, we can define natural bijections between the following three sets:

(a) unordered 2-partitions $\sigma = \{S_1, S_2\}$ of S,

(b) ordered 2-partitions $\sigma = \langle S_1, S_2 \rangle$ with the condition $s \in S_2$,

(c) subsets $T \subseteq S \setminus \{s\}$.

This is due to the fact that given the first component of an ordered pair of the above type, the second one is uniquely determined.

1.5.2 The case of $\overline{n}$. In particular, for $S = \overline{n}$ we choose n as the distinguished element and we equivalently index the generators of H^* by subsets $S \subset \overline{n-1}$ with the restriction $2 \leq |S| \leq n - 2$. (Note that this excludes the set $\overline{n-1}$ itself.) We denote the generator corresponding to such a set S:

$$D_S := D_{S, \overline{n} \setminus S},$$

for $S \subset \overline{n-1}$. The relations (i) and (ii) stated in this notation become

(i′) $D_S D_T$ if $S \cap T \neq \emptyset$ and the two sets satisfy no inclusion relation, (ii′) For any four numbers i, j, k, l,

$$\sum_{\substack{\overline{n-1}\supset T\supseteq\{i,j\}\\ k,l\notin T}} D_T + \sum_{\substack{\overline{n-1}\supset T\supseteq\{k,l\}\\ i,j\notin T}} D_T - \sum_{\substack{\overline{n-1}\supset T'\supseteq\{i,k\}\\ j,l\notin T}} D'_T - \sum_{\substack{\overline{n-1}\supset T'\supseteq\{j,l\}\\ i,k\notin T}} D'_T. \tag{1.5}$$

The expression for D_S^2 for a choice $i, j \in S$ and $k \notin S$ reads

$$D_S^2 = -\sum_{S\supset T\supseteq\{i,j\}} D_S D_T - \sum_{\substack{S\subset T\subset\overline{n-1}\\ k\notin T}} D_S D_T. \tag{1.6}$$

This is the formula (1.7) from [KMK] with i, j, k, n playing the role of i, j, k, l.

The analogs of formula (1.3) follow in the same manner.

1.5.3 Rooted trees and orientation. A rooted S-tree will be a pair (τ, v_{root}) consisting of an S-tree τ and one of its vertices v_{root} called the root. An orientation of a tree is considered to be a map $or : E_\tau \to V_\tau$, with the restriction that e is incident to $or(e)$. We will use the terminology "e is pointing towards v" to indicate $v = or(e)$ ("pointing away" will be used on the same basis). The set $or^{-1}(v)$ will be called the incoming edges; the remaining incident edges will be considered as outgoing. Furthermore, notice that an oriented edge e of a tree defines a subtree by cutting e and selecting the tree containing $or(e)$. This subtree will be called the branch of e.

1.5.4 Natural orientation for a rooted tree. For a rooted tree (τ, v_{root}) there is a natural orientation defined by setting $or(e) =$ the vertex of e, which is furthest away from the root (i.e., e is part of the unique path from this vertex to the root). Notice that in this orientation there is exactly one incoming edge to each vertex except for the root, which has none. Therefore, the restriction of or induces a one-to-one correspondence of $V(\tau) \setminus \{v_{\text{root}}\}$ and $E(\tau)$:

$$e \mapsto \text{ vertex to which } e \text{ is pointing} \qquad \textit{inversely} \qquad v \mapsto \text{ the unique incoming edge.} \tag{1.7}$$

1.5.5 Orientation for an n-tree. For a given n-tree we will fix the root to be the vertex with the flag numbered by n emanating from it. This defines a one-to-one correspondence of n-trees with rooted n-trees. Using this picture and Remark 1.5.1 we can equivalently view an n-tree (with multiplicity) as either given by the good (nice) collection of 2-partitions associated to its edges or as a good (nice) collection of subsets of $\overline{n-1}$ associated to its vertices. In the latter case we associate to each vertex the set S of the 2-partition corresponding to the incoming edge, which *does not* contain n. In this way denote, for given nice σ and $S \in \sigma$, by ν_S (resp. e_S) the vertex (resp. edge) corresponding to S.

Adopting this point of view, we can express quantities that are defined in the language of Remark 1.5.1 (c) in terms of oriented n-trees. Let σ be a collection of stable subsets of $\overline{n}$, i.e., for each $S \in \sigma$ $S \subset \overline{n-1}$ and $|S| \geq 2$. Define for any $S \in \sigma$

$$\begin{aligned} \omega_\sigma(S) &= \{T | T \subset S \text{ and maximal in this respect}\} \\ \operatorname{depth}_\sigma(S) &= |\{T | T \in \sigma \text{ and } T \supseteq S\}|. \end{aligned} \tag{1.8}$$

The definitions of (1.8) translate in the following way into tree language:

$$\begin{aligned} |S| &= |\{\text{tails marked by } i \in \overline{n-1} \text{ on the branch of } e_S\}| \\ \omega_\sigma(S) &= \{\text{outgoing edges of } \nu_S\} \\ \operatorname{depth}_\sigma(S) &= \text{the distance from } \nu_S \text{ to } \nu_{\text{root}} \end{aligned} \tag{1.9}$$

where the distance is the number of edges along the unique shortest path.

2 The intersection form

2.1 Notation

To calculate the intersection form we need a formula for two monomials of complementary degree. Recall that for a tuple (σ, m) of a nice collection of subsets of $\overline{n-1}$ and a multiplicity function $m : \sigma \mapsto \mathbb{N}$, we denote by mon$(\sigma, m)$ the monomial $\prod_{S\in\sigma} D_S^{m(S)}$. The *degree* of such a monomial is $\sum_{S\in\sigma} m(S)$. Furthermore let $\tau(\sigma, m)$ be the tuple $(\tau(\sigma), m')$ where $\tau(\sigma)$ is the tree corresponding to the good monomial $\prod_{S\in\sigma} D_S$, and $m' : E_{\tau(\sigma)} \to \mathbb{N}$ is the multiplicity function given by $e_S \to m(S)$.

2.2 Definition

A multiplicity orientation for a tree with multiplicity (τ, m) is a map mult : $F_\tau \setminus T_\tau \mapsto \mathbb{N}$ such that if ν_1 and ν_2 are the vertices of an edge e, then

$$\operatorname{mult}((\nu_1, e)) + \operatorname{mult}((\nu_2, e)) = m(e) - 1. \tag{2.1}$$

It is called good if for every $\nu \in V_\tau$ it satisfies

$$\sum_{f\in F_\tau(\nu)} \operatorname{mult}(f) = |\nu| - 3. \tag{2.2}$$

This is the analog of the good orientation in [KMK].

2.3 Lemma. For an n-tree (τ, m) in top degree (i.e., $\sum_{e \in E_\tau} m(e) = n - 3$), there exists at most one good multiplicity orientation. □

Proof. Assume that there are two good orientations mult, mult′. Consider the union of all edges on which mult $\neq$ mult′. Each connected component of this union is a tree. Choose an end edge e of this tree and an end vertex v of e. At v, the sum over all flags f of mult(f) and mult′(f) must be equal, but on (v, e) these differ. Hence, there must exist an edge $e' \neq e$ incident to v upon which mult$((v, e'))$ and mult′$((v, e'))$ differ. But this contradicts to the choice of v and e. ■

The next lemma gives a way to decide whether this good multiplicity orientation exists and if so a method to calculate it.

2.4 Lemma. Assume that an n-tree $\tau(\sigma, m)$ in top degree has a good multiplicity orientation mult. Let v_S be the vertex corresponding to $S \in \sigma$, and let f_S be the flag of the unique incoming edge; then the following formula for its multiplicity holds:

$$\mathrm{mult}(f_S) = |S| - 2 - \sum_{T \in \sigma | T \subset S} m(T). \tag{2.3}$$

□

Proof. We will use induction on the distance from the end vertices (i.e., those vertices with only one adjacent edge) in the natural orientation of n-trees given by 1.5.5; the case for the end vertices is trivial. Now let v_S be the vertex corresponding to S. By induction we can assume that for all outgoing flags (2.3) holds; i.e., for all (v, e_T) with $T \in \omega_\sigma(S)$,

$$\mathrm{mult}((v, e_T)) = m(T) - 1 - |T| + 2 + \sum_{T' \in \sigma | T' \subset T} m(T').$$

Inserting this into the condition (2.2), we arrive at

$$\begin{aligned} \mathrm{mult}(f_S) &= |v_S| - 3 - \sum_{T \in \omega_\sigma(S)} \mathrm{mult}((v, e_T)) \\ &= |S| - |\bigcup_{T \in \omega_\sigma(S)} T| + |\omega_\sigma(S)| - 2 - \sum_{T \in \omega_\sigma(S)} \left(m(T) - |T| + \sum_{T' \in \sigma | T' \subset T} m(T') + 1 \right) \\ &= |S| - 2 - \sum_{T \in \sigma | T \subset S} m(T), \end{aligned}$$

where in the last step we have used that $|\bigcup_{T \in \omega_\sigma(S)} T| = \sum_{T \in \omega_\sigma(S)} |T|$, since σ is a nice collection. ■

Consider the functional $\int_{\overline{M}_{0,S}} : H^*(\overline{M}_{0,S}) \to K$ given by

$$\mathrm{mon}(\tau) \longmapsto \begin{cases} 1, & \text{if } \deg \mathrm{mon}(\tau) = |S| - 3, \\ 0 & \text{otherwise.} \end{cases}$$

for any tree τ with $m \equiv 1$.

We put $\langle (\tau_1, m_1)\,(\tau_2, m_2) \rangle = \int_{\overline{M}_{0S}} \mathrm{mon}((\tau_1, m_1))\,\mathrm{mon}((\tau_2, m_2))$ and set to calculate this intersection index for the case when $\deg \mathrm{mon}((\tau_1, m_1)) + \deg \mathrm{mon}((\tau_2, m_2)) = |S| - 3$. Generally, we will write $\langle \mu \rangle$ instead of $\int_{\overline{M}_{0S}} \mu$.

2.5 Theorem. Let $\mathrm{mon}(\sigma_1, m_1)$ and $\mathrm{mon}(\sigma_2, m_2)$ be two monomials of complementary degree in H^*_n. If there is no good multiplicity orientation of $(\tau, m) := \tau(\sigma_1 \cup \sigma_2, m_1 + m_2)$, then $\langle \mathrm{mon}(\sigma_1, m_1)\,\mathrm{mon}(\sigma_1, m_1) \rangle = 0$. If there does exist one, then

$$\langle \mathrm{mon}(\sigma_1, m_1)\,\mathrm{mon}(\sigma_1, m_1) \rangle = \prod_{v \in V_\tau} (-1)^{|v|-3} \frac{(|v| - 3)!}{\prod_{f \in F(v)} (\mathrm{mult}(f))!^2} \prod_{e \in E_\tau} (m(e) - 1)!,$$

where mult is the unique multiplicity orientation of (τ, m) provided by Lemma 2.3, whose value is given in the formula (2.3). □

Proof. Set $E := \{e \in E_\tau | m(e) > 1\}$ and δ the subtree consisting of E with multiplicity $m|_E$ and its vertices. Consider the canonical embedding $\varphi_\tau : \overline{M}_\tau \to \overline{M}_{0S}$. We have

$$\langle \mathrm{mon}(\sigma_1, m_1)\,\mathrm{mon}(\sigma_1, m_1) \rangle = \left\langle \prod_{e \in E} \varphi^*_\tau(D^{m(e)-1}_{S(e)}) \right\rangle, \tag{2.4}$$

where the cup product in the right-hand side is taken in $H^*(\overline{M}_\tau) \cong \otimes_{v \in V_\tau} H^*(\overline{M}_{0, F_\tau(v)})$. Applying an appropriate version of the formulas (1.5), we can write for any $e \in E$ with vertices v_1, v_2,

$$\varphi^*_\tau(D_{\sigma(e)}) = -\Sigma_{v_1, e} - \Sigma_{v_2, e}, \tag{2.5}$$

where

$$\Sigma_{v_i, e} \in H^*(\overline{M}_{0, F_\tau(v_i)}) \otimes \prod_{v \neq v_i} [\overline{M}_{0, F_\tau(v)}] \tag{2.6}$$

and $[\overline{M}_{0, F_\tau(v)}]$ is the fundamental class. Later, we will choose an expression for $\Sigma_{v_i, e}$ depending on the choice of flags denoted i, j or k, l in (1.5).

Inserting (2.5) into (2.4), we get

$$\langle \mathrm{mon}(\sigma_1, m_1)\,\mathrm{mon}(\sigma_1, m_1) \rangle = \sum_{\mathrm{or}} \prod_{e \in E_\tau} (m(e) - 1)! \left\langle \prod_{\substack{(v,e) \in F_\delta \\ \mathrm{or}((v,e)) \geq 1}} \frac{1}{\mathrm{or}((v, e))!} (-\Sigma_{v,e})^{\mathrm{or}((v,e))} \right\rangle, \tag{2.7}$$

where or runs over all multiplicity orientations of δ. The summand of (2.7) corresponding to a given or can be nonzero only if for every $v \in V_\delta$ the sum of the degrees of factors equals $\dim \overline{M}_{0,F_\tau(v)} = |v| - 3$. This is what was called a good multiplicity orientation. By Lemma 2.3 there can only exist one such orientation. Now assume that one good orientation mult exists. We can rewrite (2.7) as

$$\langle \mathrm{mon}(\sigma_1, m_1)\, \mathrm{mon}(\sigma_1, m_1) \rangle = \prod_{e \in E_\tau} (m(e) - 1)! \prod_{\substack{(v,e) \in F_\delta \\ \mathrm{mult}((v,e)) \geq 1}} \frac{1}{\mathrm{mult}((v,e))!} \langle (-\Sigma_{v,e})^{\mathrm{mult}((v,e))} \rangle. \tag{2.8}$$

In view of (2.6), this expression splits into a product of terms computed in all $H^*(\overline{M}_{0,F_\tau(v)})$, $v \in V_\tau$ separately. Each such term depends only on $|v|$, and we want to demonstrate that it equals $(-1)^{|v|-3}(|v| - 3)!/\left(\prod_{f \in F(v)} (\mathrm{mult}(f))!\right)$. Put $|v| = m$, so $m \geq 3$. Let us identify F_τ with $\{1, \ldots, m\}$ and denote by $D_\rho^{(m)}$ the class of a boundary divisor in $H^*(\overline{M}_{0,m})$ corresponding to a stable partition ρ of $\{1, \ldots, m\}$, and set $d_i := \mathrm{mult}((v, e_i))$, where e_i is the edge belonging to the flag $i \in \{1, \ldots, m\}$. The contribution of v in (2.8) becomes

$$\prod_{i=1}^{m} \langle (-\Sigma_i^{(m)})^{d_i} \rangle := g(d_1, \ldots, d_m), \tag{2.9}$$

where $-\Sigma_i^{(m)}$ is the element of (2.6) and the superscript (m) is again included to keep track of the spaces involved. We will prove the following properties of the function $g(d_1, \ldots, d_m)$ identifying it as $(-1)^{m-3}(m-3)!/(d_1! \ldots d_m!)$:

(a) $g(0, 0, 0) = 1$.

(b) $g(d_1, \ldots, d_m)$ is symmetric in the d_i.

(c) If $d_m = 0$, then

$$g(d_1, \ldots, d_m) = -\sum_{i : d_i \geq 1} g(d_1, \ldots, d_i - 1, \ldots, d_{m-1}).$$

2.5.1 Remarks. Notice that up to the minus sign in (c) these are exactly the conditions satisfied by the numbers $\langle \tau_{\alpha_1} \ldots \tau_{\alpha_m} \rangle$ in genus 0 [K]. Furthermore, we can always choose the flags in such a way that the flags $1, \ldots, k$ $(k \leq m - 3)$ belong to the edges e with $\mathrm{mult}(f(v, e)) \geq 1$.

(ad a) We have by definition $\langle [\overline{M}_{0,3}] \rangle = 1$.

(ad b) The symmetricity results from the fact that the integral in question does not depend on a renumbering of the flags.

(ad c) First, we can use relation (1.5) for any k, l to write

$$-\Sigma_i^{(m)} = \sum_{\rho:\ i\rho\{k,l\}} -D_\rho^{(m)}. \tag{2.10}$$

We will calculate (2.9) inductively. Consider the projection map (forgetting the (m)th point) $p : \overline{M}_{0,m} \to \overline{M}_{0,m-1}$ and the ith section map $x_i : \overline{M}_{0,m-1} \to \overline{M}_{0,m}$ obtained via the identification of $\overline{M}_{0,m+3}$ with the universal curve. We have $p \circ x_i = \mathrm{id}$, and x_i identifies $\overline{M}_{0,m-1}$ with $D^{(m)}_{\sigma_i}$ where

$$\sigma_i = \{\{m, i\}\{1, \ldots, \widehat{i}, \ldots, m-1\}\};$$

so if we choose some $k, l \neq m$:

$$\sum_{\rho:\ i\rho\{k,l\}} -D^{(m)}_{\rho} = -p^* \left(\sum_{\rho':\ i\rho'\{k,l\}} D^{(m-1)}_{\rho'} \right) - x_{i*}([\overline{M}_{0,m-1}]). \tag{2.11}$$

We will now replace one of the Σ_i for each i with $d_i \geq 1$ using (2.10) with some arbitrary $k, l \neq m$. Then (2.9) reads

$$\prod_{i=1}^{m} \left\langle \left(-p^* \left(\sum_{\rho':\ i\rho'\{k,l\}} D^{(m-1)}_{\rho'} \right) - x_{i*}\left([\overline{M}_{0,m-1}]\right) \right) (-\Sigma^{(m)}_i)^{d_i-1} \right\rangle \tag{2.12}$$

where ρ' runs over stable partitions of $\{1, \ldots, m-1\}$. We represent the resulting expression as a sum of products consisting of several p^*-terms and several x_{i*}-terms each. If such a product contains greater than or equal to two x_{i*}-terms, it vanishes, because the structure sections pairwise do not intersect. We obtain

$$\begin{aligned} &\sum_{i:d_i\geq 1} \left\langle \prod_{j\neq i:d_j\geq 1} \left(-p^* \left(\sum_{\rho':\ j\rho'\{k,l\}} D^{(m-1)}_{\rho'} \right) (-\Sigma^{(m)}_j)^{d_j-1} \right) (-x_{i*}([\overline{M}_{0,m-1}]))(-\Sigma^{(m)}_i)^{d_i-1} \right\rangle \\ &\quad + \left\langle \prod_{i:d_i\geq 1} p^* \left(- \sum_{\rho':\ i\rho'\{k,l\}} D^{(m-1)}_{\rho'} \right) (-\Sigma^{(m)}_j)^{d_j-1} \right\rangle. \end{aligned} \tag{2.13}$$

If $d_i - 1 > 0$, then the summand containing an x_{i*}-term will vanish. To see this again, replace one of the Σ_i using (2.10), but with $k = m$ and some l. In the case $d_i - 1 = 0$ we can write the respective term in the sum in (2.13) as

$$\left\langle \left(p^* \left(- \sum_{\rho':\ j\rho'\{k,l\}} D^{(m-1)}_{\rho'} \right)^{d_j-1} \right) (-x_{i*}([\overline{M}_{0,m-1}])) \right\rangle$$

by replacing the Σ_j according to (2.11) and again using the fact that the structure sections do not pairwise intersect. Using induction on the last summand in (2.13), we arrive at the situation where all $\Sigma^{(m)}_i$'s have been replaced. And the product only contains one $p^*(\Sigma^{(m-1)}_i)$-term, but this term vanishes, because $\dim \overline{M}_{0,m-1} = m-2$. Finally, we are left with one summand for each $i : d_i \geq 1$ containing only one x_{i*}-term and p^*-terms. Using the projection formula

$$\langle p^*(X) x_{i*}([\overline{M}_{0,m+1}])\rangle = \langle X \rangle,$$

one sees that each such term equals $-g(d_1, \dots d_i - 1, \dots, d_{m-1})$. And the result follows.

■

3 A boundary divisorial basis and its tree representation

The work presented in this section is inspired by the presentation of a basis of the cohomology ring of $\overline{M}_{0n}$ given in terms of hyperplane sections in [Yu]; especially the notions of the *-operation and the order have been adapted to the present context.

3.0 Preliminaries

In order to state the basis, we make use of certain classes

$$D_S x_S^k := \pi_{f_S *}(D_S^{k+1} D_{S \amalg f_S}), \qquad k \geq 0, \tag{3.1}$$

where $\pi_{f_S *} : \overline{M}_{0,\bar{n} \amalg f_S} \to \overline{M}_{0n}$ is the forgetful map forgetting the point f_S.

Another way to present these classes is given by the following observation. Consider the following decomposition of D_S^2 using (1.6):

$$D_S^2 = D_S \left(- \sum_{\{i,j\} \subset T \subset S} D_T - \sum_{\substack{\overline{n-1} \supset T' \supset S \\ k \notin T'}} D'_T \right) =: D_S(x_S + y_S) \tag{3.2}$$

for any choice of $i, j \in S, k, l \notin S$. With the notation (3.2), we can write D_S^{k+1} in the same spirit as

$$D_S^{k+1} = D_S \left(\sum_{i=0}^{k} \binom{k}{i} x_S^i y_S^{k-i} \right). \tag{3.3}$$

In the context of the proof of Theorem 2.5, each summand of (3.3) corresponds to a choice of multiplicity orientation. In particular, the term with x_S^i corresponds to the one which satisfies $\mathrm{mult}(f_S) = i, \mathrm{mult}(f_{S^c}) = k - i$ for the flags f_S and f_{S^c} of e_S so that we can identify (3.1) with the summand corresponding to $\mathrm{mult}(f_S) = k$, $\mathrm{mult}(f_{S^c}) = 0$.

3.0.1 A tree representation. A tree representation for a class (3.1) is given by a choice of an ordered $k+1$ element subset $\langle f_1, \ldots, f_{k+1}\rangle$ of S as the sum over all assignments of the flags of $S \setminus \{f_1, \ldots, f_{k+1}\}$ to the vertices of the linear tree determined by $D_{\{f_1,f_2\}} D_{\{f_1,f_2,f_3\}} \cdots D_{\{f_1,\ldots,f_{k+1}\}}$:

$$D_S x_S^k = (-1)^k D_S \sum_{\substack{\langle S_1,\ldots,S_k\rangle \\ S_1 \amalg \ldots \amalg S_k = S\setminus\{f_1,\ldots,f_{k+1}\}}} D_{\{f_1,f_2\}\amalg S_1} D_{\{f_1,f_2,f_3\}\amalg S_2} \cdots D_{\{f_1,\ldots,f_{k+1}\}\amalg S_k}. \tag{3.4}$$

More generally, let τ given by $D_{T_1} \cdots D_{T_k}$ be any tree with $|\nu_{T_i}| = 3$ for $i = 1, \ldots, k$ and $T_1 \amalg \ldots \amalg T_k = \{f_1, \ldots, f_k\}$; then

$$D_S x_S^k = (-1)^k D_S \sum_{\substack{\langle S_1,\ldots,S_k\rangle \\ S_1 \amalg \ldots \amalg S_k = S\setminus\{f_1,\ldots,f_{k+1}\}}} D_{T_1 \amalg S_1} \cdots D_{T_k \amalg S_k}. \tag{3.5}$$

Both (3.4) and (3.5) follow from (1.5) with the appropriate choices for the flags.

3.1 The basis

Consider a class of the type

$$\mu = \pi_{n*}\left(D_{S_1} x_{S_1}^{m(S_1)} \cdots D_{S_k} x_{S_k}^{m(S_k)} D_{\overline{n-1}} x_{\overline{n-1}}^{m(\overline{n-1})}\right), \qquad m(S) \geq 0. \tag{3.6}$$

To this class we associate the underlying $(n+1)$-tree $\tau(\mu)$ determined by $D_{S_1} \cdots D_{S_k} D_{\overline{n-1}}$. The powers $m(S)$ then determine a unique multiplicity orientation in the sense of 3.0 given by $\mathrm{mult}(f_S) := m(S)$, $\mathrm{mult}(f_{S^c}) = 0$ where f_S and f_{S^c} are the flags corresponding to the edge e_S in $\tau(\mu)$.

Using the equations of the type (3.4), we can associate to each monomial μ a sum of good monomials which we will call tree(μ).

Consider the set

$$\begin{aligned}\mathcal{B}_n := \{&\pi_{n+1*}(D_{S_1} x_{S_1}^{m(S_1)} \cdots D_{S_k} x_{S_k}^{m(S_k)} D_{\overline{n-1}} x_{\overline{n-1}}^{m(\overline{n-1})}) \mid 0 \leq m(S) \leq |\nu_S| - 4 \text{ and} \\ &0 \leq m(\overline{n-1}) \leq |\nu_{\overline{n-1}}| - 3\}.\end{aligned} \tag{3.7}$$

3.1.1 Proposition. The set $\mathcal{B}_n$ is a basis for $A^*(\overline{M}_{0n})$. □

Proof. By Lemma 3.1.2 and 3.1.5. ■

3.1.2 Lemma. The set $\mathcal{B}_n$ spans $A^*(\overline{M}_{0n})$. □

Proof. From [Ke] and [KMK] we know that the good monomials span; so it will be sufficient to show that any such monomial is in the span of $\mathcal{B}_n$. Now let $\tau(\mu)$ be the tree corresponding to such a good monomial μ. If for all $\nu \in V_\tau$, $|\nu| \geq 4$, then the monomial is already in $\mathcal{B}_n$. If not, let τ_3 be a maximal subtree of τ whose vertices (except for the root—induced by the natural orientation), all have valency three; call such a tree a 3-subtree and the number of its edges its length. Furthermore, let R be the set associated with the root. Let $F_3(\tau_3)$ be the set of tails of τ_3 without the ones coming from the root. The formula (3.5) for the tree representation of $D_R x_R^l$, with the choice of $F_3(\tau_3)$ as the fixed set and τ_3 as a 3-subtree, expresses τ in terms of trees with less maximal 3-subtrees of maximal length whose vertices either comply with the conditions of $\mathcal{B}_n$ or are part of a unique maximal subtree whose root ν_R has multiplicity 0; i.e., x_R does not divide the monomial corresponding to the tree. Notice that if the root ν_R of any 3-subtree is 3-valent, then $R = \overline{n-1}$. We can now proceed by induction of the number of such maximal 3-subtrees with the maximal number of edges l.

3.1.2 The *-operation. We define the following involution on $\mathcal{B}_n \times \mathbb{Z}_2$:

$$\pi_{n+1*}\left(D_{S_1} x_{S_1}^{m(S_1)} \cdots D_{S_k} x_{S_k}^{m(S_k)} D_{\overline{n-1}} x_{\overline{n-1}}^{m(\overline{n-1})}\right) \xrightarrow{*}$$
$$\pi_{n+1*}\Big((-1)^{|\nu_{S_1}|-3} D_{S_1} x_{S_1}^{|\nu_{S_1}|-4-m(S_1)} \cdots (-1)^{|\nu_{S_k}|-3} D_{S_k} x_{S_k}^{|\nu_{S_k}|-4-m(S_k)}$$
$$(-1)^{|\nu_{\overline{n-1}}|-3} D_{\overline{n-1}} x_{\overline{n-1}}^{|\nu_{\overline{n-1}}|-3-m(\overline{n-1})}\Big). \tag{3.8}$$

This operation preserves the underlying tree $\tau(\mu)$, but changes the multiplicities in such a way that μ and μ^* have complementary dimensions. More precisely, consider μ as the push forward of the class $\bigotimes_{\nu_S \in V_{\tau(\mu)}} x_S^{m(S)} \in H^*(\overline{\mathcal{M}}_{\tau(\mu)})$ to $H^*(\overline{M}_{0n})$. Then, locally at each vertex, we have a class of degree $m(S)$. This class is replaced under the *-operation by a "dual" class of complementary degree $\dim(\overline{M}_{0,F_\tau}(\nu_S)) - m(S)$, which is provided as a summand of $\varphi_{D_S}^*(D_S x_S^{|\nu_S|-4-m(S)})$. ■

3.1.3 Lemma. For two elements μ, ν of $\mathcal{B}_n$, the integral $\int_{\overline{M}_{0n}} \mu\nu^*$ does not vanish if and only if $\tau(\mu\nu^*)$ is nonzero and if there is one good multiplicity orientation among the multiplicity orientations satisfying $\text{mult}(f_S) = m^\mu(S) + m^{\nu^*}(S) + 1$, $\text{mult}(f_{S^c}) = 0$ or $(f_S) = m^\mu(S) + m^{\nu^*}(S)$, $\text{mult}(f_{S^c}) = 1$ where f_S, f_{S^c} are the flags of the edge e_S. If such an orientation exists, it is unique and

$$\int_{\overline{M}_{0n}} \mu\nu^* = \prod_{\nu \in V_{\tau(\nu)}} (-1)^{|\nu|-3} \prod_{\nu \in V_{\tau(\mu\nu^*)}} (-1)^{|\nu|-3} \frac{(|\nu|-3)!}{\prod_{f \in F_{\tau(\mu\nu^*)}(\nu)} (\text{mult}(f))!}. \tag{3.9}$$

□

Proof. The formula (3.9) and the conditions for μ and ν, as well as the ones for the considered multiplicity orientations, follow from Theorem 2.5 by considering the summands of

$$\pi_{n+1*}\left(D_{S_1}^{\epsilon(S_1)+m(S_1)}\cdots D_{S_l}^{\epsilon(S_l)+m(S_l)}D_{\overline{n-1}}^{m(\overline{n-1})}\right)$$

corresponding via 3.0 to the given monomial

$$\mu\nu^* = \pi_{n+1*}\left(D_{S_1}^{\epsilon(S_1)}x_{S_1}^{m(S_1)}\cdots D_{S_l}^{\epsilon(S_l)}x_{S_l}^{m(S_l)}D_{\overline{n-1}}x_{\overline{n-1}}^{m(\overline{n-1})}\right)$$

with $\epsilon(S) \in \{1,2\}$. ■

Notice that in the formula (3.9) the binomial coefficients $\binom{m(e_S)-1}{\text{mult}(f_S)}$ which appear in Theorem 2.5 are absent. This is due to the fact that these factors, stemming from the expansion of $D_S^{m(e_S)}$ as in (3.3), are stripped off in the definition of the classes $D_S x_S^k$.

3.1.4 An order. Given two monomials μ, μ' of type (3.6) of the same degree we call $\mu \prec \mu'$ if, for the maximal integer k such that all sets of the depth d vertices for $1 \le d \le k$ coincide and $m(S) = m'(S)$ for all sets of the depth d' vertices for $1 \le d' < k$, one of the following conditions holds:

(a) $m(S) \le m'(S)$ for all S of depth k and the inequality is strict for at least one S, or

(b) $m(S) = m'(S)$ and $|\nu_S| \ge |\nu'_S|$ for all S of depth k and there is at least one S where the inequality is strict.

It is easy to check that this defines a half-order on $\mathcal{B}_n$.

The *-operation connects with the half-order $\prec$ in the following way.

3.1.5 Lemma. If $\mu, \nu \in \mathcal{B}_n$ are two distinct basis elements ($\mu \neq \nu$) and $\mu\nu^* \neq 0$, then $\mu \prec \nu$. □

Proof. We will use superscripts μ, ν to refer to the quantities concerning the monomials μ, ν, and take quantities without any superscript to refer to $\mu\nu^*$. So the notation $|\nu_S^\nu|$ is used for the valency of the vertex ν_S in the tree $\tau(\nu)$, and $|\nu_S|$ without any superscript is taken to be the valency of the vertex ν_S in the tree $\tau(\mu\nu^*)$. If $\mu\nu^* \neq 0$, then the underlying tree of $\mu\nu^*$ carries a unique good multiplicity orientation by Theorem 2.5. Furthermore, the underlying trees of μ and ν coincide up to depth k; this is the first condition for k. From this, together with Lemma 3.1.3, it follows that the good multiplicity orientation up to depth $k-1$ is given by $\text{mult}(f_S) = |\nu_S| - 3$. Now at depth k we must have $\text{mult}(f_S) \le |\nu_S| - 3$ and, because the multiplicity orientation is fixed for all lower depths as specified, we also have $\text{mult}(f_S) = m^\mu(S) + m^{\nu^*}(S) + \delta_{S,\overline{n-1}} = m(S) + |\nu_S^\nu| - 3 - m^\nu(S)$. Combining these two relations, we find the condition

$$m^\mu(S) - m^\nu(S) \leq |\nu_S| - |\nu_S^\nu|. \tag{3.10}$$

Furthermore, we have the inequalities $|\nu_S| \leq |\nu_S^\nu|, |\nu_S| \leq |\nu_S^\mu|$, since $\tau(\mu)$ and $\tau(\nu^*) = \tau(\nu)$ result from $\tau(\mu\nu^*)$ via contractions of edges which only increase the number of flags at the remaining vertex. So the left-hand side of (3.10) is less than or equal to zero:

$$m^\mu(S) - m^\nu(S) \leq 0. \tag{3.11}$$

Thus if the inequality is strict for some S, we arrive at condition (a). If, however, $m^\mu(S) = m^\nu(S)$ for all S of depth k, the following inequality must also hold:

$$0 \leq |\nu_s^\mu| - |\nu_s^\nu|. \tag{3.12}$$

Equality for all S in (3.12), however, would contradict the choice of k, since if $m^\mu(S) = m^\nu(S)$ and $|\nu_s^\mu| = |\nu_s^\nu|$, we have $|\nu_s^\nu| = |\nu_s| = |\nu_s^\mu|$ from the above inequalities, so that there are no contractions from $\tau(\mu\nu^*)$ to $\tau(\mu)$ and $\tau(\nu)$ up to depth $k+1$, and the sets of depth $k+1$ corresponding to the outgoing edges of ν_s^μ and ν_s^ν must also coincide. ■

3.1.6 Lemma. Consider the matrix $T = (t_{\mu,\nu})_{\mu,\nu\in\mathcal{B}_n}$ given by

$$t_{\mu,\nu} := \int_{\overline{M}_{0n}} \mu\nu^*.$$

This matrix is unipotent and the entry $t_{\mu,\nu}$ is determined by Lemma 3.1.3.

In particular, the set $\mathcal{B}_n$ is linear independent. □

Proof. For the diagonal entries, $\int \mu\mu^* \, \text{mult}(f_S) = |\nu_S| - 3$ is a good multiplicity orientation so that (3.9) renders $t_{\mu\mu^*} = 1$. Furthermore, by considering any extension of the half-order to a total order, the unipotency is proved by Lemma 3.1.5. ■

3.2 The intersection form and its inverse for the basis $\mathcal{B}_n$.

With the help of the matrix T introduced in 3.1.6, we can write the matrix M for the intersection form in the basis $\mathcal{B}_n$ as $M = TP$ where the matrix P is the matrix representation of the *-operation given by the signed permutation matrix

$$P_{\mu,\nu} = (-1)^{n-2-|E_{\tau(\mu)}|}\delta_{\mu,\mu^*}. \tag{3.13}$$

Theorem 3.2.1. The Gram-matrix $(m_{\mu\nu})$ for the basis $\mathcal{B}_n$ is given by

$$m_{\mu\nu} = (-1)^{n-2-|E_{\tau(\nu)}|} t_{\mu\nu^*} \tag{3.14}$$

and its inverse matrix $(m^{\mu\nu})$ is given by the formula

$$m^{\mu\nu} = (-1)^{n-2-|E_{\tau(\mu)}|} \left(\delta_{\mu^*\nu} + \sum_{k\geq 0} (-1)^{k+1} \sum_{\mu^* \prec \tau_1 \ldots \prec \tau_k \prec \nu} t_{\mu^*\tau_1} t_{\tau_1\tau_2} \cdots t_{\tau_{k-1}\tau_k} t_{\tau_k \nu} \right), \tag{3.15}$$

where the values for the $t_{\sigma,\sigma'}$ are given by (3.9) and the sum over k is finite. □

Proof. Formula (3.14) follows from the above decomposition $M = TP$. To prove formula (3.15), set $N := id - T$. According to Lemma 3.1.6, N is nilpotent and the inverse to the intersection form can now be written as

$$M^{-1} = PT^{-1} = P(id + N + N^2 + \cdots), \tag{3.16}$$

where the sum in (3.16) is finite. ■

4 Applications to Frobenius manifolds and quantum cohomology

4.1 Particular cases

Writing down the results of §2 and §3, we obtain the following intersection matrices M_n for small values of n.

$n = 3$: $M_3 = (1)$.

$n = 4$: For the basis $\pi_{5*}(D_{1,2,3}), \pi_{5*}(-D_{1,2,3}x_{1,2,3})$, we obtain

$$M_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$n = 5$: For the basis $\pi_{6*}(D_{1,2,3,4})$, $D_{1,2,3}$, $D_{1,2,4}$, $D_{1,3,4}$, $D_{2,3,4}$, $\pi_{6*}(D_{1,2,3,4}x_{1,2,3,4})$, $\pi_{6*}(D_{1,2,3,4}x^2_{1,2,3,4})$, the intersection matrix is

$$M_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$n = 6$: In this case, the intersection matrix also has only nonzero entries for the integrals of dual classes under the $*$-operation: $\int_{\overline{M}_{0n}} \mu\mu^*$, whose values are $(-1)^{4-|E_{\tau(\mu)}|}$.

$n \geq 7$: For the higher values of n, the structure of the matrix T is not diagonal, since entries other than those coming from the product of $*$-dual classes can also be nonzero, e.g., $\langle D_{i,j,k,l} x_{i,j,k,l} D_{i,j,k,l} x_{i,j,k,l} \rangle$ in $\overline{M}_{0,7}$. Thus, the $*$-operation fails to give the Poincaré duality for these spaces.

However, on the subspace $A^1(\overline{M}_{0n}) \oplus A^{n-4}(\overline{M}_{0n})$, the $*$-operation does provide the Poincaré duality, as can be deduced from Lemma 3.1.3. On this subspace, the matrix T is just the identity matrix so that the restriction to this subspace of M_n is given by P. In the case of small $n < 7$, this subspace is already the whole space, so that the matrices in the previous cases are just given by P.

4.2 Tensor product of higher-order operations of formal Frobenius manifolds

4.2.1 Frobenius manifolds. A formal Frobenius manifold is a triple $(H, g$, additional structure) where H is a (super-) vector space over a field K of characteristic zero, g is a nondegenerate even scalar product on H, and the additional structure is one of the following [D], [KM], [KMK]:

(i) a cohomological field theory (CohFT) (I_n),

(ii) a potential Φ for a set of abstract correlation functions (ACFs) (Y_n), satisfying the WDVV-equations, or

(iii) a structure of cyclic C_∞-algebra on H $(\circ_n)$.

The moduli space of rank-one CohFT and the respective structure of tensor product is presented in [KMK] and [KMZ].

As a brief reminder, we recall that a CohFT on (H, g) is given by a series of $\mathbb{S}_n$-equivariant maps

$$I_n : H^{\otimes n} \to H^*(\overline{M}_{0n}, K), \quad n \geq 3,$$

which satisfy the relations

$$\varphi_\sigma^*(I_n(\gamma_1 \otimes \cdots \otimes \gamma_n)) = \epsilon(\sigma)(I_{n_1+1} \otimes I_{n_2+1}) \left(\bigotimes_{j \in S_1} \gamma_j \otimes \Delta \otimes \left(\bigotimes_{k \in S_2} \gamma_k \right) \right), \tag{4.1}$$

where, for $\sigma = S_1 \amalg S_2$, φ_σ is the inclusion map of the divisor D_σ, $\varphi_\sigma : \overline{M}_{0,|S_1|+1} \times \overline{M}_{0,|S_2|+1} \to \overline{M}_{0n}$; $\Delta = \Sigma \Delta_a \otimes \Delta_b g^{ab}$ is the Casimir element; and $\epsilon(\sigma)$ is the sign of the permutation induced on the odd arguments $\gamma_1, \ldots, \gamma_n$.

4.2.2 Equivalences of the different structures. Given a CohFT, the associated system of ACF is defined as follows:

$$Y_n(\gamma_1 \otimes \ldots \otimes \gamma_n) = \int_{\overline{M}_{0n}} I_n(\gamma_1 \otimes \ldots \otimes \gamma_n). \tag{4.2}$$

The potential for a system of ACFs is given (after a choice of a basis $\{\Delta_a\}$ and dual coordinates x^a of H) as a formal power series depending on a point $\gamma = \sum x^a D_a$ by

$$\Phi(\gamma) = \sum_{n \geq 3} \frac{1}{n!} Y_n((x^a \Delta_a)^{\otimes n}). \tag{4.3}$$

The conditions (4.1) on the I_n are equivalent to the WDVV or associativity equations [KM]:

$$\sum_{ef} \partial_a \partial_b \partial_c \Phi \cdot g^{ef} \partial_f \partial_c \partial_d \Phi = (-1)^{a(b+c)} \sum_{ef} \partial_b \partial_c \partial_e \Phi \cdot g^{ef} \partial_f \partial_a \partial_d \Phi. \tag{4.4}$$

Here we use the simplified notation $(-1)^{a(b+c)}$ for $(-1)^{\tilde{x}_a(\tilde{x}_b + \tilde{x}_c)}$ where $\tilde{x}$ is the $\mathbb{Z}_2$-degree of x.

The reverse direction of (4.2), i.e., the reconstruction of a CohFT from its ACFs, is contained in the second reconstruction theorem of [KM]. In this context, the I_n can be recovered by extending the Y_n to a set of operadic ACFs, i.e., a set $\{Y(\tau) | \tau \text{ is an } n\text{-tree}\}$ satisfying

$$Y(\tau)(\gamma_1 \cdots \gamma_n) = \left(\bigotimes_{v \in V(\tau)} Y_{|v|} \right) (\gamma_1 \otimes \ldots \otimes \gamma_n \otimes \Delta^{\otimes E_\tau}), \tag{4.5}$$

where the arguments $Y_{|v|}$ are labeled by the flags of v (for the precise formalism of operadic ACFs see [KM]). The I_n themselves can be calculated via their Poincaré duals with the help of the formula

$$Y(\tau)(\gamma_1 \otimes \cdots \otimes \gamma_n) = \int_{\overline{M}_\tau} \varphi^*(I_n(\gamma_1 \otimes \cdots \otimes \gamma_n)). \tag{4.6}$$

The explicit calculation of the maps I_n given a potential Φ or a set of Y_n thus depends on the knowledge of the Poincaré duality as noted in [KMK], and is made possible by the results of §2 and §3.

The higher-order multiplications are derived from the ACFs in the following manner:

$$\circ_n := H^{\otimes n} \xrightarrow{Y_n+1} \check{H} \xrightarrow{g} H. \tag{4.7}$$

In the operadic setting given a set of higher-order multiplication, there is a natural operation associated to each n-tree τ (see [GK]) which we will call $\circ(\tau)$. Such an operation corresponds to a cyclic word with parenthesis roughly as follows. Denote the multiplication $\circ_n$ by the word $(x_1, \ldots, x_n)$ and think of it as a one-vertex tree with n incoming flags and one outgoing flag. Composing two higher multiplications corresponds to grafting two such trees in such a way that the outgoing flag of one tree is fused together with one of the incoming flags of the other tree to form an edge, e.g., the flag i for $(x_1, \ldots, x_{i-1}, (x_i, \ldots, x_{i+k}), x_{i+k+1}, \ldots, x_n)$. Continuing in this way, we obtain a tree from any such word and, vice versa, we can associate to any n-tree with the orientation of 1.5.5 an $(n-1)$-ary operation of composed higher multiplications.

4.2.3 Tensor product for Frobenius manifolds. In the language of CohFT, the tensor product of two formal Frobenius $(H', g', \{I'_n\})$ and $(H'', g'', \{I''_n\})$ is given by the tensor product CohFT on $H' \otimes H''$, which is naturally defined via the cup product in $H^*(\overline{M}_{0n}, K)$:

$$(I'_n \otimes I''_n)(\gamma'_1 \otimes \gamma''_1 \otimes \ldots \otimes \gamma'_n \otimes \gamma''_n) := \epsilon(\gamma', \gamma'') I'_n(\gamma'_1 \otimes \ldots \otimes \gamma'_n) \wedge I''_n(\gamma''_1 \otimes \ldots \otimes \gamma''_n) \quad (4.8)$$

where $\epsilon(\gamma', \gamma'')$ is the superalgebra sign.

Using (4.2)–(4.7), one can transfer this definition of the tensor product onto any of the other structures $(Y_n, Y(\tau), \Phi, \circ_n, \circ(\tau))$.

In particular, using Y'_n and Y''_n we obtain

$$(Y'_n \otimes Y''_n)(\gamma'_1 \otimes \gamma''_1 \otimes \ldots \otimes \gamma'_n \otimes \gamma''_n) = \int_{\overline{M}_{0n}} I'_n(\gamma'_1 \otimes \ldots \otimes \gamma'_n) \wedge I''_n(\gamma''_1 \otimes \ldots \otimes \gamma''_n). \quad (4.9)$$

In order to calculate the integrals on the right-hand side of (4.5), we will use the basis, the calculation of its Gram matrix, and its inverse obtained in the previous paragraph. We also utilize the operadic correlation functions corresponding to Y'_n, Y''_n (see [KM]) and use the notation $Y(\mu)$ for $Y(\mathrm{tree}(\mu))$ for a μ in $\mathcal{B}_n$. Now let $\mathcal{B}_n$ be the basis of $H^*(\overline{M}_{0n})$ given in 3.1, and $\check{\mu} = \sum_{\mu\nu} m^{\mu\nu}\nu$ the dual basis. Combining the results of §3 with the formula (4.9) and using (4.6), we obtain the following.

4.2.4 Corollary. The tensor product of two CohFT (H', g', Y') and (H'', g'', Y'') is given by

$$(Y'_n \otimes Y''_n)(\gamma'_1 \otimes \gamma''_1 \otimes \ldots \otimes \gamma'_n \otimes \gamma''_n) = \sum_{\mu,\nu \in \mathcal{B}_n} Y'(\check{\mu})(\gamma'_1 \otimes \ldots \otimes \gamma'_n) m_{\mu\nu} Y''(\check{\nu})(\gamma''_1 \otimes \ldots \otimes \gamma''_n). \quad (4.10)$$

□

4.2.5 Corollary. The tensor product of two Frobenius manifolds in terms of the higher-order multiplications is given by

$$\begin{aligned}&\circ'_n \otimes \circ''_n(\gamma'_1 \otimes \gamma''_1 \otimes \ldots \otimes \gamma'_n \otimes \gamma''_n)\\ &= \sum_{\mu,\nu\in\mathcal{B}_n} \circ'(\check{\mu})(\gamma'_1 \otimes \ldots \otimes \gamma'_n) m_{\mu\nu} \circ''(\check{\nu})(\gamma''_1 \otimes \ldots \otimes \gamma''_n).\end{aligned} \tag{4.11}$$

□

4.3 The Künneth formula in quantum cohomology

4.3.1 Quantum cohomology. The quantum cohomology of a projective manifold V will be regarded as a formal deformation of its cohomology ring with the coordinates of the space $H^*(V)$ being the parameters. The structure constants are given by a formal series Φ^V, which is defined in terms of Gromov-Witten invariants [KM]. One can regard the quantum cohomology as the structure of a Frobenius manifold on ($H^*(V)$, Poincaré pairing) with the GW-invariants playing the role of the I_n and the potential Φ^V being the potential of (4.2). The quantum cohomology of a product $V \times W$ regarded as a Frobenius manifold is just the tensor product of the Frobenius manifolds: ($H^*(V) \otimes H^*(W)$, Poincaré pairing, $\Phi^{V\times W}$), as can be shown using [B1], [B2]. Putting together (4.3) and Corollary 4.2.4, we obtain the explicit Künneth formula.

4.3.2 Corollary. The potential $\Phi^{V\times W}$ of the quantum cohomology of $V \times W$ is given by the formula

$$\Phi^{V\times W}(\gamma' \otimes \gamma'') = \sum_{n\geq 3} \sum_{\mu,\nu\in\mathcal{B}_n} Y'(\check{\mu})(\gamma'^{\otimes n}) m_{\mu\nu} Y''(\check{\nu})(\gamma''^{\otimes n}). \tag{4.12}$$

□

4.4 Examples

4.4.1 Higher-order correlation functions. Using the calculations of 4.1, we obtain the following formulas for the tensor product of the first higher-order correlation functions of (H', g', Y'_n) and (H'', g'', Y''_n). To write down the formulas, let $\sum_{a'b'} \Delta_{a'} g'^{a'b'} \Delta_{b'}$ and $\sum_{a''b''} \Delta_{a''} g''^{a''b''} \Delta_{b''}$ be the Casimir elements for g and g'.

$n = 3$:

$$(Y'_3 \otimes Y''_3)(\gamma'_1 \otimes \gamma''_1 \otimes \gamma'_2 \otimes \gamma''_2 \otimes \gamma'_3 \otimes \gamma''_3) = Y'_3(\gamma'_1 \otimes \gamma'_2 \otimes \gamma'_3) Y'_3(\gamma''_1 \otimes \gamma''_2 \otimes \gamma''_3). \tag{4.13}$$

$n = 4$:

$$\begin{aligned}
&(Y'_4 \otimes Y''_4)(\gamma'_1 \otimes \gamma''_1 \otimes \cdots \otimes \gamma'_4 \otimes \gamma''_4) \\
&= Y'_4(\gamma'_1 \otimes \cdots \otimes \gamma'_4) \sum_{a'',b''} Y''_3(\gamma''_1 \otimes \gamma''_2 \otimes \Delta_{a''}) g''^{a''b''} Y''_3(\Delta''_{b''} \otimes \gamma''_3 \otimes \gamma''_4) \\
&\quad + \sum_{a',b'} Y'_3(\gamma'_1 \otimes \gamma'_2 \otimes \Delta_{a'}) g'^{a'b'} Y'_3(\Delta'_{b'} \otimes \gamma'_3 \otimes \gamma'_4) Y''_4(\gamma''_1 \otimes \cdots \otimes \gamma''_4).
\end{aligned} \tag{4.14}$$

$n = 5$:

$$\begin{aligned}
&(Y'_5 \otimes Y''_5)(\gamma'_1 \otimes \gamma''_1 \otimes \ldots \otimes \gamma'_5 \otimes \gamma''_5) \\
&= Y'_5(\gamma'_1 \otimes \ldots \otimes \gamma'_5) \sum_{a'',b'',c'',d''} Y''_3(\gamma''_1 \otimes \gamma''_2 \otimes \Delta_{a''}) g''^{a''b''} Y''_3(\Delta''_{b''} \otimes \gamma''_3 \otimes \Delta''_{c''}) \\
&\quad \times g''^{c''d''} Y''_3(\Delta''_{d''} \otimes \gamma''_4 \otimes \gamma''_5) \\
&\quad - \sum_{l \in \{1,2,3,4\}} \sum_{\substack{a',b' \\ a'',b''}} Y'_4 \left(\bigotimes_{i \in \{1,2,3,4\} \setminus \{l\}} \gamma'_i \otimes \Delta'_{a'} \right) g'^{a'b'} Y'_3(\Delta'_{b'} \otimes \gamma'_l \otimes \gamma'_5) \\
&\quad \times Y''_4 \left(\bigotimes_{i \in \{1,2,3,4\} \setminus \{l\}} \gamma''_i \otimes \Delta''_{a''} \right) g''^{a''b''} Y''_3(\Delta_{b''} \otimes \gamma''_l \otimes \gamma''_5) \\
&\quad + \sum_{\{1,2\} \subseteq I \subset \{1,2,3,4\}} \sum_{a',b'} Y'_{|I|+1} \left(\bigotimes_{i \in I} \gamma'_i \otimes \Delta'_{a'} \right) g'^{a'b'} Y'_{6-|I|} \left(\Delta'_{b'} \bigotimes_{j \in \{1,2,3,4\} \setminus I} \gamma'_j \otimes \gamma'_5 \right) \\
&\quad \times \sum_{\{1,2\} \subseteq J \subset \{1,2,3,4\}} \sum_{a'',b''} Y''_{|J|+1} \left(\bigotimes_{i \in J} \gamma''_i \otimes \Delta''_{a''} \right) g'^{a''b''} Y''_{6-|I|} \left(\Delta''_{b''} \bigotimes_{j \in \{1,2,3,4\} \setminus J} \gamma''_j \otimes \gamma''_5 \right) \\
&\quad + \sum_{a',b',c',d'} Y'_3(\gamma'_1 \otimes \gamma'_2 \otimes \Delta_{a'}) g'^{a'b'} Y'_3(\Delta'_{b'} \otimes \gamma'_3 \otimes \Delta'_{c'}) g'^{c'd'} Y'_3(\Delta'_{d'} \otimes \gamma'_4 \otimes \gamma'_5) \\
&\quad \times Y''_5(\gamma''_1 \otimes \ldots \otimes \gamma''_5).
\end{aligned} \tag{4.15}$$

4.4.2 Higher-order multiplications. By applying Corollary 4.2.5, using the notation $(\gamma_1, \ldots, \gamma_n)$ for $\circ_n(\gamma_1 \otimes \ldots \otimes \gamma_n)$, we find the following.

$n = 2$:

$$(\gamma'_1 \otimes \gamma''_1, \gamma'_2 \otimes \gamma''_2) = (\gamma'_1, \gamma'_2) \otimes (\gamma''_1, \gamma''_2). \tag{4.16}$$

$n = 3$:

$$\begin{aligned}
&(\gamma'_1 \otimes \gamma''_1, \gamma'_2 \otimes \gamma''_2, \gamma'_3 \otimes \gamma''_3) \\
&= (\gamma'_1, \gamma'_2, \gamma'_3) \otimes ((\gamma''_1, \gamma''_2), \gamma''_3) + ((\gamma'_1, \gamma'_2), \gamma'_3) \otimes (\gamma''_1, \gamma''_2, \gamma''_3).
\end{aligned} \tag{4.17}$$

$n = 4$:

$$
\begin{aligned}
&(\gamma_1' \otimes \gamma_1'', \ldots, \gamma_4' \otimes \gamma_4'') \\
&\quad = (\gamma_1', \ldots \gamma_4') \otimes (((\gamma_1'', \gamma_2''), \gamma_3''), \gamma_4'') + (((\gamma_1', \gamma_2'), \gamma_3'), \gamma_4') \otimes (\gamma_1'', \ldots \gamma_4'') \\
&\qquad - \sum_{\{i,j,k\} \amalg \{l\} = \{1,2,3,4\}} ((\gamma_i', \gamma_j', \gamma_k'), \gamma_l') \otimes ((\gamma_i'', \gamma_j'', \gamma_k''), \gamma_l'') \\
&\qquad + \sum_{\{1,2\} \subseteq I \subset \{1,2,3,4\}} ((\gamma_I'), \gamma'_{\{1,2,3,4\} \setminus I}) \otimes \sum_{\{1,2\} \subseteq J \subset \{1,2,3,4\}} ((\gamma_J''), \gamma''_{\{1,2,3,4\} \setminus J}),
\end{aligned}
\tag{4.18}
$$

where in the last expression we have used the abbreviation (γ_I) to denote $\circ_{|I|}(\otimes_{i \in I} \gamma_i)$.

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