Around Feynman categories

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References

References

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2. with B. Ward and J. Zuniga. *The odd origin of Gerstenhaber brackets, Batalin–Vilkovisky operators and master equations*. Journal of Math. Phys. 56, 103504 (2015).
3. with I. Galvez–Carrillo and A. Tonks. *Three Hopf algebras and their operadic and categorical background*. Preprint.
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Goals

Main Objective

Provide a *lingua universalis* for operations and relations in order to understand their structure.

Internal Applications

1. Realize universal constructions (e.g. free, push–forward, pull–back, plus construction, decorated).
2. Construct universal transforms. (e.g. bar,co–bar) and model category structure.
3. Distill universal operations in order to understand their origin (e.g. Lie brackets, BV operatos, Master equations).
4. Construct secondary objects, (e.g. Lie algebras, Hopf algebras).

Applications

Applications

- Find out information of objects with operations. E.g. Gromov-Witten invariants, String Topology, etc.
- Find out where certain algebra structures come from naturally: pre-Lie, BV, ...
- Find out origin and meaning of (quantum) master equations
- Find background for certain types of Hopf algebras.
- Find formulation for TFTs.
- Transfer to other areas such as algebraic geometry, algebraic topology, mathematical physics, number theory.

Plan

Warm up I

Operations and relations for Associative Algebras

- Data: An object A and a multiplication $\mu : A \otimes A \to A$
- An associativity equation $(ab)c = a(bc)$.
- Think of μ as a 2-linear map. Let $\circ_1$ and $\circ_2$ be substitution in the 1st resp. 2nd variable: The associativity becomes

$$\boxed{\mu \circ_1 \mu = \mu \circ_2 \mu : A \otimes A \otimes A \to A}.$$

$$\mu \circ_1 \mu(a, b, c) = \mu(\mu(a, b), c) = (ab)c$$
$$\mu \circ_2 \mu(a, b, c) = \mu(a, \mu(b, c)) = a(bc)$$

- We get n–linear functions by iterating μ: $a_1 \otimes \cdots \otimes a_n \to a_1 \ldots a_n$.
- There is a permutation action $\tau\mu(a, b) = \mu \circ \tau(a, b) = ba$
- This give a permutation action on the iterates of μ. It is a free action there and there are $n!$ n–linear morphisms generated by μ and the transposition.

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Warm up II

Categorical formulation for representations of a group G.

- $\underline{G}$ the category with one object $*$ and morphism set G.
- $f \circ g := fg$.
- This is associative ✓
- Inverses are an extra structure $\Rightarrow \underline{G}$ is a groupoid.
- A representation is a functor ρ from $\underline{G}$ to $\mathcal{V}ect$.
- $\rho(*) = V$, $\rho(g) \in Aut(V)$
- Induction and restriction now are pull–back and push–forward (Lan) along functors $\underline{H} \to \underline{G}$.

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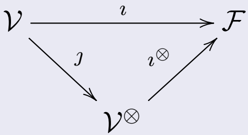
Feynman categories

Data

1. $\mathcal{V}$ a groupoid
2. $\mathcal{F}$ a symmetric monoidal category
3. $\imath : \mathcal{V} \to \mathcal{F}$ a functor.

Notation

$\mathcal{V}^{\otimes}$ the free symmetric category on $\mathcal{V}$ (words in $\mathcal{V}$).



Feynman category

Definition

Such a triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is called a Feynman category if

1. $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
2. $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal categories $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.
3. For any $* \in \mathcal{V}$, $(\mathcal{F} \downarrow *)$ is essentially small.

Hereditary condition (ii)

1. In particular, fix $\phi : X \to X'$ and fix $X' \simeq \bigotimes_{v\in I} \imath(*_v)$: there are $X_v \in \mathcal{F}$, and $\phi_v \in Hom(X_v, *_v)$ s.t. the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow \simeq & & \downarrow \simeq \\ \bigotimes_{v\in I} X_v & \xrightarrow{\bigotimes_{v\in I}\phi_v} & \bigotimes_{v\in I} \imath(*_v) \end{array} \tag{1}$$

2. For any two such decompositions $\bigotimes_{v\in I} \phi_v$ and $\bigotimes_{v'\in I'} \phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ s.t. $P_\psi^{-1} \circ \bigotimes_v \sigma_v \circ \phi_v = \bigotimes \phi'_{v'}$ where P_ψ is the permutation corresponding to ψ.
3. These are the only isomorphisms between morphisms.

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3. These are the only isomorphisms between morphisms.

Example 1

$\mathcal{F} = Sur$, $\mathcal{V} = \mathbb{I}$

- Sur be the category of finite sets and surjection with $\amalg$ as monoidal structure
- $\mathbb{I}$ be the trivial category with one object $*$ and one morphism id_*.
- $\mathbb{I}^{\otimes}$ is equivalent to the category with objects $\bar{n} \in \mathbb{N}_0$ and $Hom(\bar{n}, \bar{n}) \simeq \mathbb{S}_n$, where we think $\bar{n} = \{1, \dots, n\} = \{1\} \amalg \cdots \amalg \{1\}$, $1 = \imath(*)$.
- $\mathbb{I}^{\otimes} \simeq Iso(Sur)$.

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- $\mathbb{I}^{\otimes} \simeq Iso(Sur)$. ✓
- $T \simeq \{1, \dots, n\}$.

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ \downarrow \simeq & & \downarrow \simeq \\ \amalg_{i=1}^{|T|} f^{-1}(i) & \xrightarrow{\amalg f|_{f^{-1}(i)}} & \amalg_{i=1}^{|T|} \imath(*) \end{array}$$

$\mathcal{O}ps$ and $\mathcal{M}ods$

Definition

Fix a symmetric monoidal category $\mathcal{C}$ and $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ a Feynman category.

- Consider the category of strong symmetric monoidal functors $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F}, \mathcal{C})$ which we will call $\mathcal{F}$–ops in $\mathcal{C}$
- $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V}, \mathcal{C})$ will be called $\mathcal{V}$-modules in $\mathcal{C}$ with elements being called a $\mathcal{V}$–mod in $\mathcal{C}$.

Theorem

The forgetful functor $G : \mathcal{O}ps \to \mathcal{M}ods$ has a right adjoint F (free functor) and this adjunction is monadic.

Other versions

Enriched version

We can consider Feynman categories and target categories enriched over another monoidal category, such as $\mathcal{T}op$, $\mathcal{A}b$ or $dg\mathcal{V}ect$.

Theorem

The category of Feynman categories with trivial $\mathcal{V}$ enriched over $\mathcal{E}$ is equivalent to the category of operads (with the only iso in $\mathcal{O}(1)$ being the identity) in $\mathcal{E}$ with the correspondence given by $O(n) :=: Hom(\bar{n}, \bar{1})$. The $\mathcal{O}ps$ are now algebras over the underlying operad.

Examples of this simple stucture

Examples

Operad of surjections (corollas), non–symmetric version ordered surjections (planar corollas), simplices (Joyal dual). Operad of leaf labelled rooted trees (gluing at leaves), non–symmetric version planar rooted trees.

More

Other examples are twisted modular operads, non–sigma versions, the simplicial category, crossed simplicial groups, FI–algebras.

More constructions +–construction

In general

there is a "+" construction, like for polynomial monads, that produces a new Feynman category out of an old one.
The main theorem is that enrichments of $\mathcal{F}$ are basically in 1–1 correspondence with $\mathcal{F}^+$–$\mathcal{O}ps$.

Examples

$\mathcal{F}^+_{modular} = \mathcal{F}_{hyper}$ and twisted modular operads as algebras over the twisted triple. $\mathcal{F}^+_{surj} = \mathcal{F}_{operads}$, $\mathcal{F}^+_{monoid} = \mathcal{F}_{surj}$. (Slightly more complicated)

Algebras

The $\mathcal{F}^+$–$\mathcal{O}ps$ then give enrichments for $\mathcal{F}$ and given such an $\mathcal{O} \in \mathcal{F}^+$-$\mathcal{O}ps$ the $\mathcal{F}_{\mathcal{O}}$-$\mathcal{O}ps$ are (by definition) algebras over $\mathcal{O}$.

More constructions: $\mathcal{F}_{dec\mathcal{O}}$ joint w/ Jason Lucas

In general

Given an $\mathcal{O} \in \mathcal{F}\text{–}\mathcal{O}ps$, then there is a Feynman category $\mathcal{F}_{dec\mathcal{O}}$ which is indexed over $\mathcal{F}$. It objects are pairs $(X, dec \in \mathcal{O}(X))$ and $Hom_{\mathcal{F}_{dec\mathcal{O}}}((X, dec), (X', dec'))$ is the set of $\phi : X \to X'$, s.t. $\mathcal{O}(\phi) : dec \to dec'$.

Examples

Non–sigma operads, cyclic non–Sigma operads, non–Sigma modular operads.

Here $\mathcal{O}$ is $\mathcal{A}ssoc$, $\mathcal{C}y\mathcal{A}ssoc$, $\mathcal{M}od\mathcal{C}yc\mathcal{A}ssoc$.

There is a general theorem saying that the decoration by the push–forward exists and how such push–forwards factor. This recovers e.g. that the modular envelope of $\mathcal{C}y\mathcal{A}ssoc$ factors through non–Sigma modular operads (Result of Markl).

More $\mathcal{F}_{dec\mathcal{O}}$

Further applications

Further applications will be

1. the Westerland–Wahl A_∞ moduli space operations generalizing those of R.K. . *Moduli space actions on Hochschild Cochains*
2. The Stolz–Teichner setup for twisted field theories.
3. Kontsevich's graph comlexes.

Example 2

The Borisov-Manin category of graphs.

1. A graph Γ is a tuple $(F, V, \partial, \imath)$ of flags F, vertices V, incidence $\partial : F \to V$ and flag gluing $\imath : F^{\circlearrowleft}$. $\imath^2 = id$. We either glue two half-edges or keep a tail.
2. A graph morphism $\phi : \Gamma \to \Gamma'$ is a triple $(\phi_V, \phi^F, \imath_\phi)$, where $\phi_V : V \to V'$ is a surjection on vertices, $\phi^F : F' \to F$ is an injection and $\imath_\phi : F \setminus \phi^F(F')^{\circlearrowleft}$ a pairing (ghost edges).
3. A graph morphism from a collection of corollas Γ to a corolla $*$ has a ghost graph $\mathbb{\Gamma} = (V_\Gamma, F_\Gamma, \imath_\phi)$

$\mathfrak{F} = (\mathcal{A}gg, \mathcal{C}rl, \imath)$

$\mathcal{A}gg$ the full subcategory whose objects are aggregates of corollas.
$\mathcal{C}rl$ the category of corollas with isomorphisms.

Examples

Roughly (in the connected case and up to isomorphism)

The source of a morphism are the vertices of the ghost graph Γ and the target is the vertex obtained from Γ obtained by contracting all edges. If Γ is not connected, one also needs to merge vertices according to ϕ_V.

Composition corresponds to insertion of ghost graphs into vertices.

$$X \xrightarrow{\phi_2} Y \xrightarrow{\phi_1} *, \qquad X \xrightarrow{\phi_0} *$$

up to isomorphisms (if Γ_0, Γ_1 are connected) corresponds to inserting Γ_v into $*_v$ of Γ_1 to obtain Γ_0.

$$\amalg_v \amalg_{w \in V_v} *_w \xrightarrow{\amalg_v \Gamma_v} \amalg_v *_v \xrightarrow{\Gamma_1} *, \qquad \amalg_v \amalg_{w \in V_v} *_w \xrightarrow{\Gamma_0} *$$

Graph Examples

$\mathcal{O}ps$

We can restrict the underlying ghost graphs of maps to corollas to obtain several Feynman categories. The $\mathcal{O}ps$ will then yield types of operads or operad like objects.

Types of operads and graphs

$\mathcal{O}ps$	Graphs
Operads	rooted trees
Cyclic operads	trees
Modular operads	connected graphs (add genus marking)
PROPs	directed graphs (and input output marking)
NC modular operad	graphs (and genus marking)
Broadhurst-Connes -Kreimer	1-PI graphs
...	...

Physics connection

Feynman graphs

are the morphisms in the Feynman category. The possible vertices are the objects.

S–matrix

The external lines are given by the target of the morphism. The comma/slice category over a given target is then a graphical version of the S–matrix.

Correlation functions

These are given by the functors $\mathcal{O}$.

Open Questions

What corresponds to algebras and plus construction, functors. Possible answers via Rota–Baxter (in progress).

Universal constructions: What we can do:

1. Push–forwards and pull–backs along functors between Feynman categories.
   THINK INDUCTION/RESTRICTION/EXTENSION BY 0.
2. Co(bar) transforms and resolutions. Think (co)bar transformation/resolution for algebras as well as Feynman transforms and master equations.
   NB: THIS NEEDS MODEL CATEGORY THEORY WHICH WE PROVIDE
3. Universal operations. Lie–brackets, BV etc.
4. Hopf algebra structures (joint with I. Gálvez–Carrillo and A. Tonks).
   This includes Connes–Kreimers Renormalization Hopf algebra, Goncharov's Hopf algebra for multi–zetas (polylogs) and Baues' double cobar Hopf algebra.

Hopf algebras

Basic structures

Assume $\mathcal{F}$ is decomposition finite. Consider $\mathcal{B} = Hom(Mor(\mathcal{F}), \mathbb{Z})$. Let μ be the tensor product with unit $id_{\mathbb{I}}$.

$$\Delta(\phi) = \sum_{(\phi_0,\phi_1):\phi=\phi_1\circ\phi_0} \phi_0 \otimes \phi_1$$

and $\epsilon(\phi) = 1$ if $\phi = id_X$ and 0 else.

Theorem (Galvez-Carrillo, K , Tonks)

$\mathcal{B}$ together with the structures above is a bi–algebra. Under certain mild assumptions, a canonical quotient is a Hopf algebra

Examples

In this fashion, we can reproduce Connes–Kreimer's Hopf algebra, the Hopf algebras of Goncharov and a Hopf algebra of Baues that he defined for double loop spaces. This is a non–commutative graded version. There is a three-fold hierarchy. A non-commutative version, a commutative version and an "amputated" version.

Details I

Non–commutative version

Use Feynman categories whose underlying tensor structure is only monoidal (not symmetric). $\mathcal{V}^{\otimes}$ is the the free monoidal category.

Key Lemma

The bi–algebra equation holds due to the hereditary condition.

Unit

The unit of the co–algebra is given by $1 = id_{\emptyset}$, i.e. the identity morphism of the empty word.

Quotient by Isomorphisms

If there are any isomorphism in $\mathcal{V}$ then $\mathcal{F}$ one can quotient out the co–ideal defined by equiv. rel. generated by isomorphism diagrams of type (1). The result is called almost connected. (This is automatic if there are no isomorphism except for identities in $\mathcal{V}$).

Details II

Theorem

For the almost connected version let $\mathscr{I}$ be the ideal generated by $1 - id_X$. Then this is a co–ideal and the quotient $\mathcal{B}/\mathscr{I}$ is a connected Hopf algebra and hence a bi–algebra. Goncharov and Baues (shifted co–bar version), planar Connes-Kreimer with external lines (both tree and 1-PI).

Commutative version

For the commutative version, one looks at the co–invariants in the symmetric case. Non–planar Connes–Kreimer with external lines.

Amputated version

For this one needs a semi–cosimplicial structure, i.e. one must be able to forget external legs coherently. Then there is a colimit, in which all the external legs can be forgotten. Connes–Kreimer without external legs (e.g. the original tree version).

Details III

Generalization: co–operad with multiplication

In a sense the above examples were free. One can look at a more general setting where this is not the case. The length of an object is the replaced by a depth filtration. The algebras are then deformations of their associated graded. Main example (cooperad with multiplication) generalizes enrichment of F_{surj}.

Grading/Filtration

Co-operad with multiplication	operad degree $-$ depth
Amputated version	co-radical degree $+$ depth

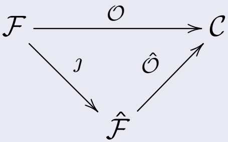
q deformation - infinitesimal version

Taking a slightly different quotient, one can get a non–unital, co–unital bi–algebra and a q–filtration. Sending $q \to 1$ recovers $\mathcal{H}$.

Universal operations

Cocompletion

Let $\hat{\mathcal{F}}$ be the cocompletion of $\mathcal{F}$. This is monoidal with Day convolution $\circledast$. If $\mathcal{C}$ is cocomplete, and $\mathcal{O} \in \mathcal{O}ps$ factors.



Theorem

Let $\mathbb{I} := \mathrm{colim}_{\mathcal{V}} \jmath \circ \imath \in \hat{\mathcal{F}}$ *and let* $\mathcal{F}_{\mathcal{V}}$ *the symmetric monoidal subcategory generated by* $\mathbb{I}$. *Then* $\mathfrak{F}_{\mathcal{V}} := (\mathcal{F}_{\mathcal{V}}, \mathbb{I}, \imath_{\mathcal{V}})$ *is a Feynman category. (This gives an underlying operad of universal operations).*

Examples

Operads

$\mathfrak{O}$ the Feynman category for operads, $\mathcal{C} = dg\mathcal{V}ect$.

- Then $\hat{\mathcal{O}}(\mathbb{I}) = \bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$ and the Feynman category is (weakly) generated by $\circ := [\sum \circ_i]$. (This is a two line calculation).
- This gives rise to the Lie bracket by using the anti–commutator. The operations go back to Gerstenhaber and Kapranov-Manin.
- It lifts to the non-Sigma case i.e. a pre–Lie structure on $\bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$.

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Universal Operations

$\mathfrak{F}$	Feynman category for	$\mathfrak{F}, \mathfrak{F}_{\mathcal{V}}, \mathfrak{F}^{nt}_{\mathcal{V}}$	weakly gen. subcat.
$\mathfrak{O}$	Operads	rooted trees	$\mathfrak{F}_{pre-Lie}$
$\mathfrak{O}^{odd}$	odd operads	rooted trees + orientation of set of edges	odd pre-Lie
$\mathfrak{O}^{pl}$	non-Sigma operads	planar rooted trees	all $\circ_i$ operations
$\mathfrak{O}_{mult}$	Operads with mult.	b/w rooted trees	pre-Lie + mult.
$\mathfrak{C}$	cyclic operads	trees	commutative mult.
$\mathfrak{C}^{odd}$	odd cyclic operads	trees + orientation of set of edges	odd Lie
$\mathfrak{M}^{odd}$	$\mathfrak{K}$–modular	connected + orientation on set of edges	odd dg Lie
$\mathfrak{M}^{nc,odd}$	nc $\mathfrak{K}$-modular	orientation on set of edges	BV

Table: Here $\mathfrak{F}_{\mathcal{V}}$ and $\mathfrak{F}^{nt}_{\mathcal{V}}$ are given as $\mathcal{F}_{\mathcal{O}}$ for the insertion operad. The former for the type of graph with unlabelled tails and the latter for the version with no tails.

Examples

Odd/anti–cyclic Operad

The universal operations are (weakly) generated by a Lie bracket. $[\ ,\ ] := [\sum_{st} \circ_{st}]$, (see [KWZ]). This actually lifts to cyclic coinvariants (non–sigma cyclic operads).
Specific examples:

- $End(V)$ for a symplectic vector space is anti–cyclic.
- Any tensor product: $\mathcal{O} \otimes \mathcal{P}(n) := (\mathcal{O}(n) \otimes \mathcal{P})(n)$ with $\mathcal{O}$ cyclic and $\mathcal{P}$ anti–cyclic is anti–cyclic.

Three geometries (Kotsevich, Conant-Vogtmann

Fix V^n n–dim symplectic $V^n \to V^{n+1}$. For each n get Lie algebras
(1) $Comm \otimes End(V^n)$ (2) $Lie \otimes End(V^n)$ (3) $Assoc \otimes End(V^n)$
Take the limit as $n \to \infty$.

Odd versions

Odd versions

Given a well-behaved presentation of a Feynman category (generators+relations for the morphisms) we can define an odd version which is enriched over $\mathcal{A}b$.

Odd Feynman categories over graphs

In the case of underlying graphs for morphisms, odd usually means that edges get degree 1, that is we use a Kozsul sign with that degree.

(Co)Bar Feynman transform

Algebra case

- C associative co–algebra. $\Omega C := Free_{alg}(\Sigma^{-1}\bar{C})+$ differential coming from co–algebra structure
- A associative algebra. $BA = T\Sigma^{-1}\bar{A}$ + co–differential from algebra structure
- ΩBA is a free resolution.
- A say finite dim or graded with finite dim pieces $\check{A}$ its dual. $FA := \Omega\check{A}$ + differential from multiplication. FFA a resolution.

We can define the same transformation for elements of $\mathcal{O}ps$ for well–presented Feynman categories

- The result of a Feynman transform is an *op* over the odd version of the Feynman category
- For the freeness we need model structures, which we give.

Master equations

The Feynman transform is quasi–free. An algebra over $F\mathcal{O}$ is dg–if and only if it satisfies the following master equation.

Name of $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$	Algebraic Structure of $F\mathcal{O}$	Master Equation (ME)
operad ,[?]	odd pre-Lie	$d(-) + - \circ - = 0$
cyclic operad [?]	odd Lie	$d(-) + \frac{1}{2}[-,-] = 0$
modular operad [?]	odd Lie + Δ	$d(-) + \frac{1}{2}[-,-] + \Delta(-) = 0$
properad [?]	odd pre-Lie	$d(-) + - \circ - = 0$
wheeled properad [?]	odd pre-Lie + Δ	$d(-) + - \circ - + \Delta(-) = 0$
wheeled prop [?]	dgBV	$d(-) + \frac{1}{2}[-,-] + \Delta(-) = 0$

Geometry and moduli spaces

Modular Operads

The typical topological example are $\bar{M}_{gn}$. These give rise to chain and homology operads.

- Gromov–Witten invariants make $H^*(V)$ and algebra over $H_*(\bar{M}_{g,n})$

Odd Modular

The canonical geometry is given by $\bar{M}^{KSV}$ which are real blowups of $\bar{M}_{gn}$ along the boundary divisors.

- We get 1-parameter gluings parameterized by S^1. Taking the full S^1 family on chains or homology gives us the structure of an odd modular operad.
- Going back to Sen and Zwiebach, a viable string field theory action S is a solution of the quantum master equation.

Next steps

- Formalize the dual pictures of primitive elements and $+$ construction as well as universal operations and PBW.
- Connect to Tannakian categories. E.g. find out the role of fibre functors or special large/small object. (Idea: special properties of $\mathcal{H}_{CK}$).
- Connect to Rota–Baxer, Dynkin-operators, B^+-operators (we can do this part) etc.
- Construct Feynman category for the open/closed version of Homological Mirror symmetry.
- Find action of Grothendieck-Teichmüller group (GT).
- ...

The end

Thank you!