Algebraic operations in geometry, topology and physics.

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Plan

References

Main papers for 2 and 3

1. with B. Ward. *Feynman categories*. Arxiv 1312.1269
2. with B. Ward and J. Zuniga. *The odd origin of Gerstenhaber, BV and the master equation* arXiv:1208.3266
3. with I. Galvez–Carrillo and A. Tonks *Three Hopf algebras and their operadic and categorical background*. Preprint.

Basic idea

Basic idea

Gain insight into the properties of difficult objects, e.g. geometrical objects, by using simpler data and operations on it.

Hierarchy of data

1. Numbers
2. Vector spaces $\rightsquigarrow$ numbers via dimension
3. Algebras
4. k-linear objects. e.g. graded, differential, modules ...
5. Families of these (deformations)
6. Many operations
7. Categories

Euler Characteristic

Euler

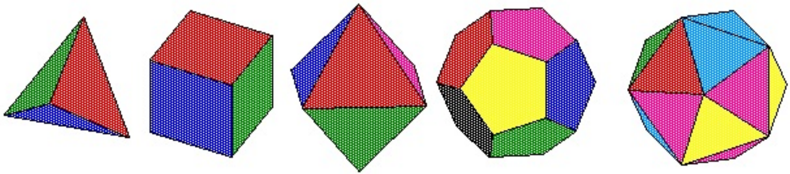
Take any convex polytope in 3d and compute.
|Vertices|-|Edges|+|Faces|. The answer is always 2.

Lhulier

|Vertices|-|Edges|+|Faces|= 2 -2| cavities|

Euler Characteristic

The five Platonic solids



The Tetrahedron The Cube The Octahedron The Dodecahedron The Icosahedron

The five regular solids discovered by the Ancient Greek mathematicians are:

The **Tetrahedron**:	4 vertices	6 edges	4 faces	each with 3 sides
The **Cube**:	8 vertices	12 edges	6 faces	each with 4 sides
The **Octahedron**:	6 vertices	12 edges	8 faces	each with 3 sides
The **Dodecahedron**:	20 vertices	30 edges	12 faces	each with 5 sides
The **Icosahedron**:	12 vertices	30 edges	20 faces	each with 3 sides

The solids are regular because the same number of sides meet at the same angles at each vertex and identical polygons meet at the same angles at each edge.
These five are the only possible regular polyhedra.

Modern version

Classification theorem

For every 2 dimensional closed orientable manifold (surface) S, the Euler characteristic (use any triangulation and a triangulation exists) $\chi(S) = 2 - 2g$ where χ is the Euler characteristic, and g uniquely fixes the surface up to homeomorphism.

Betti, Poincaré

$\chi = b_0 - b_1 + b_2$. The b_i are dimensions of vector spaces.

Finer invariants: Abelian groups

To include the Klein bottle and non–orientable surfaces, one has to actually use H_1, the first homology group. The first Betti number is the rank of the free part. One needs the torsion! E.g. $H_1(Klein) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

How much information do we get?

In 3d

Question (Poincaré): Is it enough to know that for a sphere H_1 is 0 and that it is a closed 3–manifold to uniquely characterize it.
Answer (Poincaré): No. Counterexample: Poincaré homology sphere $\rightsquigarrow$ many new invariants e.g. Casson.

Fundamental group

We can look at the possibly non–Abelian fundamental group given by all loops at a given point modulo homotopy (continuous deformation)
Question (Poincaré): Is it enough to know that for a sphere π_1 is trivial and that it is a closed 3–manifold to uniquely characterize it?
Answer (Perelman): Using Ricci–flow of Hamilton: Yes!

Other ideas

Algebra

There is a dual notion to homology called cohomology. Cohomology is a a ring. That is we can multiply objects

Forms/deRham algebra

One familiar version is given by deRham forms (what you integrate over) and their wedge product

$$\omega_1 \wedge \omega_2$$

Remarks

1. A special case is the cross product in $\mathbb{R}^3$.
2. $\omega_p \wedge \omega'_q = (-1)^{pq}\omega'_q \wedge \omega_p$ like $dx \wedge dy = -dy \wedge dx$
   (This is the sign in the cross product and also the orientation when integrating over surfaces)
3. Forms are vector spaces over $\mathbb{R}$, so no torsion.
4. The algebra is associative, unital, (super) commutative, graded, differential.
5. There is a version over $\mathbb{Z}$: singular cohomology which is a ring.

Algebra structure as a finer invariant

Additive structure

Notice that the cohomology of $\mathbb{CP}^2$ and that of $S^2 \vee S^4$ are the same.

$$H_0 = \mathbb{Z}, H_1 = 0, H_2 = \mathbb{Z}, H_3 = 0, H_4 = \mathbb{Z}$$

Ring structure

As a ring $H^*(\mathbb{CP}^2) = \mathbb{Z}[t]/(t^3)$ $deg(t) = 2$ and
$H^*(S^2 \vee S^4) = k[s,t]/(t^2, s^2, st)$ with $deg(s) = 4, deg(t) = 2$.

Theorem (Sullivan/Quillen, rational homotopy theory)

Over $\mathbb{Q}$ homotopy types are characterized by commutative differential graded algebras.

Chain level, more operations

Chain level, singular chains

There is a similar story over $\mathbb{Z}$ called singular chains $S_*(X)$ and dually singular co-chains $S^*(X)$. The multiplication on $S^*(X)$ is called $\cup = \cup_0$. It is again, graded, differential, unital, associative, but *not* strictly commutative.

E_∞

$$c \cup c' \mp c' \cup c = d(c \cup_1 c')$$
$$c \cup_i c' \mp c' \cup c = d(c \cup_{i+1} c')$$

These operations are part of a set of operations, which make the cochains into an E_∞-algebra.

Theorem (Mandell)

The E_∞ algebra structure on co–chains characterizes homotopy types of simply connected manifolds over $\mathbb{Z}$.

Based Loop spaces

Pontrjagin product

The based loop space $\Omega X = Hom_{pointed}(S_1, X)$. It has a non–associative product called

$$\mu_2 : \Omega X \times \Omega X \to \Omega X$$

simply do one loop after the other *and* re-parameterize

A_∞ structure

Due to the re-parameterization, the product is not associative. It is homotopy associative. There are higher products parameterized by Stasheff polytopes.

Theorem (Stasheff)

A connected space is homotopy equivalent to a loop space if it admits all the operations above.

A_∞

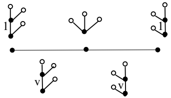
Algebraic version

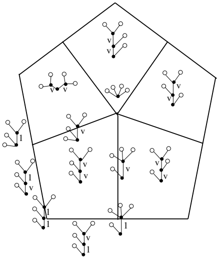
A dg–vector spaces together with $\mu_i : A^{\otimes n} \to A$ satisfying equations e.g.

$$\mu_2(\mu_2(ab), c) - \mu_2(a, \mu_2(bc)) = d\mu_3(a, b, c)$$

and so on.

Associahera/Stasheff polytopes





(cubical decomposition from paper with R.Schwell on A_∞–Deligne conjecture)

Free Loop spaces

Chas–Sullivan product

The free loop space is $LX = Hom(S^1, X)$. If X is a simply connected compact manifold of dimension d: H_{*-d} has the structure of a BV algebra. This means it has a multiplication and an (odd) Lie bracket that is compatible (Poisson). Moreover this bracket is induced by an order two differential operator Δ.

Several approaches

Cohen-Jones: Thom spectrum, Felix-Thomas: rational homotopy theory, Merkulov: Iterated integrals. Also algebraic version: K, Tradler-Zeinalian.

Theorem (K., cyclic Deligne conjecture)

For a Frobenius algebra A, $CH^*(A, A)$ admits the action of a chain model of the framed little discs operad.

Agruments so far

1. Besides having objects (numbers, groups, spaces) as invariants, considering operations on these objects help to make them more powerful.
   (Aside: The fact that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only division algebras follows from K-theory together with Adams operations. Hopf invariant one.)
2. Sometimes there are many such operations together with relations on them.

A last example: Operations from Physics/Geometric background

Correlation functions

In quantum field theory, fields form a vector spaces. A main goals it to compute correlation functions n–point functions

$$< \phi_1, \dots, \phi_n >$$

which are multilinear. The result is usually given in terms of Feynman integrals. One calculates via reduction.

Mathematical example: Cohomological Field theory [Kontsevich-Manin]

A CohFT on a vector space V with a non–degenerate quadratic form is given by multilinear maps taking values in $H^*(\bar{M}_{g,n})$ $I_{g,n} : V^{\otimes n} \to H^*(\bar{M}_{g,n})$ satisfying the equations of the cohomology classes themselves.
Short version: V is an algebra over the modular operad $H_*(\bar{M}_{g,n})$.

Gromov–Witten Invariants

Theorem (Kontsevich-Manin, Behrend-Fantechi, Ruan-Tian, ...)

Gromov–Witten invariants yield a CohFT.

Use

Geometric information. Like enumerative problems. Mirror Symmetry. CohFT gives relations that can be used to create recursion formulas. Write into power series, get PDEs. Example WDVV equation

$\forall a, b, c, d : \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = (-1)^{\tilde{a}(\tilde{b}+\tilde{c})} \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad}$

where $\Phi_{abc} = \partial_a \partial_b \partial_c \Phi$, g^{ij} is the inverse metric

Theorem (Kontsevich-Manin-K, Behrend,K)

The diagonal class yields a quantum Künneth formula.

So far

1. Find out about geometry using operations.
   - Products.
   - Module structures.
   - Coherent systems of higher order operations.
2. The operations are coded by new objects, e.g. operads.
3. Sometimes these are given by spaces/geometries again.
   - E_∞ generated by S^∞ (hemispherical decomposition)
   - A_∞ Stasheff Associahedra
   - G_∞ Cycloherda and Associaherdra.
   - $\bar{M}_{g,n}$
4. Recursions/calculational tools encoded by relations.
   - Cohomological relations.
   - Boundary relations.
   - Graph type, e.g. Whitehead moves.

Feynman categories: Goals

There are two main goals

1. Provide a *lingua universalis* for operations and relations which includes all known such gadgets as examples.
2. Do universal constructions in general.

Applications

Find out information of objects with operations. E.g. Gromov-Witten invariants, String Topology, etc.

Warm up I

Operations and relations for Associative Algebras

- Data: An object A and a multiplication $\mu : A \otimes A \to A$
- An associativity equation $(ab)c = a(bc)$.
- Think of μ as a 2-linear map. Let $\circ_1$ and $\circ_2$ be substitution in the 1st resp. 2nd variable: The associativity becomes

$$\boxed{\mu \circ_1 \mu = \mu \circ_2 \mu : A \otimes A \otimes A \to A}.$$

$$\mu \circ_1 \mu(a, b, c) = \mu(\mu(a, b), c) = (ab)c$$
$$\mu \circ_2 \mu(a, b, c) = \mu(a, \mu(b, c)) = a(bc)$$

- We get n–linear functions by iterating μ: $a_1 \otimes \cdots \otimes a_n \to a_1 \ldots a_n$.
- There is a permutation action $\tau\mu(a, b) = \mu \circ \tau(a, b) = ba$
- This give a permutation action on the iterates of μ. It is a free action there and there are $n!$ n–linear morphisms generated by μ and the transposition.

Warm up II

Categorical formulation for representations of a group G.

- $\underline{G}$ the category with one object $*$ and morphism set G.
- $f \circ g := fg$.
- This is associative and has a unit.
- Inverses are an extra structure $\Rightarrow$ $\underline{G}$ is a groupoid.
- A representation is a functor ρ from $\underline{G}$ to $\mathcal{V}ect$.
- $\rho(*) = V$, $\rho(g) \in Aut(V)$
- Induction and restriction now are pull–back and push–forward (Lan) along functors $\underline{H} \to \underline{G}$.

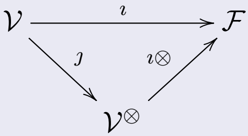
Feynman categories

Data

1. $\mathcal{V}$ a groupoid
2. $\mathcal{F}$ a symmetric monoidal category
3. $\imath : \mathcal{V} \to \mathcal{F}$ a functor.

Notation

$\mathcal{V}^{\otimes}$ the free symmetric category on $\mathcal{V}$ (words in $\mathcal{V}$).



Feynman category

Definition

Such a triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is called a Feynman category if

1. $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
2. $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal categories $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.
3. For any $* \in \mathcal{V}$, $(\mathcal{F} \downarrow *)$ is essentially small.

Hereditary condition (ii)

1. In particular, fix $\phi : X \to X'$ and fix $X' \simeq \bigotimes_{v\in I} \imath(*_v)$: there are $X_v \in \mathcal{F}$, and $\phi_v \in Hom(X_v, *_v)$ s.t. the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad\phi\quad} & X' \\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\ \bigotimes_{v\in I} X_v & \xrightarrow{\bigotimes_{v\in I}\phi_v} & \bigotimes_{v\in I} \imath(*_v) \end{array} \tag{1}$$

2. For any two such decompositions $\bigotimes_{v\in I} \phi_v$ and $\bigotimes_{v'\in I'} \phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ s.t. $P_\psi^{-1} \circ \bigotimes_v \sigma_v \circ \phi_v = \bigotimes \phi'_{v'}$ where P_ψ is the permutation corresponding to ψ.
3. These are the only isomorphisms between morphisms.

Example 1

$\mathcal{F} = Sur$, $\mathcal{V} = \mathbb{I}$

- *Sur* be the category of finite sets and surjection with $\amalg$ as monoidal structure
- $\mathbb{I}$ be the trivial category with one object $*$ and one morphism id_*.
- $\mathbb{I}^{\otimes}$ is equivalent to the category with objects $\bar{n} \in \mathbb{N}_0$ and $Hom(\bar{n}, \bar{n}) \simeq \mathbb{S}_n$, where we think $\bar{n} = \{1, \ldots, n\} = \{1\} \amalg \cdots \amalg \{1\}$, $1 = \imath(*)$.
- $\mathbb{I}^{\otimes} \simeq Iso(Sur)$
- $T \simeq \{1, \ldots, n\}$.

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ \downarrow \simeq & & \downarrow \simeq \\ \amalg_{i=1}^{|T|} f^{-1}(i) & \xrightarrow{\amalg f|_{f^{-1}(i)}} & \amalg_{i=1}^{|T|} \imath(*) \end{array}$$

$\mathcal{O}ps$ and $\mathcal{M}ods$

Definition

Fix a symmetric monoidal category $\mathcal{C}$ and $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ a Feynman category.

- Consider the category of strong symmetric monoidal functors $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F}, \mathcal{C})$ which we will call $\mathcal{F}$–ops in $\mathcal{C}$
- $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V}, \mathcal{C})$ will be called $\mathcal{V}$-modules in $\mathcal{C}$ with elements being called a $\mathcal{V}$–mod in $\mathcal{C}$.

Theorem

The forgetful functor $G : \mathcal{O}ps \to \mathcal{M}ods$ has a right adjoint F (free functor) and this adjunction is monadic.

Examples

$\mathcal{O}ps$

There is a basic Feynman category whose objects are 1-vertex graphs and whose morphisms are graphs with extra sturcture. The way we obtain several Feynman categories. The $\mathcal{O}ps$ will then yield all known types of operads or operad like objects.

Types of operads and graphs

$\mathcal{O}ps$	Graphs
Operads	rooted trees
Cyclic operads	trees
Modular operads	connected graphs (add genus marking)
PROPs	directed graphs (and input output marking)
NC modular operad	graphs (and genus marking)
...	...

Further examples

Enriched version

We can consider Feynman categories and target categories enriched over another monoidal category, such as $\mathcal{T}op$, $\mathcal{A}b$ or $dg\mathcal{V}ect$.

Theorem

The category of Feynman categories with trivial $\mathcal{V}$ enriched over $\mathcal{E}$ is equivalent to the category of operads in $\mathcal{E}$ with the correspondence given by $O(n) :=: Hom(\bar{n}, \bar{1})$. The $\mathcal{O}ps$ are now algebras over the underlying operad.

More

Other examples are twisted modular operads, non–sigma versions, the simplicial category, crossed simplicial groups, FI–algebras.

Universal constructions: What we can do:

1. Push–forwards and pull–backs along functors between Feynman categories.
   THINK INDUCTION/RESTRICTION/EXTENSION BY 0.
2. Co(bar) transforms and resolutions. Think (co)bar transformation/resolution for algebras as well as Feynman transforms and master equations.
   NB: THIS NEEDS MODEL CATEGORY THEORY WHICH WE PROVIDE
3. Universal operations. Lie–brackets, BV etc.
4. Hopf algebra structures (joint with I. Gálvez–Carrillo and A. Tonks).
   This includes Connes–Kreimers Renormalization Hopf algebra, Goncharov's Hopf algebra for multi–zetas (polylogs) and Baues' double cobar Hopf algebra.

Master equations

The Feynman transform is quasi–free. An algebra over $F\mathcal{O}$ is dg–if and only if it satisfies the following master equation.

Name of $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$	Algebraic Structure of $F\mathcal{O}$	Master Equation (ME)
operad ,[GJ94]	odd pre-Lie	$d(-) + - \circ - = 0$
cyclic operad [GK95]	odd Lie	$d(-) + \frac{1}{2}[-,-] = 0$
modular operad [GK98]	odd Lie + Δ	$d(-) + \frac{1}{2}[-,-] + \Delta(-) = 0$
properad [Val07]	odd pre-Lie	$d(-) + - \circ - = 0$
wheeled prop-erad [MMS09]	odd pre-Lie + Δ	$d(-) + - \circ - + \Delta(-) = 0$
wheeled prop [KWZ12]	dgBV	$d(-) + \frac{1}{2}[-,-] + \Delta(-) = 0$

Geometry and moduli spaces

Modular Operads

The typical topological example are $\bar{M}_{gn}$. These give rise to chain and homology operads.

- Gromov–Witten invariants make $H^*(V)$ and algebra over $H_*(\bar{M}_{g,n})$

Odd Modular

The canonical geometry is given by $\bar{M}^{KSV}$ which are real blowups of $\bar{M}_{gn}$ along the boundary divisors.

- We get 1-parameter gluings parameterized by S^1. Taking the full S^1 family on chains or homology gives us the structure of an odd modular operad.
- Going back to Sen and Zwiebach, a viable string field theory action S is a solution of the quantum master equation.

Hopf algebras

Basic structures

Assume $\mathcal{F}$ is additive and decomposition finite. Consider $Hom(\mathcal{F})$.
Let μ be the tensor product with unit $id_{\mathbb{I}}$.

$$\Delta(\phi) = \sum_{(\phi_0,\phi_1):\phi=\phi_1\circ\phi_0} \phi_0 \otimes \phi_1$$

and $\epsilon(\phi) = 1$ if $\phi = id_X$ and 0 else.

Theorem (Galvez-Carrillo, K , Tonks)

$Hom(\mathcal{F})$ together with the structures above is a bi–algebra. Under certain mild assumptions, a canonical quotient is a Hopf algebra

Examples

In this fashion, we can reproduce Connes–Kreimer's Hopf algebra, the Hopf algebras of Goncharov and a Hopf algebra of Baues that he defined for double loop spaces.

Summary

1. Lift invariants from numbers to objects (e.g. groups)
2. Consider operations to get classification/recognition
3. These operations are governed by algebraic structures, operad-like structures.
4. The operad-like structures themselves often have underlying geometries
5. The operad-like structures and their properties are axiomatized via Feynman categories
6. From Feynman categories we can distill Hopf algebras.
7. Hope: These give us geometric categories.

Next steps

- Find the connection to Tannakian categories. Find out the role of fibre functors.
- Find action of Grothendieck-Teichmüller group (GT).
- Connect to GT action (Kitchloo-Morava) on the stable symplectic category.
- Construct Feynman category for the open/closed version of Homological Mirror symmetry.
- ...

The end

Thank you!

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