On CY-LG correspondence for (0,2) toric models

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Reference

Joint work with Lev Borisov (Rutgers)

"On CY-LG correspondence for (0,2) toric models".
arXiv:1102.5444
Adv. of Math. to appear.

Outline

Main question and setup

Main question:

Can the results of L. Borisov on the LG/CY correspondence and mirror symmetry in toric N=(2,2) models be transferred to the case of (0,2) viz. heterotic theories.

Answer: Yes!

We showed this for specific case, the quintic, but our techniques generalize.
This is also the case considered by Witten in his "Phases of N=2 theories in two dimensions"

Main Idea

Method

- Use a lattice VOA whose cohomology is that of (an appropriate twisted version) Chiral de Rham complex (Malikov, Schechtman, Vaintrob).
- Move differential to get a CG/LY interpolating family.
- Technically this is (a gerbe of) Chiral Differential Operators (CDOs).

History

History

- Possibility of these CDOs (more generally as gerbes) developed by Gorbounov, Malikov, Schechtman '03/'04.
- Physics construction of CDO by M.C. Tan '07.
- Mathematical lattice VOA construction for toric case, with CY/LG and Mirror Symmetry construction by Borisov-K

Toric mirror and LG/CY after Borisov

Setup (Borisov)

- X hypersurface in a Fano toric variety.
- M_1 and N_1 dual lattices (free abelian groups)
- Δ and $\Delta^\vee$ dual reflexive polytopes in them.
- $M = M_1 \oplus \mathbb{Z}$ and $N = N_1 \oplus \mathbb{Z}$
- $K = \mathbb{R}_{\geq 0}(\Delta, 1) \cap M$ and
- $K^\vee = \mathbb{R}_{\geq 0}(\Delta^\vee, 1) \cap N$.

$\mathrm{Fock}_{M \oplus N}$ for the quintic in $\mathbb{P}^4$

Definition

The basic lattice VOA, $\mathrm{Fock}_{0 \oplus 0}$, is the vertex algebra generated by free bosonic and free fermionic fields based on the lattice $M \oplus N$ with operator product expansions

$$m^{bos}(z)n^{bos}(w) \sim \frac{m \cdot n}{(z-w)^2}, \quad m^{ferm}(z)n^{ferm}(w) \sim \frac{m \cdot n}{(z-w)}$$

and all other OPEs nonsingular.

$\mathrm{Fock}_{M\oplus N}$

Extension

$\mathrm{Fock}_{M\oplus N}$ has additional vertex operators $\mathrm{e}^{\int m^{bos}(z)+n^{bos}(z)}$ with the appropriate cocycle, the normal ordering implicitly applied and r.h.s. expanded at $z = w$.

$$\mathrm{e}^{\int m_1^{bos}(z)+n_1^{bos}(z)}\mathrm{e}^{\int m_2^{bos}(w)+n_2^{bos}(w)} = (z-w)^{m_1\cdot n_2+m_2\cdot n_1}\mathrm{e}^{\int m_1^{bos}(z)+n_1^{bos}(z)+m_2^{bos}(w)+n_2^{bos}(w)} \quad (2.1)$$

$\mathrm{Fock}^{\Sigma}_{M\oplus N}$ for the quintic in $\mathbb{P}^4$

Setup

Let Σ be the appropriate (generalized) fan in N.

- $\mathrm{Fock}^{\Sigma}_{M\oplus N}$ is the partial lattice vertex algebra defined by setting the product in (2.1) to zero if n_1 and n_2 do not lie in the same cone of Σ.
  Similarly define the vertex algebras
- $\mathrm{Fock}_{M\oplus K^\vee}$ and
- $\mathrm{Fock}^{\Sigma}_{M\oplus K^\vee}$.

Details for the quintic

Details

- $M := \{(a_0, \ldots, a_4) \in \mathbb{Z}^5, \sum a_i = 0 \mod 5\}$;
- $N := \mathbb{Z}^5 + \mathbb{Z}(\frac{1}{5}, \ldots \frac{1}{5})$
- $\deg = (1, \ldots, 1) \in M$
- $\deg^\vee = (\frac{1}{5}, \ldots \frac{1}{5})$ in N.
- Δ is the four-dimensional simplex given by the convex hull of $(5, 0, 0, 0, 0), \ldots, (0, 0, 0, 0, 5)$
- $\Delta^\vee$ is the simplex with vertices $(1, 0, 0, 0, 0), \ldots, (0, 0, 0, 0, 1)$.
- Σ has maximum-dimensional cones generated by $\deg^\vee, -\deg^\vee$ and four out of the five vertices of $\Delta^\vee$.

Lattice version for Chiral de Rham of the canonical bundle

Proposition (Borisov)

Let $W \to \mathbb{P}^4$ be the canonical bundle. Then the cohomology of the chiral de Rham complex $\mathrm{MSV}(W)$ is isomorphic to the cohomology of $\mathrm{Fock}^{\Sigma}_{M\oplus K^\vee}$ with respect to the differential

$$D_g = \mathrm{Res}_{z=0} \sum_{n\in\Delta^\vee} g_n n^{ferm}(z) \mathrm{e}^{\int n^{bos}(z)}$$

for any collection of nonzero numbers $g_n, n \in \Delta^\vee$.

Lattice version for Chiral de Rham of the quintic

Theorem (Borisov)

The cohomology of the chiral de Rham complex of a smooth quintic $F(x_0, \dots, x_4) = 0$ which is transversal to the torus strata is given by the cohomology of $\mathrm{Fock}^{\Sigma}_{M \oplus K}$ w.r.t. the differential

$$D_{f,g} = \mathrm{Res}_{z=0}\Big(\sum_{m \in \Delta} f_m m^{ferm}(z) \mathrm{e}^{\int m^{bos}(z)} + \sum_{n \in \Delta^\vee} g_n n^{ferm}(z) \mathrm{e}^{\int n^{bos}(z)} \Big)$$

where g_n are arbitrary nonzero numbers and f_m is the coefficient of F of the corresponding monomial.

The vertex algebras of mirror symmetry

Definition (Borisov)

The vertex algebras of mirror symmetry are defined as the **cohomology** of the **lattice vertex algebra** $\mathrm{Fock}_{M\oplus N}$ w.r.t. the differential

$$D_{f,g} = \mathrm{Res}_{z=0}\Big(\sum_{m\in\Delta} f_m m^{ferm}(z) \mathrm{e}^{\int m^{bos}(z)} + \sum_{n\in\Delta^\vee} g_n n^{ferm}(z) \mathrm{e}^{\int n^{bos}(z)} \Big)$$

where f_m and g_n are complex parameters.

Mirror Symmetry

Flip M, N and $\Delta, \Delta^\vee$

LG/CY Family

A family of vertex algebras of mirror symmetry

Fix F and the corresponding f_m. As the g_n vary, consider the family of vertex algebras $V_{f,g}$ which are the cohomology of $\mathrm{Fock}_{M\oplus K^\vee}$ with respect to the differential

$$D_{f,g} = \mathrm{Res}_{z=0}\Big(\sum_{m\in\Delta} f_m m^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} + \sum_{n\in\Delta^\vee} g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}\Big)$$

Limits

$(\prod_i g_{v_i})/g^5_{\deg^\vee} \to 0 \rightsquigarrow$ Cohomology of Chiral de Rham.

$g^5_{\deg^\vee} \to 0 \rightsquigarrow$ LG

$(0,2)$ for a quintic in $\mathbb{P}^4$

Setup

For $i = 1, \ldots, 5$, let $F^i = x_i R^i$ five polynomials of degree 5 s.t. $F^i|_{x_i=0} = 0$.
The equation for the quintic is $\sum_i F^i = 0$.
$\mathrm{Fock}_{M \oplus K^\vee}$ as before. Set

$$D_{(F^\cdot),g} = \mathrm{Res}_{z=0}\left(\sum_{\substack{m\in\Delta\\ 0\le i\le 4}} F^i_m m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} + \sum_{n\in\Delta^\vee} g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}\right)$$

where m_i the basis of $M_\mathbb{Q}$ which is dual to the basis of $N_\mathbb{Q}$ given by the vertices of $\Delta^\vee$, g_n are six generic complex numbers and F^i_m is the coefficient of the monomial of degree 5 of F^i that corresponds to m.

(0,2)-analogue

Definition

The vertex algebras of the (0,2) sigma model on $\sum_i F^i = 0$ are the corresponding cohomology spaces $V_{(F^\cdot),g}$

Remark 1

In the case when $F^i = x_i \partial_i f$ are logarithmic derivatives of some degree five polynomial f, we have $V_{(F^\cdot),g} = V_{f,g}$

Remark 2

Witten considered a homogeneous polynomial G of degree 5 in the homogeneous coordinates x_i on $\mathbb{P}^4$ and five polynomials G^i of degree four in these coordinates with the property $\sum_i x_i G^i = 0$. Equivalently, we consider five polynomials of degree four $R^i = \partial_i G + G^i$ and use $F^i = x_i R^i$.

Twisted Chiral deRham [GMS]

Setup

X be a smooth manifold. E a vector bundle on X s.t.
Conditions: $c_1(E) = c_1(TX)$ and $c_2(E) = c_2(TX)$ $\Lambda^{\dim X} E$ is isomorphic to $\Lambda^{\dim X} TX$, and pick a choice of such an isomorphism.

$\rightsquigarrow$ Collection of sheaves $\mathrm{MSV}(X, E)$ of vertex algebras on X

Different regluings given by $H^1(X, (\Lambda^2 TX^\vee)^{closed})$.
Locally, any such sheaf is again generated by b^i, a_i, ϕ^i, ψ_i, however ϕ^i and ψ_i now transform as sections of $E^\vee$ and E respectively.
The OPEs between the ϕ and ψ are governed by the pairing between sections of $E^\vee$ and E.

Our Special Case

Ambient Y and W

$\pi : W \to Y$ be a line bundle over an n-dimensional manifold Y with zero section $s : Y \to W$. α be a holomorphic one-form on W s.t. $\lambda \in \mathbb{C}^*$: $\lambda^* \alpha = \lambda \alpha$.

X and E

Consider the locus $X \subset Y$ of points y such that $\alpha(s(y))$ as a function on the tangent space $TW_{s(y)}$ is zero on the vertical subspace. Consider the subbundle E of $TY|_X$ which is locally defined as the kernel of $s^*\alpha$.

Condition

We will assume that it is of corank 1

Ambient and cohomology representation

Theorem (BK)

The cohomology sheaf of $\pi_* \mathrm{MSV}(W)$ with respect to $\mathrm{Res}_{z=0}\alpha(z)$ is isomorphic to a twisted chiral de Rham sheaf of (X, E).

Theorem (BK)

The cohomology of $\mathrm{Fock}^{\Sigma}_{M \oplus K^\vee}$ with respect to $D_{(F^\cdot),g}$ is isomorphic to the cohomology of a twisted chiral de Rham sheaf on the quintic $\sum_{i=0}^{i} F^i = 0$ given by R^i.

CG/CY

Interpolating Family

Consider the vertex algebras which are the cohomology of $\mathrm{Fock}_{M \oplus K^\vee}$ by $D_{(F^\cdot),g}$ as $(F^\cdot)$ is fixed and g varies.

Calabi-Yau Limit

Fix g_n for $n \neq \deg^\vee$ and $g_{\deg^\vee} \to \infty$.
In the limit the action of $D_{(F^\cdot),g} \rightsquigarrow$ action on $\mathrm{Fock}^{\Sigma}_{M \oplus K^\vee}$, after an appropriate reparametrization.

Landau-Ginzburg limit

$g_{\deg^\vee} = 0$, as in the $N = 2$ case.

Chiral Rings

Proposition

The algebra $V_{(F^\cdot),g}$ can be alternatively described as the cohomology of $\mathrm{Fock}_{M\oplus N}$ or $\mathrm{Fock}_{K\oplus N}$ by $D_{(F^\cdot),g}$.

Differential

$\mathbb{C}[(K \oplus K^\vee)_0]$ is the quotient of $\mathbb{C}[K \oplus K^\vee]$ w.r.t. the ideal spanned by monomials with positive pairing.
$d_{(F^\cdot),g}$ is defined to be the endomorphism on $\mathbb{C}[(K \oplus K^\vee)_0] \otimes \Lambda^* M_{\mathbb{C}}$ defined by

$$\sum_{i=0}^{4} \sum_{m\in\Delta} F_m^i[m] \otimes (\wedge m_i) + \sum_{n\in\Delta^\vee} g_n[n] \otimes (\mathrm{contr}.n). \qquad (3.1)$$

Chiral Rings

The Chiral Rings

are the parts of the vertex algebra where $H_A = 0$ or $H_B = 0$.
$H_A = \mathrm{Res}_{z=0} zL(z), \quad H_B = \mathrm{Res}_{z=0}(zL(z) + J(z))$.

Theorem (BK)

For generic $F^{\cdot}$ and g the eigenvalues of H_A and H_B on $V_{(F^{\cdot}),g}$ are nonnegative integers.
The $H_A = 0$ part is given as the cohomology of the corresponding eigenspace of $\mathrm{Fock}_{K \oplus K^\vee - \deg^\vee}$. As a vector space, this is isomorphic to the cohomology of $\mathbb{C}[K \oplus K^\vee] \otimes \Lambda^* M_{\mathbb{C}}$ w.r.t. $d_{(F^{\cdot}),g}$ from (3.1).
The $H_B = 0$ part comes from the corresponding eigenspace of $\mathrm{Fock}_{K - \deg \oplus K^\vee}$. As a vector space it is isomorphic to the cohomology of $\mathbb{C}[K \oplus K^\vee] \otimes \Lambda^* N_{\mathbb{C}}$ by an operator similar to (3.1) where one replaces all wedge products by contractions and vice versa.

MS and generalization

General Ansatz for the differential

Consider the lattice vertex algebra $\mathrm{Fock}_{M\oplus N}$. Pick a basis m_i of M and n_i of N. Let F^i_m and G^i_n be complex numbers for all i, $m \in \Delta$, $n \in \Delta^\vee$ such that the operator

$$D_{(F^\cdot),(G^\cdot)} = \mathrm{Res}_{z=0}\Big(\sum_i \sum_{m\in\Delta} F^i_m m_i^{ferm}(z) \mathrm{e}^{\int m^{bos}(z)}$$

$$+\sum_i \sum_{n\in\Delta^\vee} G^i_n n_i^{ferm}(z) \mathrm{e}^{\int n^{bos}(z)}\Big)$$

is a differential on $\mathrm{Fock}_{M\oplus N}$ the OPE of the above field with itself to be nonsingular.

Questions/End

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1. What is the most general setting?
2. Can we treat singularities.
3. What is the geometry that one describes.

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The End

Thank you!