Feynman categories

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References

Main paper(s)

with B. Ward and J. Zuniga.

1. *The odd origin of Gerstenhaber, BV and the master equation* arXiv:1208.3266
2. *Feynman categories: in prep*

Background

Builds on previous work by Harrelson–Zuniga–Voronov, K, K-Schwell, Kimura–Stasheff–Voronov, A. Schwarz, Zwiebach.

Outline

1. Geometry and Algebra
   - Gluing surfaces with boundary
   - The classic algebra example: Hochschild and the Gerstenhaber bracket
   - An operadic interpretation of the bracket
   - Cyclic generalization
   - Modular and general version
2. Geometry and Physics
   - KSV, Zwiebach,
   - EMOs, or S^1 gluings
3. Categorical Approach
   - Motivation and the Main Definition
   - Examples

Surfaces as boundary as a model

Basic objects

Consider a surface (topological) Σ with enumerated boundary components $\partial_\Sigma = \amalg_{i=0,\dots,n-1} S^1$.

Standard gluing

Take two such surfaces Σ, Σ' and define $\Sigma \circ_i \Sigma'$ to be the surface obtained from gluing the boundary i of Σ to the boundary 0 of Σ' and enumerate the $n + n' - 1$ remaining boundaries as

$$0, \dots, i-1, 1', \dots, (n'-1)', i+1, \dots, n-1$$

More gluings

Non–self gluing (cyclic)

Now enumerate the boundaries by a set S. Then define $\Sigma_s \circ_t \Sigma'$ by gluing the boundaries s and t. The new enumeration is by $(S \setminus \{s\}) \amalg (T \setminus \{t\})$

Self–gluing

If $s, s' \in S$ we can define $\circ_{s,s'}\Sigma$ and the surface obtained by gluing the boundary s to the boundary s'. Notice that the genus of the surface increases by one.

Variations

We can consider Riemann surfaces with possibly double point and marked points and then glue by attaching at the marked points. This introduces a new double point.

The classic algebra case: Hochschild cochains

Hochschild cohomology

- **Cochains** A associative, unital algebra
  $CH^n(A, A) = Hom(A^{\otimes n}, A)$
- **Differential** Example: $f \in CH^2(A, A)$

$$df(a_0, a_1, a_2) = a_0 f(a_1, a_2) - f(a_0 a_1, a_2) + f(a_0, a_1 a_2) - f(a_0, a_1) a_2$$

- **Cohomology** $HH^*(A, A) = H^*(CH^*(A, A), d)$

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Hochschild cohomology

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- **Cohomology** $HH^*(A, A) = H^*(CH^*(A, A), d)$

Used for deformation theory.
Used in String Topology: If M is a simply connected manifold
$H_*(LM) \simeq HH^*(S^*(M), S_*(M))$

Pre–Lie and bracket

Operad structure

Substituting g in the i–th variable of f, we obtain operations for $i = 1, \ldots, n$:

$$\begin{aligned} \circ_i : CH^n(A,A) \otimes CH^m(A,A) &\rightarrow CH^{m+n-1}(A,A) \\ f \otimes g &\mapsto f \circ_i g \end{aligned}$$

Pre-Lie product

For $f \in CH^n(A,A), g \in CH^m(A,A)$ set

$$f \circ g := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g$$

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Mind the signs!

The bracket

The bracket

If $f \in CH^n(A,A)$ set $|f| = n$ and $sf = |f| - 1$ the shifted degree.

$$\{f \bullet g\} = f \circ g - (-1)^{sf\, sg} g \circ f$$

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Theorem (Gerstenhaber '60s)

The bracket above is an odd Lie bracket on $CH^*(A, A)$ and a Gerstenhaber bracket on $HH^*(A, A)$.
Gerstenhaber:

- *odd Lie*
  - *odd anti–symmetric $\{f \bullet g\} = -(-1)^{sf\, sg}\{g \bullet f\}$*
  - *odd Jacobi (use shifted signs)*
- *and odd Poisson (derivation in each variable with shifted signs)*

Sign mnemonics

Algebra shift

If L is a Lie algebra then $\Sigma L = L[-1]$ is an odd Lie algebra.

Two ways of viewing the signs

1. shifted signs: f has degree sf
2. f has degree $|f|$ and $\bullet$ has degree 1.

Compatibility

$$-(-1)^{sf\,sg} = -(-1)^{(|f|-1)(|g|-1)} = (-1)^{|f|+|f|\,|g|+|g|}$$

Viewpoint 2 is natural

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- From a geometric point of view:

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- From a categorical point of view:

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Viewpoint 2 is natural

- From a geometric point of view: S^1 family
- From an algebraic point of view: odd operad
- From a categorical point of view: odd Feynman category

Essential Ingredients

To define $\{\bullet\}$

we need

- A (graded) collection CH^n.
- $\circ_i : CH^n \otimes CH^m \to CH^{n+m-1}$ operadic, i.e. they satisfing some compatibilities
- Need to be able to use signs and form a sum.

Generalizations we will give

- (odd) operads
- (odd) cyclic operads
- odd modular operads aka. $\mathfrak{K}$–modular operads
- odd functors from Feynman categories

Essential Ingredients

To define $\{\bullet\}$

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- A (graded) collection CH^n. (only need $\mathbb{Z}/2\mathbb{Z}$ grading)
- $\circ_i : CH^n \otimes CH^m \to CH^{n+m-1}$ operadic, i.e. they satisfing some compatibilities
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(Odd) Lie Algebras from operads

Theorem (G,GV,K,KM,..)

Given an operad $\mathcal{O}$ in Vect set $f^n \circ g^m = \sum_{i=1}^n f \circ_i g$ then

1. *$\circ$ is pre–Lie and $[f, g] = f \circ g - (-1)^{|f|\,|g|} g \circ f$ is a Lie bracket on $\bigoplus_n \mathcal{O}(n)$.*
2. *This bracket descends to $\mathcal{O}_{\mathbb{S}} := \bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$*
3. *Given an operad $\mathcal{O}$ in g–Vect set $f^n \circ g^m = \sum_{i=1}^n (-1)^{(i-1)(m-1)} f \circ_i g$ then $\circ$ is graded pre–Lie and $\{f \bullet g\} = f \circ g - (-1)^{sf\, sg} g \circ f$ is an odd Lie bracket on $\bigoplus_n \mathcal{O}(n)$.*

---

a

b

(Odd) Lie Algebras from operads

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2. *This bracket descends to* $\mathcal{O}_{\mathbb{S}} := \bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$
3. *Given an operad* $\mathcal{O}$ *in g–Vect set* $f^n \circ g^m = \sum_{i=1}^n (-1)^{(i-1)(m-1)} f \circ_i g$ *then* $\circ$ *is graded pre–Lie and* $\{f \bullet g\} = f \circ g - (-1)^{sf\, sg} g \circ f$ *is an odd Lie bracket on* $\bigoplus_n \mathcal{O}(n)$.

[a] Or in an additive category with direct sums/coproducts $\bigoplus$

[b]

(Odd) Lie Algebras from operads

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---

[a] Or in an additive category with direct sums/coproducts $\bigoplus$

[b] Here one should take the total degree

Shifting operads: How to hide the odd origin of the bracket

Operadic shift

For $\mathcal{O}$ in g–Vect: Let $(s\mathcal{O})(n) = \Sigma^{n-1}\mathcal{O} \otimes sign_n$.
Then $s\mathcal{O}(n)$ is an operad.
In the shifted operad $f \tilde{\circ}_i g = (-1)^{(i-1)(|g|-1)} f \circ_i g$ and the degree of f is sf (if $f \in \mathcal{O}(n)$ of degree 0).

Naïve shift

$(\Sigma\mathcal{O})(n) = \Sigma(\mathcal{O}(n))$.
This is *not an operad* as the signs are off. We say that it is an odd operad.

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Remark (KWZ)

$CH^*(A, A)$ is most naturally $\Sigma sEnd(A)$, i.e. an odd operad. This explains the degrees & signs, and enables us to do generalizations.

Odd vs. even

Associativity for a graded operad

$$(a \circ_i b) \circ_j c = \begin{cases} (-1)^{(|b|)(|c|)} (a \circ_j c) \circ_{i+l-1} b & \text{if } 1 \leq j < i \\ a \circ_i (b \circ_{j-i+1} c) & \text{if } i \leq j \leq i+m-1 \\ (-1)^{(|b|)(|c|)} (a \circ_{j-m+1} c) \circ_i b & \text{if } i+m \leq j \end{cases}$$

The signs come from the commutativity constraint in g–Vect

Associativity for an odd operad

$$(a \bullet_i b) \bullet_j c = \begin{cases} (-1)^{(|b|-1)(|c|-1)} (a \bullet_j c) \bullet_{i+l-1} b & \text{if } 1 \leq j < i \\ a \bullet_i (b \bullet_{j-i+1} c) & \text{if } i \leq j \leq i+m-1 \\ (-1)^{(|b|-1)(|c|-1)} (a \bullet_{j-m+1} c) \bullet_i b & \text{if } i+m \leq j \end{cases}$$

1st generalization

(Anti–)Cyclic operads.

A (anti–)cyclic operad is an operad together with an extension of the $\mathbb{S}_n$ action on $\mathcal{O}(n)$ to an $\mathbb{S}_{n+1}$ action such that

1. $T(id) = \pm id$ where $id \in \mathcal{O}(1)$ is the operadic unit.
2. $T(a^n \circ_1 b^m) = \pm(-1)^{|a||b|} T(b) \circ_m T(a)$

where T is the action by the long cycle $(1 \dots n+1)$

Typical examples

- Cyclic operad: End(V) for V a vector space with a non–degenerate symmetric bilinear form.
- Anti–Cyclic operad: End(V) for V a symplectic vector space i.e. with a non–degenerate anti–symmetric bilinear form.

Compositions and bracket

Unbiased definition

Set $\mathcal{O}(S) = [\bigoplus_{S \overset{1-1}{\leftrightarrow} \{1,\dots,|S|\}} \mathcal{O}(|S|)]_{\mathbb{S}_{n+1}}$

Then we get operations

$$_s\circ_t : \mathcal{O}(S) \otimes \mathcal{O}(T) \to \mathcal{O}((S \setminus \{s\}) \amalg (T \setminus \{t\}))$$

Bracket

For $f \in \mathcal{O}(S), g \in \mathcal{O}(T)$ set

$$[f, g] = \sum_{s \in S, t \in T} f\, {}_s\circ_t g$$

Brackets

Shifts

The operadic shift of an anti–/cyclic operad $s\mathcal{O}(n) = \Sigma^{n-1}\mathcal{O}(n) \otimes sign_{n+1}$ is cyclic/anti–cyclic.
The odd cyclic/anti–cyclic versions are defined to be the naïve shifts of the anti–cyclic/cyclic ones.

Theorem (KWZ)

Given an anti–cyclic operad $[\ ,\ ]$ *induces a Lie bracket on* $\bigoplus \mathcal{O}(n)_{\mathbb{S}_{n+1}}$ *which lifts to* $\bigoplus \mathcal{O}(n)_{C_{n+1}}$
In the odd cyclic case, we obtain an odd Lie bracket.

Brackets

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In the odd cyclic case, we obtain an odd Lie bracket.

Notice: if we take a cyclic operad, use the operadic shift and then the naïve shift, we get an odd Lie bracket.

Compatibility and examples

Compatibility

Let $N = 1 + T + \cdots + T^n$ on $\mathcal{O}(n)$. Then

$$N[f, g]_{\text{cyclic}} = [Nf, Ng]_{\text{non-cyclic}}$$

Examples: Kontsevich/Conant–Vogtman Lie algebras/New

$(\mathcal{O} \otimes \mathcal{V})(n) := \mathcal{O}(n) \otimes \mathcal{V}(n)$ with diagonal $\mathbb{S}_{n+1}$ action.
Then: cyclic $\otimes$ anti–cyclic is anti–cyclic.
Let V^n be a n dimensional symplectic vector space.
For each n get Lie algebras

$(1)\ Comm \otimes End(V) \quad (2)\ Lie \otimes End(V) \quad (3)\ Assoc \otimes End(V)$

Let V^n be a vector space with a symmetric non-degenerate form.
For each n we get a Lie algebra

$$(4) Pre-Lie \otimes End(V)$$

2nd generalization: Modular operads

Modular operads

Modular operads are cyclic operads with extra gluings

$$\circ_{ss'} : \mathcal{O}(S) \to \mathcal{O}(S \setminus \{s, s'\})$$

The operator Δ

For $f \in \mathcal{O}(S)$

$$\Delta(f) := \frac{1}{2} \sum_{(s,s)' \in S} \circ_{ss'}(f)$$

Odd version

We need the odd version. These are $\mathfrak{K}$–modular operads.

Theorems

Theorem (KWZ)

In a $\mathfrak{K}$–modular operad Δ descends to a differential on $\bigoplus \mathcal{O}(n)_{\mathbb{S}_{n+1}}$. There is also the odd Lie bracket form the underlying odd cyclic structure.
In an NC $\mathfrak{K}$–modular operad, the operator Δ becomes a BV operator for the product induced by the horizontal compositions. Moreover this algebra is then GBV.

Remarks

NC modular and NC $\mathfrak{K}$–modular, which are maybe new, means that like in PROPs there is an additional horizontal composition. There is a more general setup of when such operations exist.

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There is a more general setup of when such operations exist.
Wait a couple of minutes.

Comments

1. $\mathfrak{K}$ modular operads were defined by Getzler and Kaparnov
2. The Feynman transform of a modular operad is a $\mathfrak{K}$–modular operad $\rightsquigarrow$ Examples.
3. Restricting the twist to the triples for operads and cyclic operads gives their odd versions. For experts

$$\mathfrak{K} \simeq Det \otimes \mathfrak{D}_s \otimes \mathfrak{D}_\Sigma$$

Here Det is a cocylce with value on a (connected) graph Γ given by $Det(H_1(\Gamma))$.

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Somehow not so commenly used.

Master equation

If the “generalized operads” are dg, then one can make sense of the master equation (Details to follow).

$$dS + \Delta S + \frac{1}{2}\{S \bullet S\} = 0$$

This equation parameterizes “free” algebra structures on one hand (e.g. the Feynman transform) and compactifications on the other.

Examples

Feynman transform after Barannikov

The $\mathcal{F}_D P$–algebra structures on V are given by solutions of the master equation

$$dS + \Delta S + \frac{1}{2}\{S \bullet S\} = 0$$

on $(\bigoplus (P(n) \otimes V^{\otimes n+1})^{\mathbb{S}_{n+1}})_0$

Interpertation

The background is that $\mathcal{F}_D P$ is free as an operad, but not a free dg operad. So to specify an algebra over it, there are conditions and the Master equation is the sole equation.

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Formal Super–manifolds after Shardrin-Merkl-Merkulov

Algebra structures over a certain wheeled PROP for a formal supermanifold are given by solutions of the master equation.

The master equation in the topological setting

DM spaces

The Deligne–Mumford compactifications $\bar{M}_{g,n}$ form a modular operad. So do their homologies $H_*(\bar{M}_{g,n})$.
Remark: Gromov–Witten theory yields algebras over this operad.

KSV spaces

Let $\bar{M}^{KSV}_{g,n}$ be the real blowups of the spaces $\bar{M}_{g,n}$ along the compacitfication divisors.
Elements of these spaces are surfaces with nodes and a tangent vector at each node. More precisely, an element of $(S^1 \times S^1)/S^1$ at each node.

Master equation: String field theory

Odd/family gluings [KSV,HVZ]

Given two elements $\Sigma \in \bar{M}^{KSV}_{g,n}$ and $\Sigma' \in \bar{M}^{KSV}_{g,n}$ and a marked point on each of them, one defines a family by choosing all possible tangent vectors of the surfaces attached to each other at the marked points.

$$\Sigma_i \circ_j \Sigma' : S^1 \to \bar{M}^{KSV}_{g+g',n+n'-2}$$

These give degree–one gluings on the chain level. (Can also use correspondences on the topological level).
Moreover, the set of fundamental classes form a solution to the master equation.

Master equation: String field theory

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Moreover, the set of fundamental classes form a solution to the master equation.

The master equation drives the compactification

Operads with S^1 action EMOs and the like

Definition/List

We call a modular operad, ...in the topological category an S^1 equivariant operad, if on each $\mathcal{O}(n)$, ...the $\mathbb{S}_n$ action, ..., is augmented to an $\mathbb{S}_n \wr S^1$, ...action and the action is balanced, that is $\rho_i(l)a_i \circ_j b = a_i \circ_j \rho_j(-l)b$

Proposition

Given and S^1 equivariant modular operad, ...has an induced $\mathfrak{K}$–modular, odd ...structure on its S^1 equivariant chains.

Example

The *Arc* operad of KLP and the various operads/PROPs of K used in string topology are further examples.

We need a global view for The Zoo: an inventory of species

Types of operads and graphs

Type	Graphs
Operads	connected rooted trees
Cyclic operads	connected trees
Modular operads	connected graphs
PROPs	directed graphs
NC modular operad	graphs

- This means that for each such diagram there is a unique composition after decorating the vertices of such a graph with "operad" elements.
- These type of graphs will correspond to morphisms in the categorical setup.
- The original setup was algebras over a triple. [Markl, Getzler-Kapranov, ...] although NC modular is in principle totally new.

Twisted (modular) operads

Twisted/odd operads

In the same setting.

Type	Graphs
odd operad	connected rooted trees with orientation on the set of all edges
anti–cyclic	connected trees with orientation on each edge
odd cyclic	connected trees with orientation on the set of all edges
$\mathfrak{K}$–modular operads	connected graphs with an orientation on the set of all edges
$\mathfrak{K}$–modular NC operads	graphs with an orientation on the set of all edges

What we want and get

Goal

We need a more general theory of things like operads so encompass all the things we have seen so far.

Generality

We think our approach is just right to fit what one needs.
Our definition fits in between the Borisov–Manin and the Getzler approaches. It is more general than Borisov–Manin as it includes odd and twisted modular versions, EMOs, etc.. It is a bit more strict than Getzler's use of patterns, but there one always has to first prove that the given categories are a pattern.

The main new character: Feynman categories

Definition

A one comma generating subcategory $\mathcal{V}$ of a symmetric monoidal category $(\mathcal{F}, \otimes)$ is a subcategory $\mathcal{V}$ such that,

1. *$\mathcal{V}$ is a groupoid, that is every morphism is an isomorphism.*
2. *$\mathcal{V}$ has the full automorphism groups, that is $Hom_{\mathcal{V}}(*, *) = Hom_{\mathcal{F}}(*, *)$.*
3. *$\mathcal{V}$ freely generates the objects: (a) for each $X \in \mathcal{F}$ there exists an isomorphism $\phi : X \to \otimes_{v \in I} *_v$ with $*_v \in \mathcal{V}$ for a finite index set I. And (b) the decomposition is essentially unique: For any two such isomorphisms there is a bijection of the two index sets $\psi : I \to J$ and a diagram*

$$\bigotimes_{v \in I} *_v \xleftarrow[\simeq]{} X \xrightarrow[\simeq]{} \bigotimes_{w \in J} *'_w, \qquad \bigotimes_{v \in I} *_v \xrightarrow{\simeq \bigotimes \phi_v} \bigotimes_{w \in J} *'_w \tag{1}$$

where $\phi_v \in Hom_{\mathcal{V}}(*_v, *'_{\psi(v)})$ are isomorphisms. $|I| :=$ the length of X.

Feynman categories

Definition (continued)

4. The comma category $(\mathcal{F} \downarrow \mathcal{V})$ generates the morphisms. Any morphism $X \to X'$ is part of a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow \simeq & & \downarrow \simeq \\ \bigotimes_{v \in I} X_v & \xrightarrow{\bigotimes_{v\in I} \phi_v} & \bigotimes_{v \in I} *_v \end{array} \tag{2}$$

where the $*_v \in \mathcal{V}$, $X_v \in \mathcal{F}$ and $\phi_v \in Hom(X_v, *_v)$

5. For any $* \in \mathcal{V}$, the comma category $(\mathcal{F} \downarrow *)$ is essentially small, viz. it is equivalent to a small category.

Feynman categories

Definition

A Feynman graph category (FGC) is a pair $(\mathcal{F},\mathcal{V})$ of a monoidal category $\mathcal{F}$ whose objects are sometimes called clusters or aggregates and a comma generating subcategory $\mathcal{V}$ whose objects are sometimes called stars or vertices.

Definition

Let $\mathcal{C}$ be a symmetric monoidal category. Consider the category of strict monoidal functors $\mathcal{F}\text{–}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F},\mathcal{C})$ which we will call $\mathcal{F}$–ops in $\mathcal{C}$ and the category of functors $\mathcal{V}\text{–}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V},\mathcal{C})$ which we will call $\mathcal{V}$–modules in $\mathcal{C}$.
If $\mathcal{C}$ and $\mathcal{F}$ respectively $\mathcal{V}$ are fixed, we will only write $\mathcal{O}ps$ and $\mathcal{M}ods$.

Monadicity

Theorem

If $\mathcal{C}$ is cocomplete then the forgetful functor G from $\mathcal{O}ps$ to $\mathcal{M}ods$ has a left adjoint F which is again monoidal.

Corollary

$\mathcal{O}ps$ is equivalent to the algebras over the triple FG.

Morphisms

Given a morphism (functor) of Feynman categories $i : \mathcal{F} \to \mathcal{F}'$ there is pullback i^* of $\mathcal{O}ps$ and if $\mathcal{C}$ is cocomplete there is a left Kan extension which is, as we prove, monoidal. This gives the push–forward i_*.
The free functor is such a Kan extension. So is the PROP generated by an operad and the modular envelope. The passing between biased and unbiased versions is also of this type.

Examples

Basic Example

$(\mathcal{A}gg, \mathcal{C}rl)$: $\mathcal{C}rl$ is the category of S–corollas with the automorphisms $Aut(S)$ and $\mathcal{A}gg$ are disjoint unions of these with morphisms being the graph morphisms between them.

Ordered/Ordered Examples

If a Feynman category has morphisms indexed by graphs, we can define a new Feynman category by using as morphisms pairs of a morphism and an order/orientation of the edges of the graphs.

Ab-version

Enriching over the category of Abelian groups, we get the odd versions.

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If a Feynman category has morphisms indexed by graphs, we can define a new Feynman category by using as morphisms pairs of a morphism and an order/orientation of the edges of the graphs. There is a technical version of this called a Feynman category indexed over $\mathcal{A}gg$.

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Enriched versions

Proposition

If $\mathcal{C}$ is Cartesian closed, each coboundary (generalizing the term as used by Getzler and Kapranov) defines an Feynman category enriched over $\mathcal{C}$ (or Ab) such that the $\mathcal{O}ps$ are exactly the twisted modular operads.

Categorical version of EMOs and the like

1. First we can just add a twist parameter to each edge and enrich over Top. The $\mathcal{O}ps$ will then have twist gluings.
2. Instead of taking S–corollas, we can take objects with an $Aut(S) \wr S^1$ action that is balanced. Then there is a morphism of Feynman categories, from this category to the first example.

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Note, we do not have to restrict to dgVect here. There is a more general Theorem about these type of enriched categories

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General version

Theorem (KWZ)

Let $\mathcal{C}$ be cocomplete.

1. *If we have a Feynman category indexed over $\mathcal{A}gg$ with odd self–gluings then for every $\mathcal{O} \in \mathcal{O}ps$ the object $colim_{\mathcal{V}}\mathcal{O}$ carries a differential Δ which is the sum over all self–gluings.*
2. *If we have a Feynman category indexed over $\mathcal{A}gg$ with odd non–self–gluings then for every $\mathcal{O} \in \mathcal{O}ps$ the object $colim_{\mathcal{V}}\mathcal{O}$ carries an odd Lie bracket.*
3. *If we have a Feynman category indexed over $\mathcal{A}gg$ with odd self–gluings and odd non–self–gluings as well as horizontal (NC) compositions, then the operator Δ is a BV operator and induces the bracket.*

More,...

Further results

1. There is also a generalization of Barannikov's result. Using the free functor and the dual notion of Co-$\mathcal{O}ps$, i.e. contravariant functors.
2. There is one more generalization, which replaces $\mathcal{A}gg$ with a category with unary and binary generators with quadratic relations for the morphisms.
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Thank you!