Condensed matter, C^*–geometry and topological invariants

Ralph Kaufmann

Purdue University

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References

R. Kaufmann, S. Khlebnikov, and B. Wehefritz-Kaufnann

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1. "Notes on topological insulators". arXiv:1501.02874

Plan

Condensed matter and C^*

Disclaimer

This will be a very short glimpse which is not intended to be complete, exhaustive or anything else of that sort.
There are excellent reviews of this subject starting with Bellissard, Schulz-Baldes and van Elst, to Prodan more recently ('14).

Basic appearance of C^*

Condensed matter/ Lattice/ Tanslational symmetry

We consider a condensed matter system, which has a crystal structure. This means that it is a structure that is invariant under a translational symmetry. (Recall disclaimer).

Mathematical version

We start with a graph $\Gamma \subset \mathbb{R}^d$ which has a symmetry group $L \simeq \mathbb{Z}^d$ that acts on $\mathbb{R}^d$ and leaves Γ invariant. $L(\Gamma) = \Gamma$.
Set $\bar{\Gamma} = \Gamma/L$.

Adding translation operators

Hilbert space

Let Λ be the vertices of Γ and $\bar{\Lambda}$ those of $\bar{\Gamma}$.
$\mathscr{H} = \ell^2(\Lambda) = \bigoplus_{\bar{v}\in\bar{\Lambda}} H_{\bar{v}}$ where $\mathscr{H}_{\bar{v}} = \ell^2(\pi^{-1}(\bar{v}))$

Action of L

L acts via translation operators on $\mathscr{H}$:
For $l \in L$: $T_l(\phi)(v) = \phi(v - l)$.
This action is by isometries and it maps: $\mathscr{H}_{\bar{v}} \to \mathscr{H}_{\bar{v}}$.

Action of T (free Abelian) subgroup of $\mathbb{R}^n$ generated by the edge vectors by partial isometries.

then the translation yields an operator $T_{\vec{e}} : \mathscr{H}_{\bar{w}} \to \mathscr{H}_{\bar{v}}$. This extends to an operator $\hat{T}_{\vec{e}}$ on $\mathscr{H}$ via $\hat{T}_{\vec{e}} = i_{\bar{v}} T_{\vec{e}} P_{\bar{w}}$ where $i_{\bar{v}} : \mathscr{H}_{\bar{v}} \to \mathscr{H}$ is the inclusion and $P_{\bar{w}} : \mathscr{H} \to \mathscr{H}_{\bar{w}}$ is the projection.

Magnetic field the appearance of NCG

Projective 2–cocycle

We may also use a 2–cocycle $\alpha \in Z^2(T, U(1))$ and use projective translation operators or magnetic translation operators.

Constant magnetic field

Fix 2–form $\hat{\Theta} = \Theta_{ij} dx_i \wedge dx_j$ given by a skew symmetric matrix Θ. We let $B = 2\pi\hat{\Theta}$. We obtain a two–cocycle $\alpha_B \in Z^2(\mathbb{R}^n, U(1))$: $\alpha_B(u, v) = \exp(\frac{i}{2} B(u, v))$, and its restriction to Γ.

Magnetic translations

Let A be a potential for B (on $\mathbb{R}^n$). The magnetic translation partial isometry is now given by

$$U_{l'}\psi(l) = e^{-i\int_l^{(l-l')} A}\,\psi(l - l').$$

Haper Hamiltonian

Physics action

Use Weyl quantization and Peierls substitution for one particle action. In the magnetic case the magnetic translations were introduced by Wannier. And the magnetic field gives rise to a projective representation whose commutators include the fluxes of the magnetic field.

Harper Hamiltonian

If $\vec{e}$ is a directed edge whose image under π is from $\bar{v}$ to $\bar{w}$, The (magnetic) Harper operator is

$$H = \sum_{e \in E} \hat{U}_{\vec{e}} + \hat{U}_{-\vec{e}}$$

C^*–geometry

Connes–Bellissard–Harper approach to electronic properties of a Γ wire system

Consider a C^*–algebra $\mathscr{B}$ which is the smallest algebra containing the Hamiltonian and the symmetries.
Here Hamiltonian is the Harper Hamiltonian, which acts on the Hilbert space $\mathscr{H} = \ell^2(\Lambda)$ where Λ are the vertices.

Base algebra and cover

The translations alone generate a C^*–subalgebra $\mathscr{A} \subset \mathscr{B}$. This inclusion is the *effective* geometry

Examples

The main examples

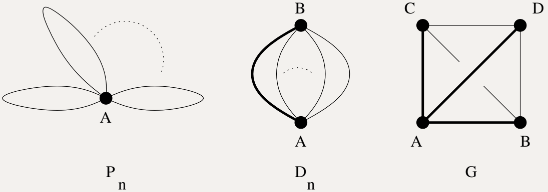


Figure: Graphs with rooted spanning trees. The root is A. The petal graphs P_n the graphs D_n and the graph G

Remarks

The P_n graph arises from the square lattice $\mathbb{Z}^n$, D_2 corresponds to the honeycomb lattice, D_3 to the Diamond lattice and G to the Gyroid lattice. $\mathbb{Z}^2$ is the geometry for the QHE, and D_2 is the geometry for graphene.

Results [KKWK] non-commutative

Expectation

Generically expect that $\mathcal{B} = M_k(\mathbb{T}_\Theta) \sim_{Morita} \mathbb{T}_\Theta$. $k = \# vertices$.

Theorem [KKWK]

This is true for P , D_2, D_3 and G cases and we classified the locus where $\mathscr{B}$ is a proper subalgebra. Also at rational B–field there are only finitely many gaps in the spectrum (Hofstadter's butterfly).

Commutative case [KKWK]

X is a branched cover of T. For the lattice case $T = T^n$ and the cover is generically unramified. We gave the ramification locus and branching for P D_3 G and the honeycomb.

Discussion

Remarks

1. The gaps are important for gap–labeling by K-theory. Here the gap is labelled by the projector $P_{\leq E}$ which projects to Eigenstates of energy $\leq E$. It is assumed that E is in a gap.
2. Notice for $\mathbb{Z}^2$ there is no gap in the commutative case. QHE only works in the presence of B–field. Get quantization. The Kubo formula says that the relevant quantity is the first Chern class [BvES-B].
3. For D_2, D_3 and G the commutative singular geometry is interesting. Graphene D_2 and the Gyroid have Dirac points. This means that there is a linear dispersion relation near these points and hence relativistic quasi–particles.
   (Nice characterization using singularity theory [KKWK])
4. The choice of rooted spanning tree gives rise to a re–gauging groupoid, which captures all additional symmetries.

Example 1: The Bravais lattice case aka. $\mathbb{Z}^n$

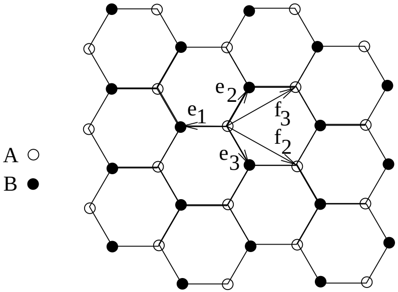
Setup

- $T = L = \mathbb{Z}^n$
- Magnetic translations: $U_i := U_{e_i}$ generate. Relations $U_i U_j = e^{2\pi i \Theta_{ij}} U_j U_i$.
- $H = \sum_i U_{e_i} + U_{e_i}^*$: $H \in$ algebra generated by the magnetic translations.

Result

The Bellissard-Harper algebra is $\mathscr{B} = \mathbb{T}^n_\Theta$. The non–commutative n–torus.

Example 2: The Honeycomb lattice aka. Graphene



Setup

The honeycomb lattice is a subset of the lattice generated by $-e_1 := (1,0)$ and $e_3 := \frac{1}{2}(1, -\sqrt{3})$. Set $e_2 = -e_1 - e_3 = \frac{1}{2}(1, \sqrt{3})$.

- $L \simeq \mathbb{Z}^2$ generated by $f_2 := e_2 - e_1 = \frac{1}{2}(-3, \sqrt{3})$ and $f_3 := e_3 - e_1 = \frac{1}{2}(3, \sqrt{3})$.
- T is generated by the e_i

The Honeycomb lattice II

The Harper Operator

$\mathscr{H} = \mathscr{H}_A \oplus \mathscr{H}_B$ and $U_{e_i} : \mathscr{H}_B \to \mathscr{H}_A$. Fix the magnetic field by $\phi = \hat{\Theta}(-e_1, e_2)$, $\chi := e^{i\pi\phi}$.

Set $\hat{U}_i := \begin{pmatrix} 0 & 0 \\ U_{e_i} & 0 \end{pmatrix}$, $\quad \hat{U}_{-i} := \begin{pmatrix} 0 & U_{-e_i} \\ 0 & 0 \end{pmatrix}$

where U_{e_i} and $U_{-e_i} = U_{e_i}^{-1} = U_{e_i}^*$ are the isomorphisms between $\mathscr{H}_A$ and $\mathscr{H}_B$.

The Harper Hamiltonian now reads:

$$H = \sum_{i=1}^{3} \hat{U}_i + \hat{U}_i^{-1} = \begin{pmatrix} 0 & U_{e_1}^* + U_{e_2}^* + U_{e_3}^* \\ U_{e_1} + U_{e_2} + U_{e_3} & 0 \end{pmatrix}$$

The Honeycomb lattice III

The Matrix Harper Operator

Fixing bases, we obtain the matrix expression:

$$H = \begin{pmatrix} 0 & 1 + U^* + V^* \\ 1 + U + V & 0 \end{pmatrix} \in M_2(\mathbb{T}^2_\theta)$$

where we have used the operators $U := \chi U_{f_2}$ and $V = \bar{\chi} U_{f_3}$ which satisfy $UV = qVU$ with $q := e^{2\pi i \theta} = \bar{\chi}^6$ where $\theta = \hat{\Theta}(f_2, f_3)$

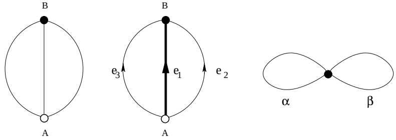


Figure: The graph $\bar{\Gamma}$, a choice of oriented edges and a spanning tree τ, $\bar{\Gamma}/\tau$

The algebra $\mathscr{B}$ in the honeycomb case

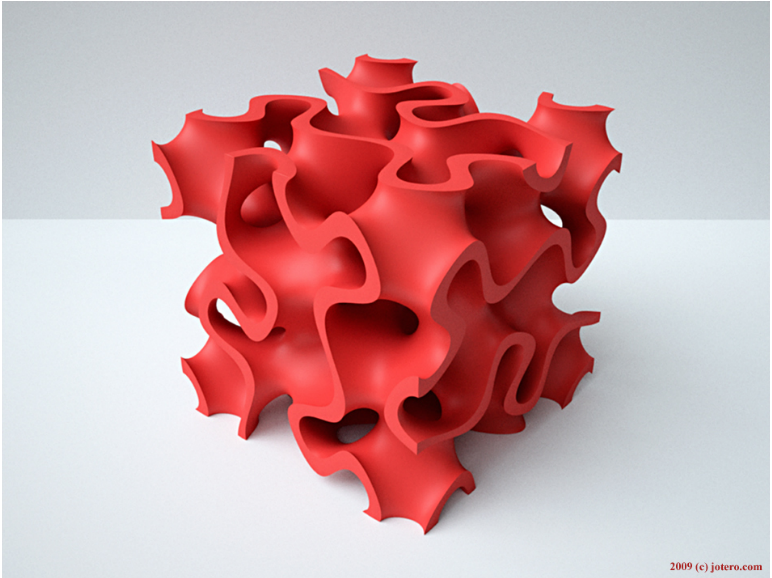
Theorem

If $q \neq \pm 1$ or $q = -1$ and $\chi^4 \neq 1$ then $\mathscr{B}_\Theta = M_2(\mathbb{T}^2_\theta)$ and hence is Morita equivalent to $\mathbb{T}^2_\theta$.
If $q = -1$ and $\chi^4 = 1$ or if $q = 1$ and $\chi \neq \pm 1$ then $\mathscr{B}_\Theta$ is a proper subalgebra of $M_2(\mathbb{T}^2_{\frac{1}{2}})$ (which we know).
If $q = 1$ and $\chi = \pm 1$ then $\mathscr{B}_\Theta = C^*(X)$ where X is the double cover of the torus $S^1 \times S^1$ ramified at the points $(e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}})$ and $(e^{2\pi i \frac{2}{3}}, e^{2\pi i \frac{1}{3}})$.

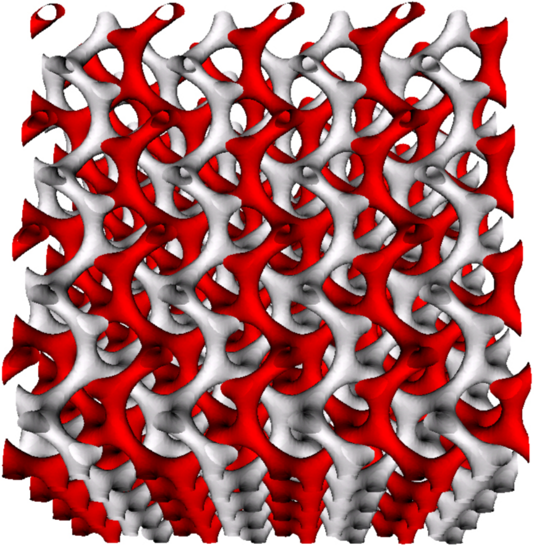
Remark

The two ramification points play a special role in graphene where they are known as Dirac points.

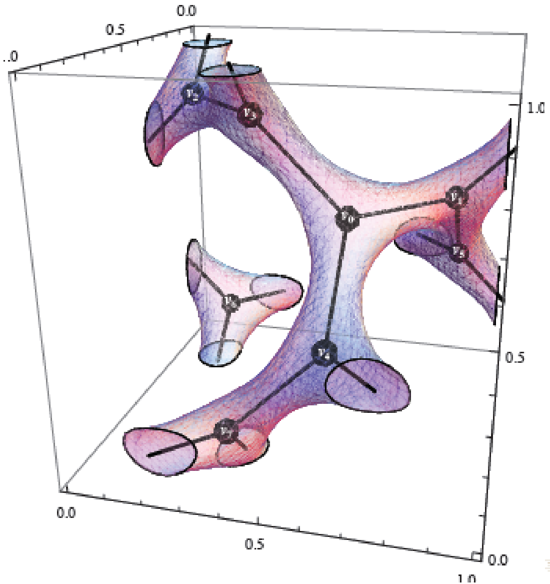
The fat surface F for the Gyroid



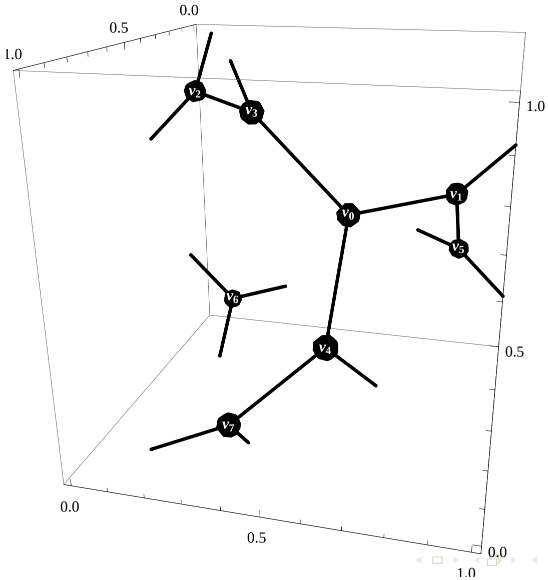
The two channel systems C_+, C_-



The Channel C_+ with its skeletal graph Γ_+



The skeletal graph Γ_+



Example 3: The Gyroid case

Data

- L for Γ_+ is the bcc lattice spanned by the vectors f_i or g_i.
- T is the fcc lattice spanned by the edge vectors e_4, e_5, e_6.
- In Hilbert space decomposition the Graph Harper Operator H becomes the 4×4 matrix

$$H = \begin{pmatrix} 0 & U_1^* & U_2^* & U_3^* \\ U_1 & 0 & U_6^* & U_5 \\ U_2 & U_6 & 0 & U_4 \\ U_3 & U_5^* & U_4^* & 0 \end{pmatrix}$$

- Magnetic Field Parameters:

$$\theta_{12} = \frac{1}{2\pi} B \cdot (g_1 \times g_2), \theta_{13} = \frac{1}{2\pi} B \cdot (g_1 \times g_3), \theta_{23} = \frac{1}{2\pi} B \cdot (g_2 \times g_3)$$

The Gyroid case

The matrix Harper Operator

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & U_1^* U_6^* U_2 & U_1^* U_5 U_3 \\ 1 & U_2^* U_6 U_1 & 0 & U_2^* U_4 U_3 \\ 1 & U_3^* U_5^* U_1 & U_3^* U_4^* U_2 & 0 \end{pmatrix} =: \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & A & B^* \\ 1 & A^* & 0 & C \\ 1 & B & C^* & 0 \end{pmatrix}$$

The coefficients can be expressed in terms of the operators of the magnetic translation operators of the bcc lattice. Set $U := U_{f_1}, V := U_{f_2}$ and $W := U_{f_3}$.

$$A = aV^* W, \quad B = bWU^*, \quad C = cW^* UV \tag{1}$$

with a, b, c given explicitly in terms of the magnetic field. A, B, C span a $\mathbb{T}^3_\Theta$:

$$AB = \alpha_1 BA, \quad AC = \bar{\alpha}_2 CA, \quad BC = \alpha_3 CB$$

Results for the Gyroid

Theorem

If $\Phi \neq 1$ or $\Phi = 1$ and at least one $\alpha_i \neq 1$ and all ϕ_i are different then $\mathscr{B}_\Theta = M_4(\mathbb{T}^3_\Theta)$ and $K(\mathscr{B}_\Theta) = K(\mathbb{T}^3)$.
If $\phi_i = 1$ for all i (commutative case) then $K(\mathscr{B}_\Theta) = K(X)$ where X is a ramified cover of the 3–torus with explicitly given ramification locus (consisting of four isolated points).
In all other cases $\mathscr{B}_\Theta \subsetneq M_4(\mathbb{T}^3_\Theta)$.

Parameters

$\alpha_1 := e^{2\pi i\theta_{12}}, \bar{\alpha}_2 := e^{2\pi i\theta_{13}}, \alpha_3 := e^{2\pi i\theta_{23}}$
$\phi_1 = e^{\frac{\pi}{2} i\theta_{12}}, \quad \phi_2 = e^{\frac{\pi}{2} i\theta_{31}}, \quad \phi_3 = e^{\frac{\pi}{2} i\theta_{23}}, \quad \Phi = \phi_1\phi_2\phi_3$

Questions

Empirical data

In all cases, the degenerate points are the ones one can compute from the projective action of graph symmetries. There seems to be no *a priori* proof however. Not even for the dimension of this locus.

Duality?

In all cases, the (maximal) dimension of the locus of enhanced symmetries in the commutative case coincides with the dimension of the locus of points where $\mathscr{B}_\Theta$ is not the full matrix algebra.

Basic setup

A family of Hamiltonians.

$$H : T \to Herm^k$$

(Usually, $T = T^d$ a d–dimensional torus, and the family is generically non–degenerate and smooth).

Structures

1. Universal action $Herm^k \times \mathbb{C}^k \to \mathbb{C}^k$.
2. Eigenvalue geometry. Branched covers. Singularities at branch points $\rightsquigarrow$ singularity theory.
3. Eigenbundle geometry. Line bundles. $\rightsquigarrow$ Chern classes/topological charges.
4. NCG of Eigenvalue geometry is $\mathscr{B}$. NCG of Eigenbundle geometry not so clear. Numerics.

Results for Examples:

P_n

This produces the trivial self cover $T^n \to T^n$. It becomes interesting in the projective setting.

D_3 Honeycomb/Graphene

In the commutative case of there are two degenerate points in the spectrum, which are cone–like/viz. Dirac. These are the famous graphene Dirac points

D_4 Diamond

Here there are three circles of double degeneracies that mutually touch in two points

Gyroid: A_3 singularity and its strata

Singularities

- two cusps: in stratum of type A_2
- double point: in stratum of type (A_1, A_1)

Theorem [KKWK]

The singular points of X for the Gyroid are given exactly by the above (analytically). And the four A_1 singularities are all Dirac points.

The spectrum of the Gyroid Harper Hamiltonian along the diagonals

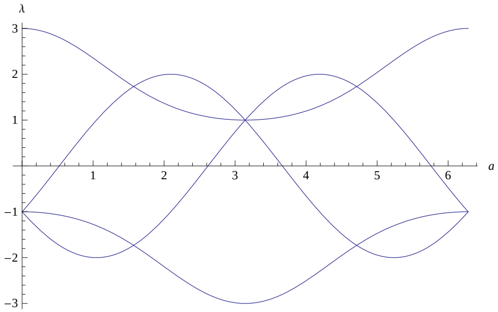


Figure: Spectrum of Harper Gyroid Hamiltonian for $a = b = c$

Bundle geometry

Bundle geometry

- Trivial vector bundle $T \times \mathbb{C}^k \to T$.
- T_{deg} be the locus of points s.t. $H(t)$ has multiple Eigenvalues. $T_0 := T \setminus T_{deg}$.
- $$\begin{array}{ccccc} T \times \mathbb{C}^k & \hookleftarrow & T_0 \times \mathbb{C}^k & \xrightarrow{\sim} & \bigoplus_{i=1}^k \mathscr{L}_i \\ \downarrow & & \downarrow & \swarrow & \\ T & \hookleftarrow & T_0 & & \end{array}$$

  Need that Eigenvalues are real.
- $c_1(\mathscr{L}_i)$ are the charges corresponding to the Berry phases. Integral over Berry curvature ω [Berry, Simon].

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  Need that Eigenvalues are real.
- $c_1(\mathscr{L}_i)$ are the charges corresponding to the Berry phases. Integral over Berry curvature ω [Berry, Simon].
- There are versions for higher degeneracies involving higher Chern–classes. Not today.

Chern classes

2d

If T is two–dimensional compact. Then the Chern classes are given by $\int_T \omega$. This is what happens in the quantum Hall effect. Here $T = T_0 = T^2$. Notice that if $T = T^2$ but $T_{deg} \neq \emptyset$, then all $c_1(\mathscr{L}_i) = 0$. This is the case for graphene $\rightsquigarrow$ Dirac points not topologically protected.

3d

The Chern classes are determined by their pairing with $H_2(T_0, \mathbb{Z})$. If $T = T^3$ there is nice method to encode this using slicing.

Slicing

Setup

- $\pi_i : T^3 = S^1 \times S^1 \times S^1 \to S^1$ the i–th projection.
- $\imath(t) : T^2 = S^1 \times S^1 \to T^3 = S^1 \times S^1 \times S^1$ inclusion $(t_1, t_2) \mapsto (t_1, t_2, t)$.
- $c^i(t) := \int_{T^2} \imath(t)^* c_1(\mathscr{L}_i)$ for $t \notin \pi_3(T_{deg})$.
- For $t \in \pi_3(T_{deg})$ set $c^i(t) := 0$. This is also the result of pulling back the Chern class to $T^2 \setminus \imath(t)^{-1}(T^{deg})$.
- There are of course similar definitions for the other two inclusions and higher dimensions.

Proposition

If T_{deg} is discrete, one can arrange that the $c^i(t)$ for all three projections completely determine the line bundles $\mathscr{L}_i$. (In fact slightly less is needed.)

Chern jumps and local charges

Local charges/jumps

T three dimensional, p isolated point in T_{deg}. The local charges at p are $c^i_{loc}(p) = \int_{S^2(p)} c_1(\mathscr{L}_i)$ where $S^2(p)$ is a little sphere centered at p.

A local model (Berry, Simons, ...) in 3d for an isolated $2k+1$–dimensional crossing

$H(\mathbf{x}) := \mathbf{x} \cdot \mathbf{L} = xL_x + yL_y + zL_z$ where $L_{x,y,z}$ is a k dimensional representation of spin m.

The local charges are $c^i_{loc} \in \{-m, \dots, m\}$.

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Jumps for T^3

Assume for convenience that π_3 is locally bijective at p. By Stokes:
$c^i(\pi_3(p) + \epsilon) - c^i(\pi_3(p) - \epsilon) = c^i_{loc}(p)$

Questions

Local models

For a double crossing/Dirac point, the above model is the only model. What are the other local models for higher degeneracies? Phase diagram?

Global properties

1. Depending on properties of $H(t)$ can one say something directly about the $\mathscr{L}_i$ or the c^i?
2. How much does this determine them? Examples:
   $\sum_i c^i(t) \cong 0$ always.
   If there is time reversal symmetry $c^i(t) = -c^i(-t)$.
3. How much does knowing the local models determine the global structure?
4. What is the behavior under perturbations?

Our favorite Example, the Gyroid. Newest results

Local Models

The A_1 singularities have the spin local model as needed, but also the A_2 singularities are locally diffeomorphic to the spin 1 case.

Local to global

The local structure of singularities and time reversal symmetry completely determines the functions c^i.

Deformations preserving time reversal symmetry

Numerically, the Dirac points are stable as expected. The A_2 singularities split into four A_1 singularities. This is *a priori* unexpected. *A posteriori* it can be explained as the minimal possible splitting, using the global structure and preserved time reversal symmetry.

Plots

Undeformed case

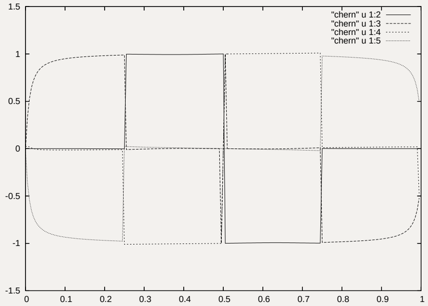

Figure: Slicing along z numerically, can prove anaytically. Corollary: Dirac points in Gyroid are stable

Plots

Deformed case

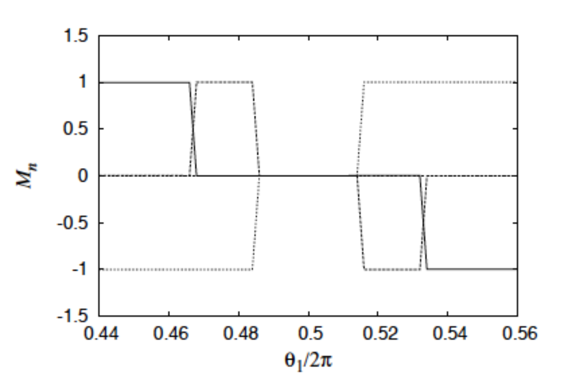

Figure: Slicing along z numerically near the old A_2. This breaks up into *four* A_1 points

New geomtry from time reversal symmetry TRS, joint with D. LI and B. KW.

Basic remarks

1. The global results where possible because of TRS.
2. If $\mathscr{T}$ reverses time then $\mathscr{T}^{-1}H\mathscr{T} = \bar{H}$. That is the vector bundles and Eigenbundles are in KR.
3. Furthermore there is no gap in the Honeycomb! But there is Spin QHE. For this one needs to upgrade $\mathscr{L}_i$ to spinors.
   1. This is possible, and one actually adds a term to the action: Spin-Orbit coupling $\rightsquigarrow$ gap (Haldane, Kane-Mele)
   2. There are topological invariants associated to this. These are not the Chern classes as they are zero. They are $\mathbb{Z}/2\mathbb{Z}$ valued invariants. (Kane-Mele,Balents–Moore, Kitaev, Moore-Freed).
   3. These have several incarnations. Such as winding numbers, odd Chern characters, Chern-Simons mod 2 or simply KR, KO, KH. (Not that easy to sort out.)

Time Reversal

General setup

The time reversal operator Θ is an anti-unitary operator, i.e.,

$$\langle \Theta\psi, \Theta\phi \rangle = \langle \phi, \psi \rangle, \qquad \Theta(a\psi + b\phi) = \bar{a}\Theta\psi + \bar{b}\Theta\phi$$

For a spin-$\frac{1}{2}$ particle such as an electron, it has the property

$$\Theta^2 = -1 \tag{2}$$

which results in the Kramers degeneracy, i.e., all energy levels are doubly degenerate in a time reversal invariant electronic system.

Kramers degeneracy meant that the vector bundle of states may only split

$$\mathscr{V} \simeq \bigoplus V_n \to T^d$$

with $rk(V_n) = 2$ and $c_1(V_n) = 0$.

Time reversal invariants

Invariant models

A time reversal invariant model is required to have $[H(\mathbf{r}), \Theta] = 0$, or in the momentum representation

$$\Theta H(\mathbf{k})\Theta^{-1} = H(-\mathbf{k}) \tag{3}$$

Time reversal invariant (TRI) points

By the above Θ induces an action on T^d (parameterizing k). The fixed points for this action are called TRI points. Notice T^2 has 4 such points with coordinates 0 or π and T^3 has 8 such points.

Spin orbit

SO-Hamiltonian Kane-Mele

$$H_{KM} = \sum_{i=1}^{5} d_i(\mathbf{k})\Gamma_i + \sum_{1=i<j}^{5} d_{ij}(\mathbf{k})\Gamma_{ij} \qquad (4)$$

where the gamma matrices are

$$\mathbf{\Gamma} = (\sigma_x \otimes s_0, \sigma_z \otimes s_0, \sigma_y \otimes s_x, \sigma_y \otimes s_y, \sigma_y \otimes s_z)$$

with the Pauli matrices s_i representing the electron spin and

$$\Gamma_{ij} = \frac{1}{2i}[\Gamma_i, \Gamma_j]$$

The time reversal operator

$$\Theta = i(\sigma_0 \otimes s_y)K \qquad (5)$$

Kane–Mele Invariant

Local matrix representation on $V_n \to T^d$ (rk 2 bundle)

$$w_n(\mathbf{k}) = (< u_n^s(-\mathbf{k}), \Theta u_n^t(\mathbf{k}) >) = \begin{pmatrix} 0 & -e^{-i\chi_n(\mathbf{k})} \\ e^{-i\chi_n(-\mathbf{k})} & 0 \end{pmatrix} \in U(2) \tag{6}$$

Kane-Mele Fomula

At the TRI points w is skew–symmetric.

$$(-1)^\nu = \prod_{\Gamma_i \in \mathbf{\Gamma}} \frac{\sqrt{\det w_n(\Gamma_i)}}{pfw_n(\Gamma_i)} \tag{7}$$

for the fixed points $\mathbf{\Gamma}$ of the time reversal symmetry.

Other interpretations

There are a lot more ways to define this invariant (reason for paper w. D. Li and B K-W.

1. Via determinant line bundles.
2. Via polarization.
3. Via $\nu \equiv \mathbf{n} - \mathbf{h} \pmod 2$ where $\mathbf{n}$ is a half winding number and $\mathbf{h}$ is a holonomy.
4. Maslov index/η invariant.
5. In 3d it is related to Chern–Simons theory, the odd Chern character, the mod 2 index theorem and (next).
6. Parity anomaly.
7. Via homotopy/K-theory.

Chern-Simons

Idea

Think of w as a $U(2)$–gauge transformation g, and H as Dirac operator D.

Chern-Simons invariant

$$\upsilon \equiv \frac{1}{24\pi^2}\int_{\mathbb{T}^3} d^3k\, tr(w^{-1}dw)^3 \pmod 2 \tag{8}$$

Spectral flow

$$sf(D, g^{-1}Dg) = \frac{1}{\sqrt{\pi}}\int_0^1 tr(\dot{D}_t e^{-D_t^2})dt \tag{9}$$

$D_t = (1-t)D + tg^{-1}Dg, \quad \dot{D}_t = g^{-1}[D, g]$

Index theorem

Paring

$$index(PgP) = \langle [D], [g] \rangle = -sf(D, g^{-1}Dg) \tag{10}$$

where $P := (1 + D|D|^{-1})/2$ is the spectral projection.

Toeplitz index theoerm

$$sf(D, g^{-1}Dg) = \int_M \hat{A}(M) \wedge ch(g) \tag{11}$$

where $\hat{A}$ is the A-roof genus and $ch(g)$ is the odd Chern character of $g \in K^{-1}(M)$, M underlying spin manifold.

$$ch(g) := \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} tr[(g^{-1}dg)^{2k+1}] \tag{12}$$

3d situation

3–torus

In particular, we have $\hat{A}(T^3) = 1$ since $\hat{A}$ is a multiplicative genus and $\hat{A}(S^k) = 1$ for spheres. Hence the degree of g can be computed as the spectral flow on the 3d Brillouin torus,

$$sf(D, g^{-1}Dg) = -\left(\frac{i}{2\pi}\right)^2 \int_{\mathbb{T}^3} ch(g) = \deg g \qquad (13)$$

Putting all together

$$\upsilon \equiv sf(H_e, w^{-1}H_e w) \mod 2 \qquad (14)$$

Main identity (Wang-Qi-Zhang,Freed-Moore)

$$\upsilon = \nu$$

Symmetries and K-theory

Three types of discrete (pseudo)symmetries

Time reversal symmetry $\mathcal{T}$, the particle-hole symmetry $\mathcal{P}$ and the chiral symmetry $\mathcal{C}$ (Wigner-Dyson, Altland and Zirnbauer, Kitaev).

H is TRI if $\mathcal{T}H\mathcal{T}^{-1} = H$, and $\mathcal{T}^2 = \pm 1$ depending on the spin being integer or half-integer,

$$TRS = \begin{cases} +1 & \text{if} \quad \mathcal{T}H(\mathbf{k})\mathcal{T}^{-1} = H(-\mathbf{k}),\ \mathcal{T}^2 = +1 \\ -1 & \text{if} \quad \mathcal{T}H(\mathbf{k})\mathcal{T}^{-1} = H(-\mathbf{k}),\ \mathcal{T}^2 = -1 \\ 0 & \text{if} \quad \mathcal{T}H(\mathbf{k})\mathcal{T}^{-1} \neq H(-\mathbf{k}) \end{cases} \tag{15}$$

Similarly, the particle hole symmetry (PHS) also gives three classes,

$$PHS = \begin{cases} +1 & \text{if} \quad \mathcal{P}H(\mathbf{k})\mathcal{P}^{-1} = -H(\mathbf{k}),\ \mathcal{P}^2 = +1 \\ -1 & \text{if} \quad \mathcal{P}H(\mathbf{k})\mathcal{P}^{-1} = -H(\mathbf{k}),\ \mathcal{P}^2 = -1 \\ 0 & \text{if} \quad \mathcal{P}H(\mathbf{k})\mathcal{P}^{-1} \neq -H(\mathbf{k}) \end{cases} \tag{16}$$

Chiral symmetry

Chiral symmetry

The chiral symmetry can be defined by the product $\mathcal{C} = \mathcal{T} \cdot \mathcal{P}$, sometimes also referred to as the sublattice symmetry. Since $\mathcal{T}$ and $\mathcal{P}$ are anti-unitary, $\mathcal{C}$ is a unitary operator.

Special case

If both $\mathcal{T}$ and $\mathcal{P}$ are non-zero, then the chiral symmetry is present, i.e., $\mathcal{C} = 1$. On the other hand, if both $\mathcal{T}$ and $\mathcal{P}$ are zero, then $\mathcal{C}$ is allowed to be either 0 (type A or unitary class) or 1 (type AIII or chiral unitary class).

10 fold way

In sum, there are $3 \times 3 + 1 = 10 = 8 + 2$ symmetry classes. In particular, the half-spin Hamiltonian with time reversal symmetry falls into type AII or symplectic class, which is the case we are mostly interested in.

K–theories

Symmetries

The symmetries are related to KR, KH, KO according to the action on the base and the fibers. Notice, that $\pi : (V_i, \Theta) \to (T^d, \mathcal{T})$ is a quaternionic bundle since Θ is the lift of $\mathcal{T}$ such that $\Theta^2 = -1$.

Twisted equivariant matter (Freed–Moore)

Generalization of the above classification with possible twists.

Summary/Questions

1. C^*–geometry from condensed matter system. (NCG and CG)
2. Extra topological information by slicing. Stability under TRI perturbations. Q: What is the NCG of this?
3. Several versions of $\mathbb{Z}/2Z$. Q: Which one is good/useful in NCG?
4. This is also related to Bulk/boundary correspondence. Q: can we get something for NCG?

The end

Thank you!