CFT from the arc point of view and structural relations to planar algebras.

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References

Survey

Arc Geometry and Algebra: Foliations, Moduli Spaces, String Topology and Field Theory.
To appear in Handbook of Teichmueller Theory.
Preprint available on my webpage.
`http://www.math.purdue.edu/~rkaufman/pubs.html`

References

Geometry/Topology

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Actions/Cell models

- *Moduli–space actions on the Hochschild co-chain complex I: Cell models*. Journal of Noncommutative geometry, Vol 1, pp. 333–384,
- *Moduli–space actions on the Hochschild co-chain complex II: Correlators*. Journal of Noncommutative Geometry 2, 3 (2008), 283-332.
- *Open/Closed String Topology and Moduli Space Actions via Open/Closed Hochschild Actions*. SIGMA 6 (2010) 036, 33 pages.

General Overview

Main facts

There are two graphical versions which capture (some aspects) of CFT.

1. V.F.R. Jones's planar algebras
2. The $\mathcal{ARC}$ operad and its cousins.

Main boxed statement

There must be a relationship between these two shadows

Main goal

Make this more precise and use it to cross–fertilize.

Outline

Arc CFT

CFT as algebras over an operad

Just as TFTs are algebras over a certain PROP, that of Frobenius algebras, CFTs are can be thought of as algebras over the Segal PROP.
One can equivalently think about functors from cobordism categories.

Several Models

There is a slight problem when going from the topological to the conformal case as the gluing data gets complicated. One way out is to use $B\Gamma$, where Γ is the mapping class group. Other models have been used by Segal, Kriz, Stolz-Teichner, etc.

Our Model

We use the combinatorial model of Moduli space.

Context

Relation to TFT/CFT

Geometry	data (roughly)	Theory
Topological surfaces w/ boundary Cobordism	$(\sigma, \partial\Sigma)$	TFT
Surface w/ conformal structure/boundary "Segal operad/category" $M_{g,n}$	$(\sigma, \partial\Sigma, [g])$	CFT
Complex curve w/ marked points/ $\bar{M}_{g,n}$	$(C, p_1, \dots, p_n)$	CohFT GW invariants
Foliations	$(\Sigma, \partial\Sigma, p_i \in \partial_i\Sigma, [\alpha])$	Hyp CFT π_0 gives TFT.

Levels of the construction

Aspects of the $\mathcal{ARC}$ theory

1. Continuous
   - Topological level: Operad, PROP. CFT, π_0 gives TFT
   - Chain level: Operators, Algebra up to homotopy, e.g. BV up to homotopy A_∞.
   - Homology level: Operators, Algebras. BV, Gerstenhaber structure
2. Discrete
   - Discrete partial suboperad ($\mathbb{N} \subset \mathbb{R}_{>0}$)
   - Combinatorial indexing on cell level
   - Discretization for action.

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   - Homology level: Operators, Algebras. BV, Gerstenhaber structure
2. Discrete This is what links to planar algebras
   - Discrete partial suboperad ($\mathbb{N} \subset \mathbb{R}_{>0}$) No signs
   - Combinatorial indexing on cell level
   - Discretization for action. Signs

Actions

Types of Actions

1. continuous: loop space recognition
2. discrete: several different versions
   - On tensor algebra (cyclic bar complex) of a (Frobenius) algebra.
     Our main line of applications so far.
   - Open/closed version on double sided bar complex.
   - On tensor algebra of a vector space.
     Exists. Has direct connection to planar diagrams.

Main difference

Module variable at marked points

Correlators

Physics

We should have some algebra of fields M and correlation functions

$$\langle \phi_1, \dots, \phi_n \rangle_\Sigma$$

for $\phi_i \in M$ and Σ a surface with conformal structure.

Chain level

We will give a chain level structure, that is. The is an for an (open) cellular decomposition of (open) moduli space, whose cells are indexed by graphs Γ on a topological surface F (with extra data). We will give correlation functions

$$\langle \phi_1, \dots, \phi_n \rangle_{\Gamma,F}$$

here the algebra of fields will be $M = CH^*(A, A)$ or in the open/closed case $B(N, A, N)$

Physics: Moving Strings and their Interactions

String Slogan

As strings move they sweep out a surfaces

Physics: Moving Strings and their Interactions

String Slogan

As strings move they sweep out a surfaces

Our Slogan

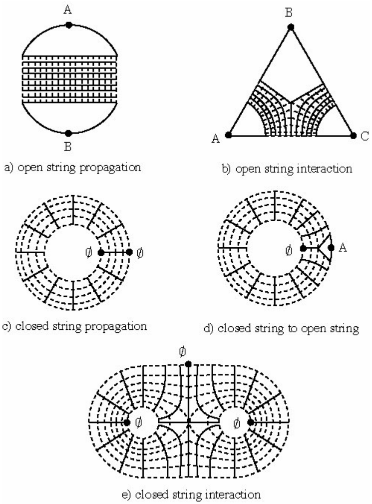
As strings move they sweep out a partially measured foliation

Moving strings and Foliations

Dynamic picture

- Think of strings as parameterized S^1s and intervals.
- The endpoints of the intervals are labeled by the brane labels. For the circles we label the image of 0 by $\emptyset$.
- As the strings move, separate and recombine they sweep out a surface.
- the image of the moving strings gives us a foliation perpendicular to the strings. Leaves are the trajectories of points.
- The parameterization of the string gives us a transversal measure.

Moving strings and Foliations



- Partially measured foliations — solid lines
- Transversal ("squeezed") string foliation — dashed lines
- A, B, C are D-brane labels
- $\emptyset$ indicates closed string
- Singular leaves included

Moving strings and Foliations

Geometric encoding

- A surface with boundary and brane–labeled points on the boundary, together with a partially measured foliation not hitting the marked points.
- Notice that this foliation does not have to fill the surface. We can squeeze the leaves together to form bands of a given width
- So we could also replace a band by the data of one (non–singular) leaf and a real number, viz. the width.

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Basic characters

Surfaces with arcs

The basic building block are surfaces

- with enumerated boundary components
- a window in each boundary component
- arcs running from window to window

considered up to homotopy and action of the mapping class group

Extra structures

1. (projective) positive weights $\rightsquigarrow$ topological version
2. positive integer weights $\rightsquigarrow$ actions
3. nothing $\rightsquigarrow$ graphs indexing cells and combinatorics.

Data for open/closed

Can/will do open/closed version.

This means

1. Add more points on the boundary.
2. Label points with a set of D–brane labels $\mathcal{B}$
3. $\emptyset$ will mean "closed string"
4. For the gluing structure we will use power set $\mathcal{P}(\mathcal{B})$. Think "intersections of branes".

Data (F, β)

- A surface $F^s_{g,r}$ genus g, r boundaries, s punctures.
- Points $p_i, i \in I$ on the boundaries (at least one per boundary)
- a brane labelling: $\beta, \{p_i\} \to \mathcal{P}(\mathcal{B})$.
  $\emptyset$-label only possible if p_i only point on the boundary.
  $n = \#$ of $\emptyset$ labels and $m = \#$ other labels.

Moving strings and Foliations: some families

Bands vs. graph

Bands indicated by one non-singular leaf.
Width of the band given be a positive number, also called weight.

Rules

1. no crossings
2. not incident to the marked points on the bounday
3. not parallel to each other
4. not parallel to the boundary
   Brane labelled point *not* part of boundary.
   $\emptyset$ labelled point part of the boundary.

(Conditional) Gluing

The gluing procedure

- Fix two windows, can be on the same surface or on different surfaces.
- If there is a marking by $\emptyset$ then both boundaries must be marked by $\emptyset$. Only glue closed to closed and open to open.
- When the widths agree, match the bands and cut along them according to the common partition.
- Remove any closed leaves.

Scaling version

If we do not allow self gluing, then we could scale all weights by a common factor. This was done in [KLP] for closed to closed. One gets the same answer on homology for the non-self gluings.

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- Remove any closed leaves.
  This looks like $\delta = 0$ or $\delta = 1$ (later)

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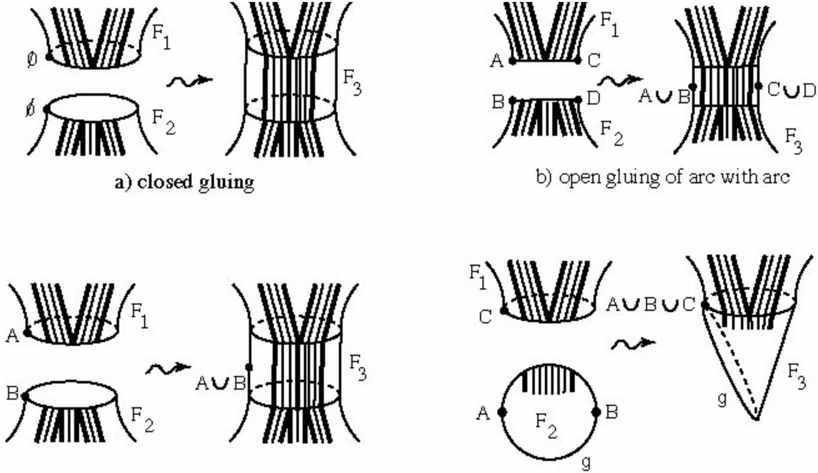
Types of Gluings



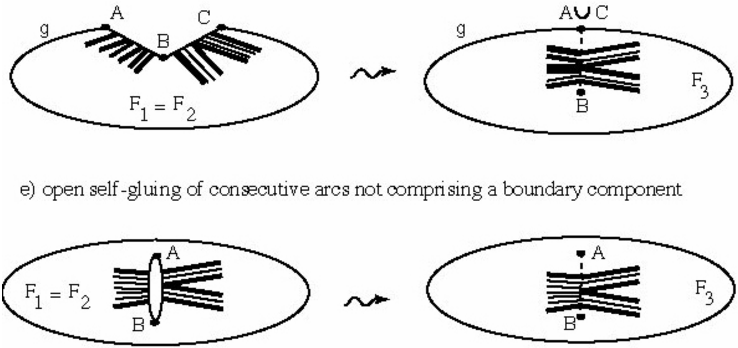
a) closed gluing

b) open gluing of arc with arc

c) open gluing of curve with curve

d) open gluing of arc with curve

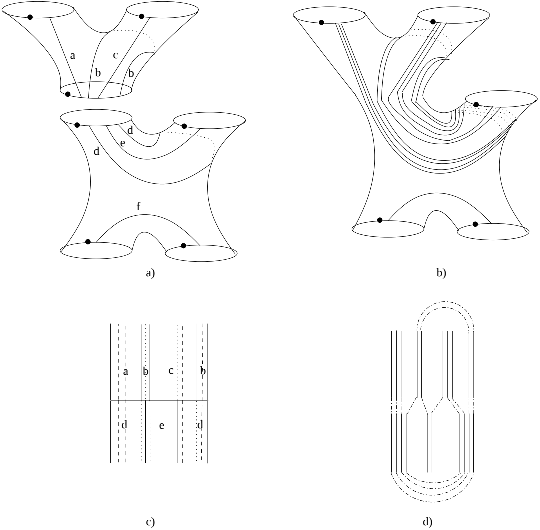
Gluing



e) open self-gluing of consecutive arcs not comprising a boundary component

f) open self-gluing of consecutive arcs comprising a boundary component

Local Gluing/Global Effects



Theorems

Topological

1. The gluings give the structure of a topological operad. Using the scaling action this is a cyclic operad. Using the $\mathbb{R}_{>0}$ color, it is colored modular.
2. In the o/c version we get a c/o structure. This basically means bi–colored, $\mathbb{R}_{>0}$-colored modular.
3. π_0 gives a new proof of minimality of Cardy–Lewellen axioms, using Whitehead moves.

Remarks

- To get an unconditional gluing, all boundaries must be hit.
- The closed theory is a suboperad. (This is the original $\mathcal{ARC}$)
- Can modify gluing and change the space. Careful!

Theorems

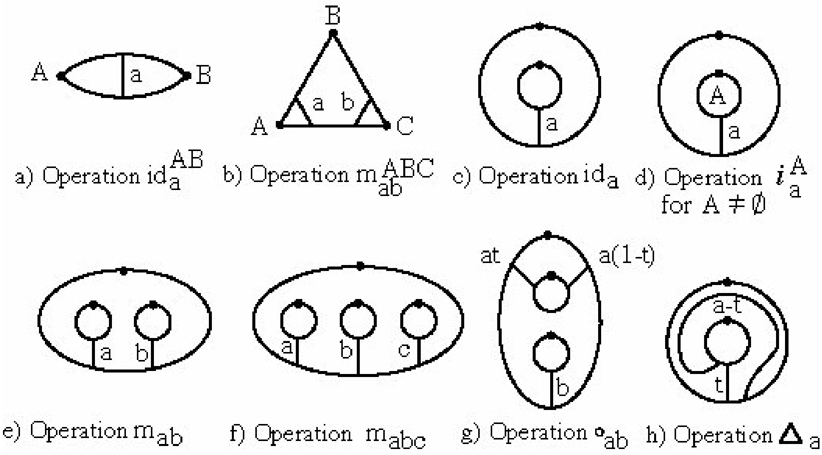
Homology level

1. We get a bi–modular operad in the open/closed case. Modular in the closed case.
2. This can be restricted to the cyclic case where it coincides with the cyclic operad from the scaling version.

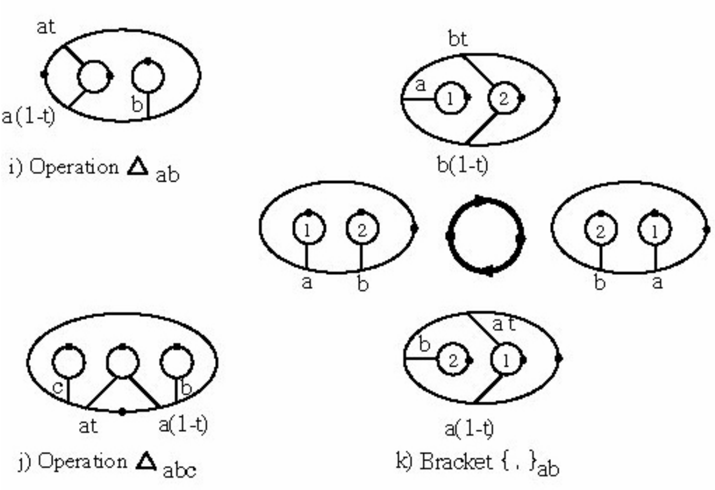
Chain level/family gluing

1. We get a chain level operad/PROP. This uses intricate flows for the $\mathbb{R}_{>0}$ colored version. This is what is used in the proofs about homology.
2. This can be thought of as family operations, that is operators with moduli, which glue.
3. Important for applications to String Topology (Chas Sullivan).
4. Also this is where the pre–Lie product, the Gerstenhaber bracket and the BV operator live. All these are of degree 1.

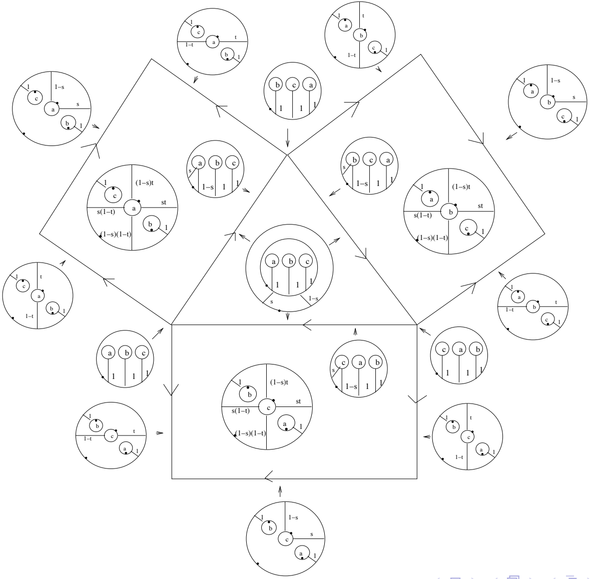
Operators: Degree 0 and BV



Operators: Degree 1



Relations: The BV equation



TFT and CFT

Moduli space

- The moduli space of n-punctured surfaces with a tangent vector at each sits inside this space of the $\mathcal{ARC}$ operad. (Closed strings only). It is the locus where the arcs quasi-fill, i.e. cut up the surface into polygons. These are the spaces for our hyperbolic/combinatorial CFT.
- Also including the locus of open strings with the same condition, we get a definition of open/closed hyperbolic/combinatorial CFT.

Moduli space/CFT

Theorem

1. *The subspace of quasi–filling arc graphs on $F^0_{g,n}$ is homeomorphic to the moduli space of genus g surfaces with n punctures and one tangent space at each puncture.*
2. *These spaces form a cyclic rational operad (densely defined compositions).*
3. *They induce a cell level action where the cells are labelled by quasi–filling graphs of arcs w/o weights.*

Remarks

- The cell complex computes the cohomology of moduli space
- The corresponding cell level action for this is on the associated graded. This means $\delta = 0$ in the sense that graphs with closed loops are codim 2 and projected out.

String topology I, Sullivan PROP

Sullivan chord diagrams

Divide boundaries into Ins and Outs, and allow arcs only running from In to Out. Moreover all In boundaries are hit.

Theorem

1. *The weighted arc graphs of the above type form a quasi–PROP (associative only up to homotopy).*
2. *The have a CW–model and the induced structure on the cellular chains is a PROP.*
3. *The chains are again indexed by the graphs of the above type.*

Hochschild actions

Motivation using the logic of Kontsevich-Kapustin-Rozansky

- The closed string states are deformations of the open string states.
- The open string states are represented by a category of D-branes.
- Hence the closed strings should be elements of the Hochschild co–chains of the endomorphism algebra of this category.
- Now thinking on the worldsheet, we can insert closed string states. That is, for a world sheet, we should get a correlator by inserting, say n closed string states.
- This is what we have done, if one simplifies to a space filling D-brane and twists to a TCFT.
- In one includes open strings, then one should look at bi–modules.

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Hochschild actions

Theorems

1. Deligne's conjecture, first proof of cyclic Deligne conjecture.
2. Definition of Chas–Sullivan quasi–PROP, CW chain level PROP and rigorous algebraic Chas–Sullivan string topology action.
3. There is indeed an action on the Hochschild co–chains of a Frobenius algebra by the relevant moduli space.

Remarks

The action is through correlators and these are given by discretizing the foliations.
The moduli action uses an associated graded construction. $\delta = 0$.

The action in the closed case

Correlators

Using the isomorphisms for a Frobenius algebra $A \simeq \check{A}$ over the ground field F the operations are in

$$
\begin{aligned}
& Hom(CH^*(A,A)^{\otimes k}, CH^*(A,A)^{\otimes l}) \\
\simeq & \bigoplus_{\mathbf{n},\mathbf{m}} Hom(A^{\otimes n_1} \otimes \cdots \otimes A^{\otimes n_{k+1}}, A^{\otimes m_1} \otimes \cdots \otimes A^{\otimes m_{l+1}}) \\
\simeq & Hom(CH^*(A,A)^{\otimes k+l}, F) \\
\simeq & Hom((TA)^{\otimes k+l}, F)
\end{aligned}
$$

Remarks

We will give the homogenous components corresponding to a surface. We have to be careful however that these identifications are not dg for the Hom complex. (more on this later)

The operations

Angle markings

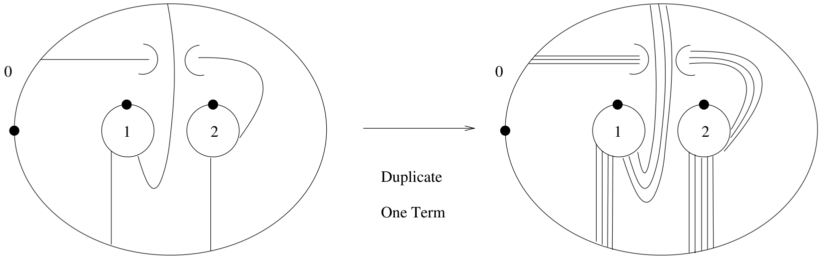
We will consider surfaces with arcs and fixed angle markings. That is a map of the flags of the graph of arcs to $\{0, 1\}$

The step by step instructions

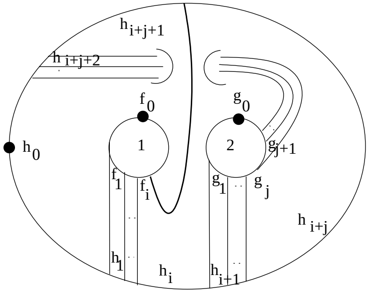
For a surface with arcs S with N boundaries. Fix $(n_1, \dots, n_N)$ for homogenous $\phi_i \in TA^{\otimes n_i}$.

1. Duplicate the arcs on the i-th boundary, such that there are n_i angles with decoration 1.
2. The new angles are all decorated by 1.
3. If this is not possible the operation is 0.
4. else decorate the angles marked by 1 by given elements ϕ_i.
5. Sum over all $n_1, \dots, n_N$

Example



Example: All angles marked by 1



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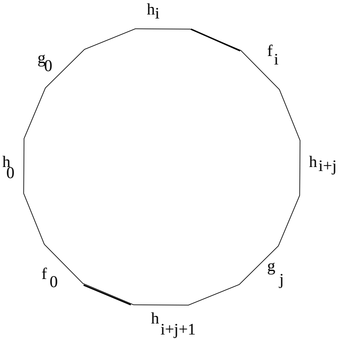


II

The operations

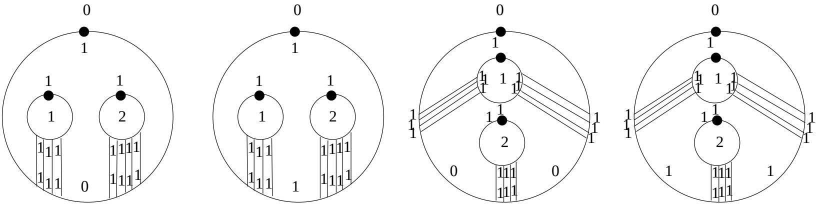
The formula

Fix F and an arc family Γ. Notice: The complementary regions of the arcs are surfaces with boundary. Let $\langle \quad \rangle$ be the paring for A and set $\int a =< a, 1 >$

$$\langle \phi_1, , \phi_N \rangle_{F,\Gamma,n_1,\dots,n_N} = \prod_{\text{complementary regions } P} \int \prod_{\angle\text{: angle decorated by } 1} a_\angle e^{-\chi(P)+1}$$

where $a_\angle$ are the tensor factors of the ϕ_i and $\int a = \langle a, 1 \rangle$.

Differences between the cases



1. $f^n \cup g^m(a_1, \dots, a_{n+m-1}) = f^n(a_1, \dots, a_n) g^m(a_{n+1}, \dots a_{n+m})$
2. $f^n \sqcup g^m(a_1, \dots a_{n+m+1}) =$ $f^n(a_1, \dots, a_n) a_{n+1} g^m(a_{n+2}1, \dots, a_{n+m+1})$
3. $f^n \circ g^m = f(a_1, \dots, a_{n+m-1}) =$ $\sum_i \pm f^n(a_1, \dots, a_{i-1}, g^m(a_i, \dots a_{i_{m+i-1}}), a_{i+m}, \dots, a_{n+m+1})$.
4. $f^n \square g^m(a_1, \dots, a_{n+m+2}) =$ $\sum_i \pm f^n(a_1, \dots, a_{i-1}, a_i g^m(a_{i+1}, \dots a_{i_m}) a_{i+m+1}, a_{i+m+2}, \dots, a_{n+m+2}$

String topology

Standard Marking

Decorate all inner In angles by 1, all inner Out angles by 0 and all outer angles by 0.

Theorem

If A is commutative Frobenius algebra then the correlators for the standard marking yield a dg-PROP action on the reduced Hochschild co-chains

Remark

This generalizes our proofs of Deligne's and the cyclic Deligne conjecture.

CFT action

Standard marking

Mark all angles by 1

Theorem

Marking all angles by 1 yields operadic correlation functions, that is these maps induce the structure of an algebra over the cyclic chain operad of moduli space.

Remark

This marking corresponds to the Functor Operads of McClure and Smith. To get to the other marking, one can use the co–simplicial structure.

Starting Point

Older fact

The fact that the composition is operadic/PROPic in both cases comes from integer weighted familiies via a discretization map that assigns all possible integer weights $(n_1, \dots, n_N)$.
This map is operadic.

New observations after Vaughan F. R. Jones' talk at Purdue

This operadic map corresponds to an underlying combinatorial partial operad map, which glues multi–arcs if their number agrees.
This is like planar algebras.
Gives new perspective of closed leaves.

Vector action

Procedure

1. Duplicate edges as before
2. Decorate ends of arcs or flags with *dual* elements of V.
   This amounts to just taking $\langle v_{in}, v_{out} \rangle$ on each of the multiple arcs.
3. Notice no multiplication.

Closed leaves

1. Do not appear for the strict Sullivan–PROP
2. Can be algebraically quotiented out by using a filtration on a subspaces of operations.

Planar conection

Observation

- Up to closed leaves this is the same as in the planar algebra case (with hindsight).
- For a closed leaf one gets a contribution of $\delta = dim(V)$. Summing over all cabling diagrams, we need $0 \leq \delta \leq 1$ if we would like to include closed leaves in the sum.
- If $\delta = 0$ the terms with closed leaves would be absent from the sums. This is what we did when we take the associated graded.
- If $\delta = 1$ we can throw out the loops in the diagrams. This is what we did when we removed the loops.

Different aspects

Aspects

- Sums of diagrams vs. individual ones.
- Algebra vs. vector version:
  Module variable V_0.
- Families, higher order operations:
  Gerstenhaber, BV higher $\cup_i$.
- Hochschild/cosimplicial vs. planar algebra sequences.
  Relative tensor products $M \otimes_N M$.
- Different traces/states/pairings
- Shaded vs. open/closed

Comparison: what can we learn?

Arc $\to$ Planar

- Operations like G–bracket and BV
- Higher genus
- Module variable
- Open/closed version
- $\delta = 0$, filtration

Planar $\to$ Arc

- Relative version of action $N \subset M$, $M \otimes_N M \ldots$
- Going beyond Hochschild. Links to other theories
- Shading.
- More internal operations/relative theories
- $0 < \delta < 1$

Cosimplicial structure

Observation

The cosimplicial structure looks the same. But we use a module variable.

Import

The planar diagrams for the $s_i, \sigma_i, d_i, \delta_i$ from planar algebras. These give $\int, \mu, \Delta, 1$ (in two ways.)

Export

We get non–trivial Hochschild differential because of the module variables. The degeneracies (viz, the equation in the Delphi notes) "force" us to use the reduced Hochschild complex.

Shaded vs. open/closed

Observation

The action in the shaded regions is that of an open/closed theory.

Differences

In the $\mathcal{ARC}$ theory, "N has non–trivial dynamics."

In the planar theory one uses relative products.

Action

Technical things

1. We fix algebras Frobenius algebras $A_S, S \in \mathcal{B}$, we set $C := A_\emptyset$ (can weaken this)
2. We fix $r_S : C \to A_S$ inclusions, this makes A_S into a C–module
3. The action is on the collection of $CH^*(C, A_S \otimes A_T^{op})$, actually on the isomorphic double bar complex $B(A_T, C, A_S)$.
4. Bi–module structure works as expected. E.g. the multiplication $B(A_S, C, A_T) \otimes B(A_T, C, A_U) \to B(A_S, C, A_U)$ factors through $B(A_S, C, A_T) \otimes_{A_T} B(A_T, C, A_U)$

Open/Closed Action

The action is again given in three steps

1. Duplicate: Duplicate each edge a fixed number of times
2. Decorate: Decorate each boundary piece between edges by elements of C and by elements of the relevant algebra next to the markings
3. Integrate around the boundary of the polygons using weights and markings

Decoration and Weights

The boundary of the underlying surface minus the discrete representative of the graph is a disjoint union of intervals called *boundary pieces.* There are three types:

1. those not containing a marked point
2. those containing a marked point β–labelled by $\emptyset$
3. those containing a marked point with β–label not $\emptyset$.

Type	Decoration	Weight $\omega(s)$
(1) s without marked point	$a \in A_\emptyset$	a
(2) s with marked point marked by $\emptyset$	$a \in A_\emptyset$	a
(3) s marked point marked by S	$(a_S^{(1)}, a_S^{(2)})$, $a_S^i \in A_S$	$r_S^\dagger(a_S^{(1)} a_S^{(2)})$

Tabelle: General Weights

The Formula for the action

For a homogeneous
$\mathbf{a} = \bigotimes_{w \in \text{Windows of } \alpha} a_w \in \bigotimes_{w \in \text{Windows of } \alpha} B(\beta(w))$ such that $a_w \in B_{\alpha(w)-1}(\beta(w))$, we decorate as above and define

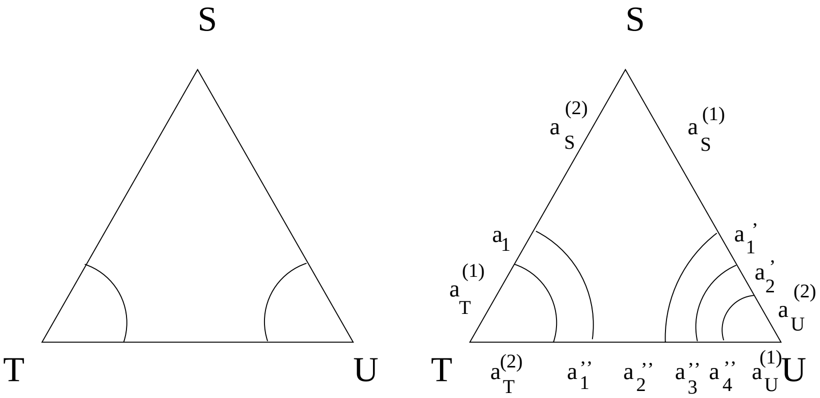
$$Y_{S_i}(\mathbf{a}) = \int e^{-\chi(S)+1} \prod_{\substack{\text{Decorated sides} \\ s \text{ of } S_i}} \omega(s) \prod_{\substack{\text{Punctures } p \\ \text{inside } S_i}} r^{\dagger}_{\emptyset\beta(p)}(e_{\beta(p)}) \tag{1}$$

If $\mathbf{a}$ is as above but there is some $a_w \notin B_{\alpha(w)-1}(\beta(w))$ we set $Y_{S_i}(\mathbf{a}) = 0$.
We then define

$$Y_{(\Gamma,w))}(\mathbf{a}) := \prod_i Y_{S_i}(\mathbf{a}) \tag{2}$$

and extend by linearity.

Action



Euler Condidtion

Definition

We say that a basic $\mathcal{B}$–FA satisfies the condition of commutativity (C) if $A_\emptyset$ is commutative.
And we say that a $\mathcal{B}$–FA satisfies the the Euler compatibility condition or the condition (E) if for all $B \in \mathcal{B}$, $a^{(1)}, a^{(2)} \in A_B$.

$$(E) \qquad \sum_{ij} r_B^\dagger(a^{(1)}\Delta_i^B)\, g_B^{ij}\, r_B^\dagger(\Delta_i^B a^{(2)}) = e_\emptyset r_B^\dagger(a^{(1)}a^{(2)})$$

Observation

Main Observation

The planar operation within a black shaded region is precisely the one obtained above. A white hole corresponds to an internal marked point.

Caveat

Although very similar, the Hochschild action is not just cabling. (1) we use non–relative products. (2) The module variables enter the game.

Relative theory

Import

- RELATIVE VERSION OF STRING TOPOLOGY. Consider $\pi : Z = X \times Y \to Y$ and the relative product $Z \times_Y Z$.
- The Hochschild complex usually appears as follows. Say X is simply connected, compact and one has $\phi : \Delta \to LX$ Then the inclusion $\mu_n \in S^1$ gives $\phi_n : \delta \to X^n$. Using Eilenberg-Zilber we hence get a chain in $S_*(X) \otimes \dots S_*(X)$.
- NOW, HOW CAN WE GET THE RELATIVE PRODUCT? If $\phi : \Delta \to LZ$ is such that $\pi_i \circ \phi_n = \pi_j \circ \phi_n$ then ϕ_n lands in the relative fibers and by Eilenberg–Moore in the relative tensor product over the cochains (after suitably dualizing).
- This means that the loops are all vertical. $\pi\phi(t, \theta) = y(t)$. Hence using Eilenberg–Moore, we should get a relative theory like in planar algebras.

Constraints

Export

- Using V.F.R. Jones' words. Looking at relative fiber products $X \times_Y X$ we do get constrained systems.
- This is very clear for say the product of two affine varieties over a third one.
- This lets on think in terms of in non–commutative geometry.
- The spins on a line constraint could be viewed as $\mathbb{R}^2 \times_{\mathbb{R}^1} \mathbb{R}^2$ which indeed removes one degree of freedom.

Rêveries

Spin systems

Maybe the module variable is the Auxiliary Space used in Bethe-Ansatz.

Limit

In topology, there is a way to go back from chains to the topological level using the cosimplicial structure. The functor is called total space $Tot(X^\bullet)$.
Maybe Tot corresponds to the thermodynamic limit for spin systems. It does give the loop space from its μ_n sampling.

∞–structures

Arcs from an output to itself give ∞ structures. A_∞ with R. Schwell, cyclic A_∞ B. Ward. Maybe this leads to ∞ structures in subfactors.

The End

Thank you!