Applied Knot Theory

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Purdue University

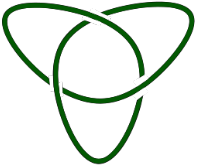
Student Colloquium
October 21, 2020

Knot Theory Basics

Introductory Definitions

Definition

A subset K of $\mathbb{R}^3$ (or S^3) is a knot if $K \cong S^1$



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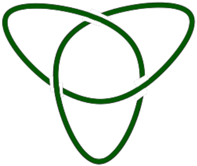

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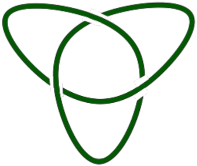

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Definition

A knot K is the unknot if it is equivalent to the boundary of some 2-dimensional disk

Equivalence

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Knots K and K' are equivalent if there is an orientation-preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K) = K'$

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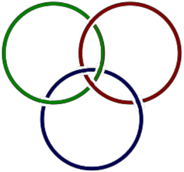
Alternate Definition

K and K' are equivalent if there is an ambient isotopy $h : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ with $h_0 = \text{id} : \mathbb{R}^3 \to \mathbb{R}^3$, $h_1(K) = K'$, and $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ a homeomorphism for all $t \in [0, 1]$

Related Objects

Definition

A subset L of $\mathbb{R}^3$ is a link if $L \cong S^1 \sqcup S^1 \sqcup \ldots \sqcup S^1$



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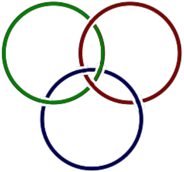
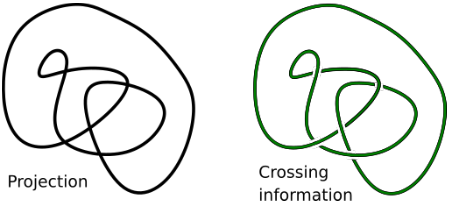


Definition

A subset T of some closed ball $B \subset \mathbb{R}^3$ is a tangle if $T \cong [0,1] \sqcup \ldots \sqcup [0,1]$ with endpoints at fixed points in ∂B and the remainder of $T \subset B^{\mathrm{o}}$

Reidemeister Moves

Knot diagram: (sensible) projection to $\mathbb{R}^2$, together with crossing information



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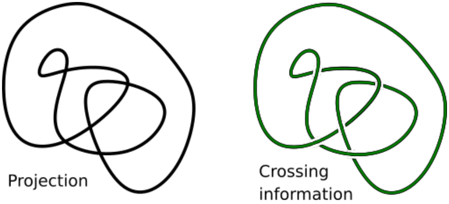
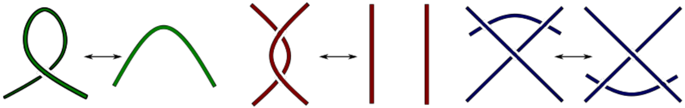
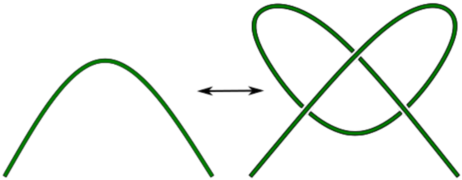


Reidmeister Moves

Diagrams of equivalent links related by some sequence of the following three moves:



Examples



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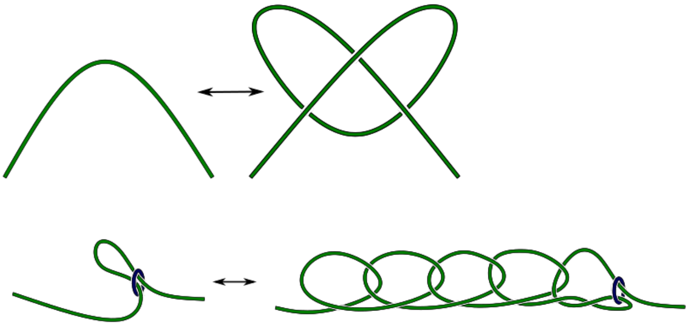

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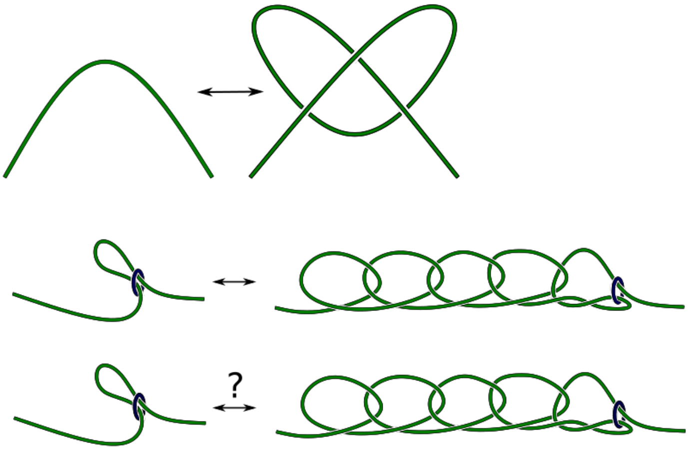

The Application

Motivation

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Tasks

1. Translate crocheting into knot theory terms
2. Confirm swatches crocheted to different patterns are non-equivalent

What is Crocheting?

- starting

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- basic stitches
  - chain (ch)
  - slip stitch (sl st)
  - single crochet (sc)
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  - chain (ch)
  - slip stitch (sl st)
  - single crochet (sc)
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- finishing off
- more complicated stitches
  - sc2tog, clusters, front post/back post stitches, etc.

Existing Work

Knitting and Knot Theory

Matsumoto, Markande (physics and applied mathematics)

- Utilize periodic nature of knitted fabric

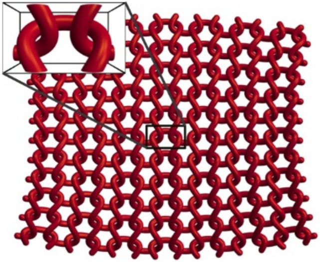

Image Source: "Knotty knits are tangles on tori," Markande and Matsumoto

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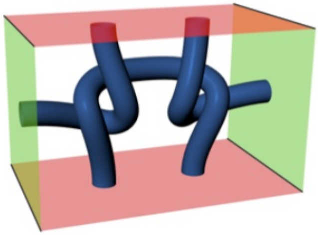

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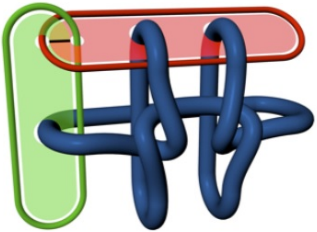

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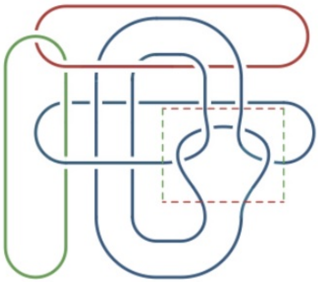

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- Consider individual stitch inside thickened torus $\mathbb{T}^2 \times I$
- Translate to S^3
- Translate to knot diagram
- Rigorize adding stitches for more complex patterns

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Translation to Crocheting

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- Local, not global, analysis
- Less suited to wider variety of stitch types
- Periodicity assumption less valid in crochet

Definition Attempts

How do you Crochet (mathematically)?

Test 1: Draw inspiration from physical process

Working Definition

A crocheted swatch is a tangle obtained from the trivial tangle by adding a starting stitch followed by some finite number of stitches, followed by one finishing stitch

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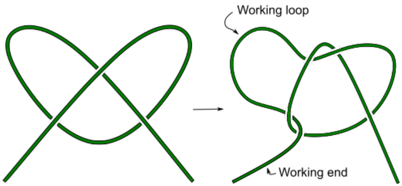
Definition (starting stitch)

For now, all swatches start with 'pretzel knot' (can expand later)



Stitches

Define stitch by stitch? $\times$

Test 2: Define abstractly using stitch ingredients

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What makes up a crochet stitch (physically)?

- Start with one working loop on hook
- Do some combination of the following
  1. loop the working end over the crochet hook (add one working loop)
  2. pull a loop through a "hole" in the swatch (add one working loop)
  3. given $n \geq 1$ working loops on hook, if the last thing you have done isn't (1), pull a bight of the working end through $1 \leq k \leq n$ working loops (reduce working loops by $k - 1$)
- The stitch ends when the number of working loops is 1.

Stitches, Mathematically

Need to define working end, working loop, etc
Test 3: Define in terms of knot diagrams, add orientation, rigorize idea of crochet hook

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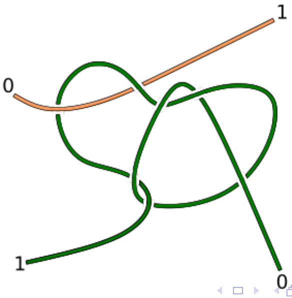

Stitches, Mathematically

Need to define working end, working loop, etc
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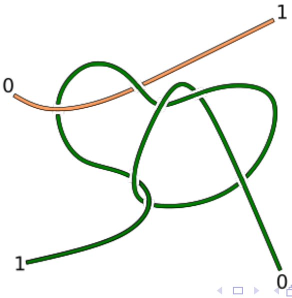
Definition

A swatch diagram is a tangle diagram obtained from a starting diagram (below) by applying some number of stitch moves, and then identifying the 1 end of the 'yarn' (green) with the 0 end of the 'hook' (apricot)



Working ends/loops

Definition

The working end in a swatch diagram is the segment of the yarn between the yarn's final self-crossing and the 1 end

Working loop (colloquially) is a hole through which the hook passes

Definition

The number of working loops (for a given diagram) is

$$\frac{1}{2} \times \# \text{ of crossings between yarn and hook}$$

Stitches

Definition

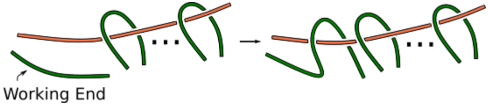
A stitch move is a sequence of diagram modifications of types 1-3 (defined next) s.t.

- the initial diagram was constructed from a starting stitch and a finite number of stitch moves (so has one working loop)
- a modification of type 1 is never followed by a modification of type 3
- the number of working loops after each intermediate modification is > 1
- the number of working loops after the final modification is 1

Stitch Diagram Modifications

Assume the rest of the diagram remains unchanged

1. add a loop from working end:



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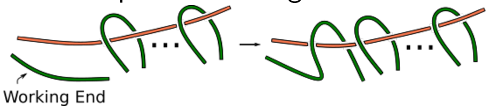

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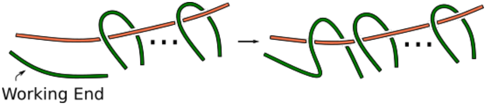

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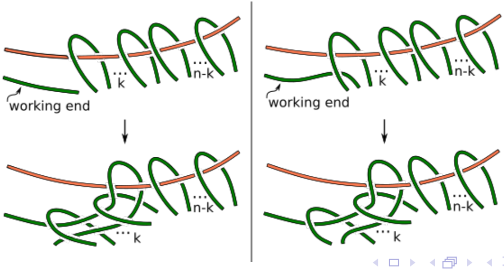
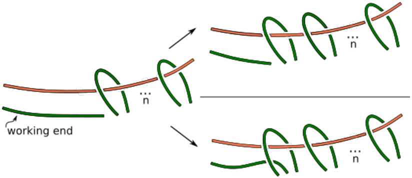
3. pull working end through k loops ($k \leq n$ # of working loops)



Second Modification: Detail

A stitch diagram modification of type 2 must have the following properties

- Alteration to working loops:



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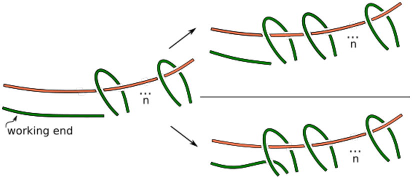

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  1. new crossings $c_1, \ldots, c_m$ with swatch
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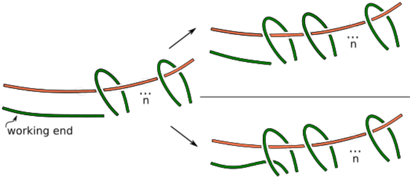

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- crossing number of new tangle $\geq$ 2+crossing number of old tangle (?)

Proofs of Non-Triviality

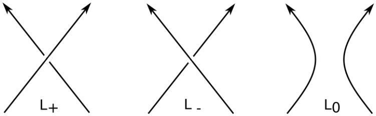
Definition/Characterization

The Jones polynomial invariant is a function

$$V : \{\text{Oriented links in } S^3\} \to \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

such that

1. $V(\text{unknot}) = 1$
2. If oriented links L_+, L_-, L_0 are identical except for one crossing where we have



then

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0$$

The Jones Polynomial

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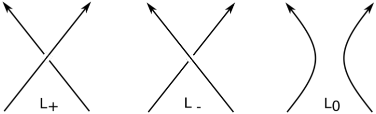

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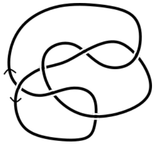
Proposition

$$V(\bigcirc\ \boxed{\text{knot}}) = (-t^{-1/2} - t^{1/2})V(\boxed{\text{knot}})$$

Starting Stitch

Let V be the Jones polynomial of the following (starting stitch, finished off, made into a knot).



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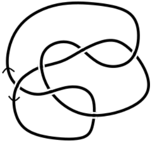
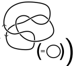


- $t^{-1}V($$) - tV + (t^{-1/2} - t^{1/2})V($$) = 0$

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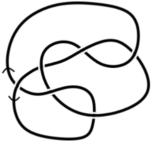
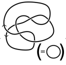
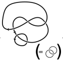
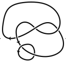
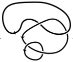


- $t^{-1}V($  $) - tV + (t^{-1/2} - t^{1/2})V($  $) = 0$
- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$

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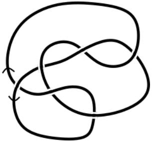
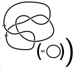
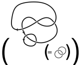
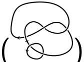
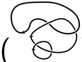
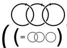
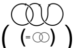


- $t^{-1}V($  $) - tV + (t^{-1/2} - t^{1/2})V($  $) = 0$
- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$
- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$

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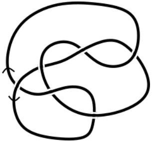
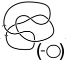
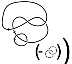
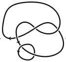
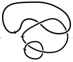
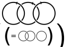
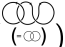
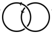
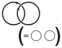
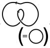
Let V be the Jones polynomial of the following (starting stitch, finished off, made into a knot).



- $t^{-1}V($  $) - tV + (t^{-1/2} - t^{1/2})V($  $) = 0$
- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$
- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$
- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$

Starting Stitch (cont.)

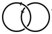
- $V(\bigcirc) = 1$

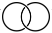
Starting Stitch (cont.)

$$V(\bigcirc\ \boxed{\text{knot}}) = (-t^{-1/2} - t^{1/2})V(\boxed{\text{knot}})$$

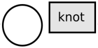
- $V(\bigcirc) = 1$
- $V(\bigcirc\bigcirc) = -t^{-1/2} - t^{1/2}$

Starting Stitch (cont.)

$t^{-1}V($$) - tV($$) + (t^{-1/2} - t^{1/2})V($$) = 0$

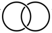
- $V($$) = 1$
- $V($$) = -t^{-1/2} - t^{1/2}$
- $V($$) = -t^{5/2} - t^{1/2}$

Starting Stitch (cont.)

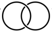
$V($

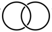
$) = (-t^{-1/2} - t^{1/2}) V($

$)$

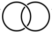
- $V($$) = 1$
- $V($$) = -t^{-1/2} - t^{1/2}$
- $V($$) = -t^{5/2} - t^{1/2}$
- $V($$) = (-t^{-1/2} - t^{1/2})(-t^{5/2} - t^{1/2})$
  $= t^3 + t^2 + t + 1$

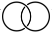
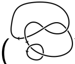
Starting Stitch (cont.)

$t^{-1}V($$) - tV($$) + (t^{-1/2} - t^{1/2})V($$) = 0$

- $V($$) = 1$
- $V($$) = -t^{-1/2} - t^{1/2}$
- $V($$) = -t^{5/2} - t^{1/2}$
- $V($$) = (-t^{-1/2} - t^{1/2})(-t^{5/2} - t^{1/2})$
  $= t^3 + t^2 + t + 1$
- $V($$) = -t^{5/2} - t^2 + 2 + t^{-2}$

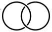
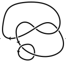
Starting Stitch (cont.)

$$t^{-1}V($$  $$) - tV($$  $$) + (t^{-1/2} - t^{1/2})V($$  $$) = 0$$

- $V($  $) = 1$
- $V($  $) = -t^{-1/2} - t^{1/2}$
- $V($  $) = -t^{5/2} - t^{1/2}$
- $V($  $) = (-t^{-1/2} - t^{1/2})(-t^{5/2} - t^{1/2})$
  $= t^3 + t^2 + t + 1$
- $V($  $) = -t^{5/2} - t^2 + 2 + t^{-2}$
- $V($  $) = -t^{3/2} - 2t^{-1/2} + t^{-3/2} - t^{-5/2} + t^{-7/2}$

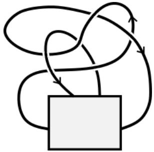
Starting Stitch (cont.)

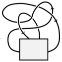
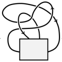
$t^{-1}V($$) - tV + (t^{-1/2} - t^{1/2})V($$) = 0$

- $V($$) = 1$
- $V($$) = -t^{-1/2} - t^{1/2}$
- $V($$) = -t^{5/2} - t^{1/2}$
- $V($$) = (-t^{-1/2} - t^{1/2})(-t^{5/2} - t^{1/2})$
  $= t^3 + t^2 + t + 1$
- $V($$) = -t^{5/2} - t^2 + 2 + t^{-2}$
- $V($$) = -t^{3/2} - 2t^{-1/2} + t^{-3/2} - t^{-5/2} + t^{-7/2}$
- $V = t^{-5} - 2t^{-4} + 2t^{-3} - 2t^{-2} + 2t^{-1} - 1 + t \neq 1$

Chain stitch

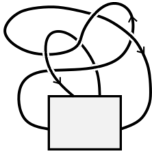
Let L' be some crocheted swatch, and let L be L' with a chain stitch added at the end

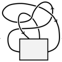
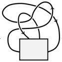
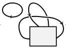
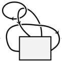


- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$

Chain stitch

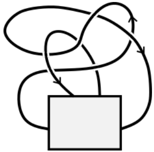
Let L' be some crocheted swatch, and let L be L' with a chain stitch added at the end

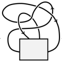
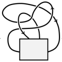
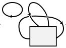
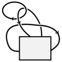


- $t^{-1}V($$) - tV($$) + (t^{-1/2} - t^{1/2})V($$) = 0$
- $t^{-1}V($$) - tV($$) + (t^{-1/2} - t^{1/2})V($$) = 0$

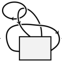
Chain stitch

Let L' be some crocheted swatch, and let L be L' with a chain stitch added at the end



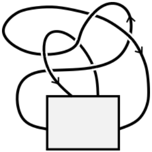
- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$
- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$

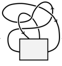
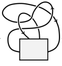
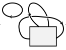
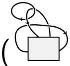
$\Rightarrow$

- $V($  $) = (-t^{-5/2} - t^{-1/2})V($  $))$
  $= (-t^{-5/2} - t^{-1/2})V(L')$

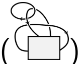
Chain stitch

Let L' be some crocheted swatch, and let L be L' with a chain stitch added at the end



- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$
- $t^{-1}V($  $) - tV($  $) + (t^{-1/2} - t^{1/2})V($  $) = 0$

$\Rightarrow$

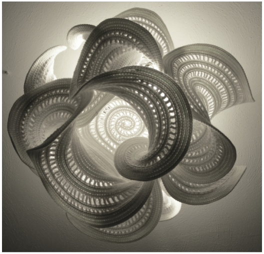
- $V($  $) = (-t^{-5/2} - t^{-1/2})V($  $))$
  $= (-t^{-5/2} - t^{-1/2})V(L')$
- $V(L) = t^2 + (t^{-2} - t^{-1} + 1 - t)V(L')$

Conclusion

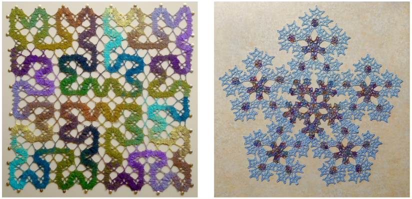
How to Apply Math to Crochet

(mostly not mine)

Matthew Wright (http://mrwright.name/crochet/)



Gabriele Meyer (http://gallery.bridgesmathart.org/exhibitions/2014-joint-mathematics-meetings/gabriele_meyer)



Lana Holden (http://gallery.bridgesmathart.org/exhibitions/2017-joint-mathematics-meetings/lanaholden,

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