Continuity and Derivative Mini-Exercises

MA 504 Problem Session

November 3, 2022

1. Let X and Y be metric spaces and say that $f : X \to Y$ is some function. Rank the following properties in order of strength. Are any equivalent?

   - f is continuous
   - f is uniformly continuous
   - For some $\lambda \in [0, 1)$, $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$ [If $Y = X$, f is a contraction mapping.]
   - f is such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$
   - There exists a $c \in \mathbb{R}$ with $c \geq 0$ such that $d(f(x), f(y)) \leq c\, d(x, y)$ for all $x, y \in X$. [This property is known as Lipschitz continuity, though the textbook does not discuss it.]
   - Any other property that you want to include in your schematic

2. (Fall 2015 504 Exam 2) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let C, U, K, and B all be subsets of $\mathbb{R}$, and assume that C is closed, U is open, K is compact, and B is bounded. Answer True or False for each statement. [The exam required no proof or explanation, but it might be beneficial to cite a theorem/brief justification or find a counterexample.]

(a) $f(C)$ is closed	T	F
(b) $f^{-1}(C)$ is closed	T	F
(c) $f(K)$ is compact	T	F
(d) $f^{-1}(K)$ is compact	T	F
(e) $f(U)$ is open	T	F
(f) $f^{-1}(U)$ is open	T	F
(g) f is bounded on K	T	F
(h) f is bounded on C	T	F
(i) f is bounded on U provided U is also bounded	T	F
(j) f is bounded on B	T	F

3. (Fall 2015 504 Exam 2)

   (a) Let $A \subset \mathbb{R}$ be bounded, and let $f, g : A \to \mathbb{R}$ be uniformly continuous functions. Prove that fg is also uniformly continuous on A.

   (b) Show that $f(x) = x$ and $g(x) = \sin(x)$ are both uniformly continuous on $\mathbb{R}$, but their product is not. You may use any properties of $\sin(x)$ that you know.

4. Let $f(x)$ be differentiable on the compact interval $[a, b]$. Is $f(x)$ bounded? What about $f'(x)$?