Monotonicity and Totally Nonnegative Spaces

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Advanced Topics Examination
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Introduction

Total Nonnegativity

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Can extend definition to any split semi-simple algebraic group over $\mathbb{R}$.

- B, B_- opposite Borel subgroups
- U (resp U_-) unipotent radical of B (resp B_-)
- $x_i(t) = \exp(te_i)$ (e_i Chevalley generators of the Lie algebra of U, $t \in \mathbb{R}$)

Y (totally nonnegative elements of U) mulitpicative submonoid of U generated by $x_i(t)$, $t \geq 0$

Example: $G = SL(n, \mathbb{R})$

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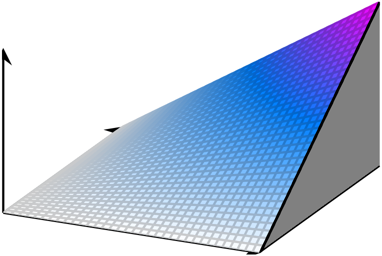

Coxeter Groups

Definition

Let W be a group and $S \subset W$. If W has a presentation of the form

- Generators: S
- Relations:
  - $s^2 = e$ for all $s \in S$
  - others of the form $(ss')^{m(s,s')} = e$ for $s \neq s' \in S$, $m(s,s') \geq 2$

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Example

$W = S_n$: S set of adjacent transpositions $s_i = (i \;\; i+1)$ for $1 \leq i \leq n-1$

Words in Coxeter Systems

Let $w \in W$, $S = \{s_i\}$

$$w = s_{i_1} \cdots s_{i_k}$$

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Let $u, v \in W$. If there is a reduced word for u that is a subword of a reduced word for v, then $u \leq v$ in the Bruhat order

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Proposition

If W is finite, there exists a unique element $w_0 \in W$ so that $w \leq w_0$ for all $w \in W$

Fomin Shapiro Conjecture

Stratification of Y

Let W be the Weyl Group of G

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$u \leq v$ in the Bruhat order iff $Y_u^o \subset \overline{Y_v^o}$

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Notice

$u \leq v$ in the Bruhat order iff $Y_u^{\circ} \subset \overline{Y_v^{\circ}}$

Proposition (Lusztig)

Let $(i_1, \ldots, i_d)$ be a reduced word for $w \in W$. Then the map

$$(t_1, \ldots, t_d) \mapsto x_{i_1}(t_1) \cdots x_{i_d}(t_d)$$

is a homeomorphism between $\mathbb{R}^d_{>0}$ and Y_w°

Example: $G = SL(3, \mathbb{R})$

$$x_1(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad x_2(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

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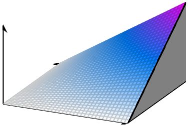
- $Y^{o}_{(1,2,1)} = Y^{o}_{(2,1,2)} = \{(x, y, z) \mid x, y > 0, 0 < z < xy\}$



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- $Y^{\circ}_{id} = \{(0,0,0)\}$
- $Y^{\circ}_{(1)} = \{(x,0,0) \mid x > 0\}$
- $Y^{\circ}_{(2)} = \{(0,y,0) \mid y > 0\}$
- $Y^{\circ}_{(2,1)} = \{(x,y,0) \mid x > 0, y > 0\}$
- $Y^{\circ}_{(1,2)} = \{(x,y,xy) \mid x,y > 0\}$
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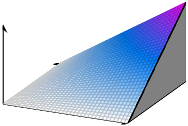

Links of Strata

Notation: Let $Y_w = \overline{Y_w^{\circ}}$.

Definition

Let $Y_u^{\circ} \subset Y_v$ ($\Leftrightarrow u \leq v$). Let

- $p \in Y_u^{\circ}$ arbitrary
- N a smooth manifold with $N \cap Y_u^{\circ} = \{p\}$ and N transverse to Y_u°
- $B_\delta(p)$ ball of radius δ centered at p

Then $\mathrm{Lk}(u, v) = Y_v \cap N \cap \partial B_\delta(p)$

Notation: Let $Y_w = \overline{Y_w^o}$.

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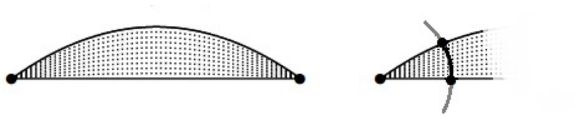

Figure: $\mathrm{Lk}((0),(1,2,1))$ and $\mathrm{Lk}((1),(1,2,1))$ for $SL(3,\mathbb{R})$

Definition

A set $U \subset \mathbb{R}^m$ is an *m*-cell if $U \cong (B^m)^{\mathrm{o}}$. U is a regular *m*-cell if the pair $(\overline{U}, U) \cong (B^m, (B^m)^{\mathrm{o}})$

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Conjecture

For all $u, v \in W$ with $Y_u^{\mathrm{o}} \subset Y_v$, $\mathrm{Lk}(u, v)$ is a regular cell complex (decomposes into regular cells).

Fomin Shapiro Conjecture

Definition

A set $U \subset \mathbb{R}^m$ is an m-cell if $U \cong (B^m)^o$. U is a regular m-cell if the pair $(\overline{U}, U) \cong (B^m, (B^m)^o)$

Conjecture

For all $u, v \in W$ with $Y_u^o \subset Y_v$, $\mathrm{Lk}(u, v)$ is a regular cell complex (decomposes into regular cells).

Motivation

Björner: $[u, v]$ a Bruhat interval $\Rightarrow$ there exists a regular cell complex with face poset isomorphic to $[u, v]$.
Goal: find naturally arising construction.

Resolution by Hersh

Some Notation

Let $(i_1, \ldots, i_d)$ be a word for $w \in W$. Denote

$$f_{(i_1,\ldots,i_d)} : \mathbb{R}^d_{\geq 0} \cap S^{d-1} \to Y_w$$
$$(t_1, \ldots, t_d) \mapsto x_{i_1}(t_1) \cdots x_{i_d}(t_d)$$

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Notation Change

Henceforth, for $w = (i_1, \ldots, i_d)$

- $Y_w^{\circ} = f_{(i_1,\ldots,i_d)}(\mathbb{R}^d_{>0} \cap S^{d-1})$
- $Y_w = f_{(i_1,\ldots,i_d)}(\mathbb{R}^d_{\geq 0} \cap S^{d-1}) \cong \mathrm{Lk}((0), (i_1, \ldots, i_d))$

Cell Collapses

$(i_1, \ldots, i_d)$ reduced: $f_{(i_1,\ldots,i_d)}$ homeomorphism on interior, not necessarily injective on boundary

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Example

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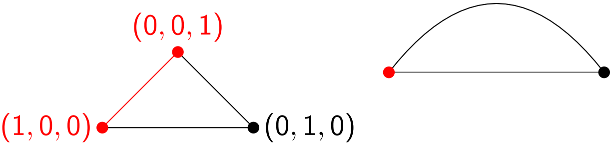

Resolution

Theorem (Hersh)

Let $(i_1, \ldots, i_d)$ be a reduced word for $w \in W$. Let $\sim$ be the identifications given by any series of face collapses on $\mathbb{R}^d_{\geq 0} \cap S^{d-1}$ such that

1. $x \sim y \Rightarrow f_{(i_1,\ldots,i_d)}(x) = f_{(i_1,\ldots,i_d)}(y)$
2. the series of collapses eliminates all regions whose words are not reduced

Then $\overline{f_{(i_1,\ldots,i_d)}} : \mathbb{R}^d_{\geq 0} \cap S^{d-1} / \sim \to Y_w$ is a homomorphism which preserves cell structure

Monotonicity

Coordinate Cones

In this section, sets and functions are definable in some o-minimal structure over $\mathbb{R}$

Let $L_{j,\sigma,c} = \{\mathbf{x} \in \mathbb{R}^n \mid x_j \sigma c\}$ for $\sigma \in \{<, =, >\}$, $c \in \mathbb{R}$

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Definition

A coordinate cone is a set of the form

$$C = L_{j_1,\sigma_1,c_1} \cap \ldots \cap L_{j_m,\sigma_m,c_m} \subset \mathbb{R}^n$$

with the j_i distinct elements of $\{1, \ldots, n\}$. Similarly, an affine coordinate subspace has the form

$$S = L_{j_1,=,c_1} \cap \ldots \cap L_{j_m,=,c_m} \subset \mathbb{R}^n$$

Semi-monotone Sets

Definition/Theorem

An open bounded set $X \subset \mathbb{R}^n$ is semi-monotone if for each coordinate cone C, $X \cap C$ is connected (equivalently, if for every affine coordinate subspace S, $X \cap S$ is connected)

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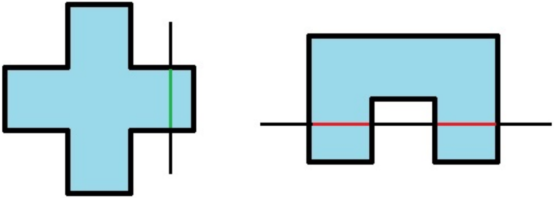

semi-monotone   not semi-monotone

Monotone Functions

Let $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$ nonempty and semi-monotone, and let F be the graph of f

Definition

f is submonotone if it is bounded, upper semi-continuous, and for all $b \in \mathbb{R}$, $\{\mathbf{x} \in X \mid f(\mathbf{x}) < b\}$ is semi-monotone. f is supermonotone if $-f$ is submonotone.

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Definition

f is monotone if it is both sub and supermonotone and either strictly increasing in, strictly decreasing in, or independent of x_j for all $1 \leq j \leq n$

Monotone Functions (Characterization)

Let $f : X \to \mathbb{R}$ be bounded and continuous, with $X \subset \mathbb{R}^n$ open, bounded, and nonempty

Theorem

Let f be strictly increasing in, strictly decreasing in, or independent of each x_j, $1 \leq j \leq n$. Then the following are equivalent

I. f is monotone

II. $F \cap C$ is connected for each coordinate cone C

III. $F \cap S$ is connected for each affine coordinate subspace S

Monotone Maps

Let $\mathbf{f} = (f_1, \ldots, f_k) : X \to \mathbb{R}^k$, $X \subset \mathbb{R}^n$ nonempty and semi-monotone, F the graph of f.

Definition

Let $H = \{x_{j_1}, \ldots, x_{j_\alpha}, y_{i_1}, \ldots, y_{i_\beta}\} \subset \{x_1, \ldots, x_n, y_1, \ldots, y_k\}$ where $\alpha + \beta = n$. H is a basis if $(x_{j_1}, \ldots, x_{j_\alpha}, f_{i_1}, \ldots, f_{i_\beta}) : X \to \mathbb{R}^n$ is injective

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Definition

$\mathbf{f} : \mathbb{R} \to \mathbb{R}^k$ is monotone if f_i is monotone for all i
Inductively, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ is monotone if for each f_i not independent of x_j

1. For each $b \in \mathbb{R}$, $F \cap \{y_i = b\}$ is the graph of a monotone map $\mathbf{f}_{i,j,b}$ from a semi-monotone subset of $\text{span}\{x_1, \ldots, \hat{x}_j, \ldots, x_n\}$ to $\text{span}\{y_1, \ldots, y_{i-1}, x_j, y_{i+1}, \ldots, y_k\}$
2. The system of basis sets associated with $\mathbf{f}_{i,j,b}$ does not depend on b

Monotone Maps (Characterization)

Let $\mathbf{f} : X \to \mathbb{R}^k$ be bounded and continuous, with $X \subset \mathbb{R}^n$ open, bounded, and nonempty, F the graph of f.

Definition

$\mathbf{f}$ is quasi-affine if for any $T = \text{span}\{x_{j_1}, \ldots, x_{j_\alpha}, y_{i_1}, \ldots, y_{i_\beta}\}$, $\alpha + \beta = n$, the projection $\rho_T : F \to T$ is injective iff the image $\rho_T(F)$ is n dimensional

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Theorem

Let $\mathbf{f}$ be quasi-affine. Then the following are equivalent

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II. $F \cap C$ is connected for each coordinate cone C

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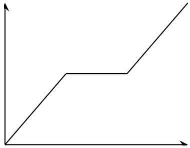
Examples



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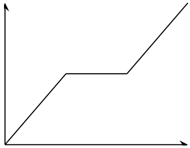

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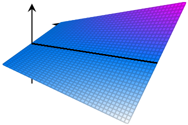

$z = xy$ on
$0 < x < 1$, $-1 < y < 1$
not monotone

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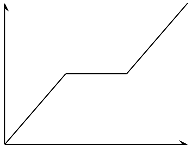

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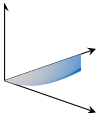
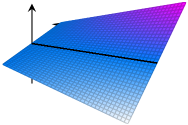


$z = x^2 + y^2$ on

$0 < x < 1$, $0 < y < 1 - x$

not monotone



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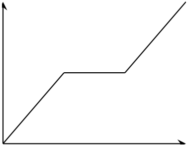

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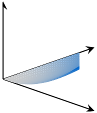

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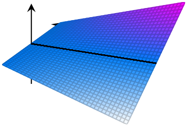

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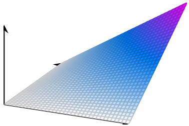

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monotone

Regular Cells

Theorem (Basu, Gabrielov, Vorobjov)

The graph $F \subset \mathbb{R}^{n+k}$ of a monotone map $\mathbf{f} : X \to \mathbb{R}^k$ on a semimonotone set $X \subset \mathbb{R}^n$ is a regular n-cell.

Application: Toric Cubes

Definition

A toric cube is the image of a map of the form

$$f_{\mathcal{A}} : [0,1]^d \to [0,1]^n$$
$$\mathbf{t} = (t_1, \dots, t_d) \mapsto (\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n})$$

where $\mathcal{A} = \{\mathbf{a_1}, \dots, \mathbf{a_n}\} \subset \mathbb{R}^d$ and for $\mathbf{a}_i = (a_{i,1}, \dots, a_{i,d})$, $\mathbf{t}^{\mathbf{a}_i}$ denotes $(t_1^{a_{i,1}}, \dots, t_d^{a_{i,d}})$. An open toric cube is the image of the restriction of such an $f_{\mathcal{A}}$ to $(0,1)^d$.

Theorem (Basu, Gabrielov, Vorobjov)

An open toric cube is the graph of a monotone map, and hence is a regular cell.

Application: Vandermonde Varieties

Let **R** be a real closed field

Definition

The Weyl chamber in $\mathbf{R}^k$ is

$$\mathcal{W}^{(k)} = \{(X_1, \ldots, X_k) \in \mathbf{R}^k \mid X_1 \leq \ldots \leq X_k\}$$

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Definition

Let $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbf{R}^d$. The Vandermonde variety $V_{d,\mathbf{y}}^{(k)} \subset \mathbf{R}^k$ is the variety defined by $p_1^{(k)} = y_1, \ldots, p_d^{(k)} = y_d$ where

$$p_j^{(k)} = \sum_{i=1}^{k} X_i^j$$

Application: Vandermonde Varieties

Proposition (Basu, Riener)

For all $\mathbf{y} \in \mathbf{R}^d$, $d \leq k$, either $V_{d,\mathbf{y}}^{(k)} \cap \mathcal{W}^{(k)}$ is empty or a point, or $V_{d,\mathbf{y}}^{(k)} \cap \mathcal{W}^{(k)} = \overline{V_{d,\mathbf{y}}^{(k)} \cap \mathcal{W}^{(k),\circ}}$ and $V_{d,\mathbf{y}}^{(k)} \cap \mathcal{W}^{(k),\circ}$ is a semi-monotone set, and hence a regular cell

Fibers of Maps

Route to an Alternate Proof

Theorem (Davis, Hersh, Miller)

Let $v \in W$ with $(i_1, \ldots, i_d)$ a reduced word for v. Then if for all $w \in W$, $w \leq v$, we have $f^{-1}_{(i_1,\ldots,i_d)}(p)$ is contractible for $p \in Y^{o}_{w}$, then Y_w is a regular cell complex for each $w \leq v$.

(Key ingredient in proof)

Let $\sim$ be an equivalence relation on the closed ball B^n so that

- all equivalence classes are contractible
- $S^{n-1}/\sim \,\cong S^{n-1}$
- if $x \sim y$ with $x \in S^{n-1}$, then $y \in S^{n-1}$
- if $x \sim y$ with $x \notin S^{n-1}$, then $y = x$

Then $B \cong B/\sim$

Reduced Words and Injectivity

Definition/Proposition

The Demazure product on W is the unique associative map $\delta : W \times W \to W$ such that for $w \in W$ and $s \in S$,

$$\delta(w, s) = \begin{cases} ws & l(ws) > l(w) \\ w & l(ws) < w \end{cases}$$

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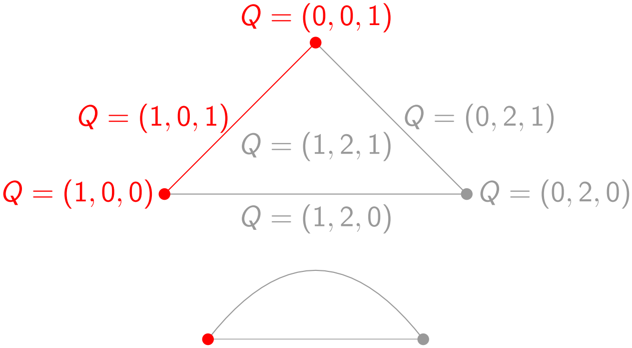
Theorem

Let $(i_1, \ldots, i_d)$ be a reduced word for v and let $p \in Y_w$. Then $f^{-1}_{(i_1,\ldots,i_d)}(p)$ is stratified via the standard decomposition of the simplex. Let Q be a subword of $(i_1, \ldots, i_d)$. $f^{-1}_{(i_1,\ldots,i_d)}(p) \cap \mathbb{R}^{Q}_{>0}$ is nonempty iff Q multiplies to w under the Demazure product, and is non-trivial iff the expression is not reduced.

Example: $G = SL(3, \mathbb{R})$

$v = (1,2,1)$, $p \in Y_{(1)}$, $f^{-1}_{(1,2,1)}(p)$ in red.



The Goal

Conjecture

Let $G = SL(n, \mathbb{R})$, let $(i_1, \ldots, i_d)$ be a reduced word for $v \in W$, let $w \leq v$, and let $p \in Y_w^{\circ}$. Then the strata of $f^{-1}_{(i_1,\ldots,i_d)}(p)$ are graphs of monotone maps, and hence this stratification is a regular cell decomposition.

This holds in the cases $n = 3$ and $n = 4$, by computation

Example 1

$$f_{(1,3,2,1,3,2)} = \begin{bmatrix} 1 & t_1 + t_4 & (t_1 + t_4)t_6 + t_1 t_2 & t_1 t_3 t_5 \\ 0 & 1 & t_3 + t_6 & t_3 t_5 \\ 0 & 0 & 1 & t_2 + t_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$p = (a, b, c, ab, 0, 0)$$
$$\in Y^{\mathrm{o}}_{(3,1,2)} = \{(x, y, z, xy, 0, 0) \mid x, y, z > 0, x + y + z = K\}$$

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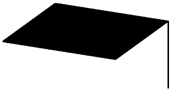
$$f^{-1}(p) = \{(t_1, t_2, 0, a - t_1, c - t_2, b) \mid 0 \leq t_1 \leq a, 0 \leq t_2 \leq c\} \\ \cup \{(a, c, t_3, 0, 0, b - t_3) \mid 0 \leq t_3 \leq b\}$$



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$(1,3,2,0,0,0)$	$\{(a,c,b,0,0,0)\}$	point
$(1,3,0,0,0,2)$	$\{(a,c,0,0,0,b)\}$	point
$(1,3,2,0,0,2)$	$\{(a,c,t_3,0,0,b) \mid 0 < t_3 < b\}$	line
$(0,3,0,1,0,2)$	$\{(0,c,0,a,0,b)\}$	point
$(1,3,0,1,0,2)$	$\{(t_1,c,0,a-t_1,0,b) \mid 0 < t_1 < a\}$	line
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$(0,3,0,1,3,2)$	$\{(0,t_2,0,a,c-t_2,b) \mid 0 < t_2 < c\}$	line
$(1,3,0,1,3,2)$	$\{(t_1,t_2,0,a-t_1,c-t_2,b) \mid 0 < t_1 < a, 0 < t_2 < c\}$	square

Example 2

$$f_{(1,3,2,1,3,2)} = \begin{bmatrix} 1 & t_1 + t_4 & (t_1+t_4)t_6 + t_1 t_2 & t_1 t_3 t_5 \\ 0 & 1 & t_3 + t_6 & t_3 t_5 \\ 0 & 0 & 1 & t_2 + t_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{aligned} p = (a, b, 0, d, 0, 0) &\in Y^{\circ}_{(2,1,2)} \\ &= \{(x, y, 0, u, 0, 0) \mid x, y > 0, 0 < u < xy, x + y = K\} \end{aligned}$$

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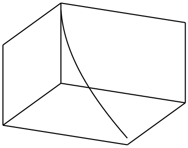
$$f^{-1}(p) = \{(t_1, 0, \frac{ab - d}{a - t_1}, a - t_1, 0, \frac{d - t_1 b}{a - t_1}) \mid 0 \leq t_1 \leq d/b\}$$



Example 2

$$f = f_(1,3,2,1,3,2),\ p \in Y^{\mathrm{o}}_{(2,1,2)}$$

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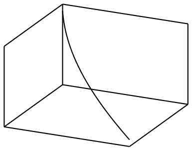
$(1,0,2,1,0,0)$	$\{(\frac{d}{b}, 0, b, a-\frac{d}{b}, 0, 0\}$	point
$(0,0,2,1,0,2)$	$\{(0,0,b-\frac{d}{a}, a, 0, \frac{d}{a})\}$	point
$(1,0,2,1,0,2)$	$\{(t_1, 0, \frac{ab-d}{a-t_1}, a-t_1, 0, \frac{d-t_1 b}{a-t_1}) \mid 0 < t_1 < \frac{d}{b}\}$	



Changes of Coordinates

For $W = S_n$: $(t_1, t_2, t_3 > 0)$

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$$x_i(t_1)x_i(t_2) = x_i(t_1 + t_2)$$

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- braid moves

$$x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2 t_3}{t_1 + t_3}\right) x_i(t_1 + t_3) x_{i+1}\left(\frac{t_1 t_2}{t_1 + t_3}\right)$$

Structure of Fibers (preliminary)

Fiber $f^{-1}_{(i_1,\ldots,i_d)}(p)$, $p \in Y^{\circ}_w$, $w = (j_1, \ldots, j_k)$ reduced, $Q = (i'_1, \ldots, i'_{d'})$ a subword of $(i_1, \ldots, i_d)$ multiplying to w under the Demazure product, not reduced

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- Case 0: Q multiplies to $(j_1, \ldots, j_d)$ without braid moves
  - $x_{i'_1}(t_1) \cdots x_{i'_d}(t_d) \mapsto$ $x_{j_1}(t_{1,1}+\ldots+t_{1,n_1}) \cdots x_{j_k}(t_{k,1}+\ldots+t_{k,n_k}) \overset{\text{inj.}}{\mapsto} (a_1, \ldots, a_k)$
  - fiber a cross product of simplicies, graph of monotone map

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  - fiber a cross product of simplicies, graph of monotone map
- Case 1: Q Multiplies to $(j_1, \ldots, j_k)$ via one braid move followed by a modified nil-move
  - $\ldots x_i(t_p) x_{i+1}(t_{p+1}) x_i(t_{p+2}) x_{I+1}(t_{p+3}) \ldots \mapsto$ $\ldots x_{i+1}\left(\frac{t_{p+1} t_{p+2}}{t_p + t_{p+2}}\right) x_{i+1}(t_p + t_{p+2}) x_{i+1}\left(\frac{t_p t_{p+1}}{t_p + t_{p+2}} + t_{p+3}\right) \ldots$ $\overset{\text{inj.}}{\mapsto} (a_1, \ldots a_k)$
  - fiber graph of $t_p \mapsto \left(\frac{a'_1 a'_2}{a'_2 - t_p}, a'_2 - t_p, \frac{a'_2 a'_3 - (a'_1 + a'_3) t_p}{a'_2 - t_p}\right)$ semi-monotone

Outline of Results

- Conjectured: strata of $f^{-1}_{(i_1,\ldots,i_d)}(p)$ monotone for all $v, w \in S_n$, $w \leq v$ ($v = (i_1, \ldots, i_d)$, $p \in Y_w^{o}$).

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- (Davis, Hersh, Miller) The face poset of the stratification of $f^{-1}_{(i_1,\ldots,i_d)}(p)$ is isomorphic to the face poset of the interior dual block complex of the subword complex $\Delta((i_1, \ldots, i_d), w)$
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Future Directions

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Totally nonnegatvie part of flag variety $(G/P)_{\geq 0}$

- Conjecture: regular cell complex homeomorphic to a ball
- Evidence:
  - contractible
  - cell poset that of a regular cell complex homeomorphic to a ball
  - regular cell complex up to homotopy equivalence
- Special case: totally nonnegative Grassmanian $(Gr_{n,k})_{\geq 0}$

References I

George Lusztig. "Introduction to Total Positivity". In: *Positivity in Lie Theory: Open Problems*. Vol. 26. De Gruyter Expositions in Mathematics. Berlin and New York: Walter De Gruyter, 1998, pp. 133–145. ISBN: 3-11-016112-5.

Sergey Fomin and Michael Shapiro. "Stratified Spaces Formed by Totally Positive Varieties". In: *Michigan Math J.* 48 (2000), pp. 253–270.

Anders Björner. "Posets, Regular CW Complexes, and Bruhat Order". In: *Europ. J. Combinatorics* 5 (1984), pp. 7–16.

Patricia Hersh. "Regular Cell Complexes in Total Positivity". In: *Inventiones Mathematicae* 197 (2014), pp. 57–114.

References II

Saugata Basu, Andrei Gabrielov, and Nicolai Vorobjov. "Semi-monotone Sets". In: *J. Eur. Math. Soc.* 15 (2013), pp. 635–657.

Saugata Basu, Andrei Gabrielov, and Nicolai Vorobjov. "Monotone Functions and Maps". In: *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas* 107 (2013), pp. 5–33.

Saugata Basu, Andrei Gabrielov, and Nicolai Vorobjov. "Toric Cubes are Closed Balls". In: ().

Saugata Basu and Cordian Riener. "Vandermonde Varieties, Mirrored Spaces, and the Cohomology of Symetric Semi-algebraic Sets". In: (2019).

References III

James F. Davis, Patricia Hersh, and Ezra Miller. "Fibers of Maps to Totally Nonnegative Spaces". In: (2019).

Konstanze Rietsch and Lauren Williams. "Discrete Morse Theory for Totally Non-Negative Flag Varieties". In: *Advances in Mathematics* 223 (2010), pp. 1855 –1884.