Monotonicity and Totally Nonnegative Spaces

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Student Colloquium
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Total Nonnegativity

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Here,

- U upper triangular matrices with 1's on the diagonal
- Y totally nonnegative part of U.

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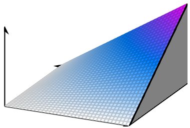
$$Y = \{(x, y, z) \in \mathbb{R}^3 \mid x, y \geq 0, 0 \leq z \leq xy\}$$



Coxeter Groups

Definition

Let W be a group and $S \subset W$. If W has a presentation of the form

- Generators: S
- Relations:
  - $s^2 = e$ for all $s \in S$
  - others of the form $(ss')^{m(s,s')} = e$ for $s \neq s' \in S$, $m(s,s') \geq 2$

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Example

- $W = S_n$ symmetric group
- $S = \{s_i = (i \;\; i+1) \mid 1 \leq i \leq n-1\}$
- Relations $(s_i s_{i+1})^3 = e$, $(s_i s_j)^2 = e$ for $|i - j| > 1$

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$$w = s_{i_1} \cdots s_{i_k}$$

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Proposition

If W is finite, there exists a unique element $w_0 \in W$ so that $w \leq w_0$ for all $w \in W$

Example: S_3

Bruhat order for S_3 (note $s_1 s_2 s_1 = s_2 s_1 s_2$)

$$
\begin{tikzcd}
& s_1 s_2 s_1 = (1\ 3) & \\
s_1 s_2 = (1\ 2\ 3) \arrow[ur] & & s_2 s_1 = (1\ 3\ 2) \arrow[ul] \\
s_1 = (1\ 2) \arrow[u] \arrow[urr] & & s_2 = (2\ 3) \arrow[u] \arrow[ull] \\
& \mathrm{id} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

The Connection

Let $W = S_n$, $s_i = (i \;\; i+1)$ for $1 \leq i \leq n-1$. Define maps

$$x_i : \mathbb{R} \longrightarrow SL_n(\mathbb{R})$$
$$t \longmapsto \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ & & 1 & t & \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{bmatrix}$$

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$$\mathbb{R}^k \to SL_n(\mathbb{R})$$
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If $(t_1, t_2, t_3) \in \mathbb{R}^3_{\geq 0}$,

$$\text{Image} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \middle| x, y \geq 0, 0 \leq z \leq xy \right\}$$

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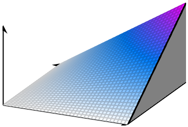

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$$\begin{aligned} f_{(i_1,\ldots,i_d)} : \Delta_{d-1} &\to SL_n(\mathbb{R}) \\ (t_1, \ldots, t_d) &\mapsto x_{i_1}(t_1) \cdots x_{i_d}(t_d) \end{aligned}$$

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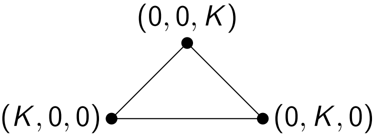
- $f_{(i_1,\ldots,i_d)}(\Delta_{d-1} \cap \mathbb{R}^d_{>0}) = Y^{\circ}_w$
- $f_{(i_1,\ldots,i_d)}(\Delta_{d-1}) = Y_w \; (= \overline{Y^{\circ}_w})$

Consequences

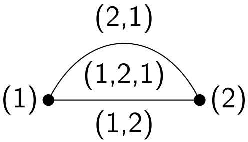
Let $w_0 \in S_n$ denote the longest word

Note

- $Y_{w_0} = \bigcup_{u \in S_n} Y_u^{\circ}$
- This decomposition is a stratification of Y_{w_0}



The simplex Δ_2



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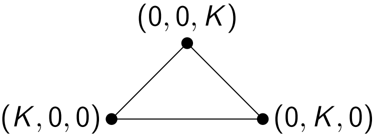

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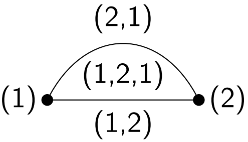

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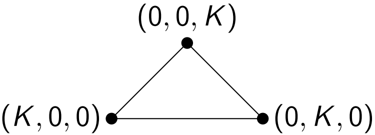
- $u \leq v$ in S_n iff $Y_u^{\circ} \subset Y_v$



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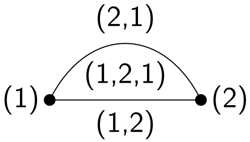

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A Moment of Topology

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Definition

A set $U \subset \mathbb{R}^m$ is an *m*-cell if $U \cong (B^m)^{\mathrm{o}}$. U is a regular *m*-cell if the pair $(\overline{U}, U) \cong (B^m, (B^m)^{\mathrm{o}})$

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A Hausdorff space X is a (finite) cell complex if it can be decomposed into a (finite) collection of cells e_α such that

1. For each e_α there exists a continuous $f_\alpha : B^m \to X$ such that f_α maps $(B^m)^{\text{o}}$ homeomorphically to e_α and maps ∂B^m to a finite union of cells of dimension $< m$.
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Definition

A cell complex is regular if it has a cell decomposition so that f_α is also a homeomorphism on ∂B^m for each α.

Fomin Shapiro Conjecture

Conjecture (S. Fomin, M. Shapiro)

For each $u \in S_n$, the stratum Y_u^{o} is a regular cell, and hence Y_{w_0} is a regular cell complex (as is Y_w for each $w \in W$).

Resolution: Cell Collapses

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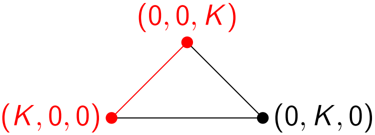
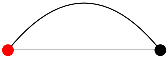
Problem: f_w generally not a homeomorphism on the boundary.

Eg: $n = 3$

$$f_{(1,2,1)}(t_1, t_2, t_3) = \begin{bmatrix} 1 & t_1 + t_3 & t_1 t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$





Theorem (Hersh)

Let $(i_1, \ldots, i_d)$ reduced. Let $\sim$ be the identifications given by any series of face collapses on Δ_{d-1} such that

1. $x \sim y \Rightarrow f_{(i_1,\ldots,i_d)}(x) = f_{(i_1,\ldots,i_d)}(y)$
2. the series of collapses eliminates all regions whose words are not reduced

Then $\overline{f_{(i_1,\ldots,i_d)}} : \Delta_{d-1}/\sim \to Y_w$ is a homomorphism which preserves cell structure

Monotonicity: Coordinate Cones

In this section, sets and functions are definable in some o-minimal structure over $\mathbb{R}$

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Definition

A coordinate cone is a set of the form

$$C = L_{j_1,\sigma_1,c_1} \cap \ldots \cap L_{j_m,\sigma_m,c_m} \subset \mathbb{R}^n$$

with the j_i distinct elements of $\{1, \ldots, n\}$.

Similarly, an affine coordinate subspace has the form

$$S = L_{j_1,=,c_1} \cap \ldots \cap L_{j_m,=,c_m} \subset \mathbb{R}^n$$

Semi-monotone Sets

Definition/Theorem

An open bounded set $X \subset \mathbb{R}^n$ is semi-monotone if for each coordinate cone C, $X \cap C$ is connected (equivalently, if $X \cap S$ is connected for every affine coordinate subspace S)

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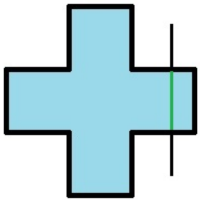

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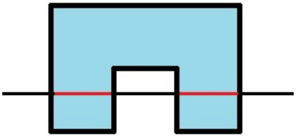

not semi-monotone

Monotone Functions

Let $f : X \to \mathbb{R}$ be bounded and continuous, $X \subset \mathbb{R}^n$ open, bounded, and nonempty, F the graph of f

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- one of the following (equivalent) conditions holds
  - $F \cap C$ connected for each coordinate cone C
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$\mathbf{f}$ is quasi-affine if for any $T = \text{span}\{x_{j_1}, \ldots, x_{j_\alpha}, y_{i_1}, \ldots, y_{i_\beta}\}$, $\alpha + \beta = n$, the projection $\rho_T : F \to T$ is injective iff the image $\rho_T(f)$ is n dimensional

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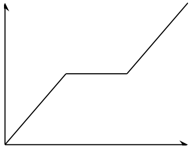
Examples



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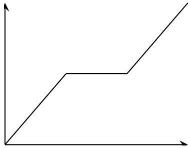

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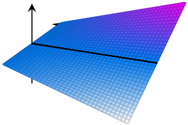

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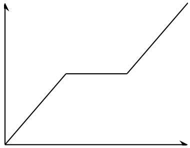

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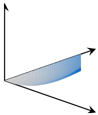

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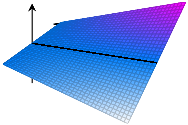

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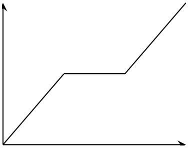

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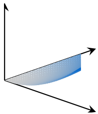

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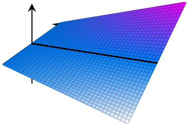
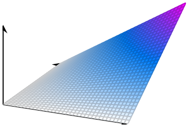
$0 < x < 1,\ 0 < y < 1 - x$
not monotone



$z = xy$ on
$0 < x < 1,\ -1 < y < 1$
not monotone



$z = xy$ on
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Monotonicity and Regularity

Theorem (S. Basu, A. Gabrielov, N. Vorobjov)

The graph $F \subset \mathbb{R}^{n+k}$ of a monotone map $\mathbf{f} : X \to \mathbb{R}^k$ on a semimonotone set $X \subset \mathbb{R}^n$ is a regular n-cell.

Theorem (J. Davis, P. Hersh, E. Miller)

Let $v \in W$ with $(i_1, \ldots, i_d)$ a reduced word for v. Then if for all $w \in W$, $w \leq v$, we have $f^{-1}_{(i_1,\ldots,i_d)}(p)$ is contractible for $p \in Y^{o}_{w}$, then Y_w is a regular cell complex for each $w \leq v$.

Route to an Alternate Proof

Theorem (J. Davis, P. Hersh, E. Miller)

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(Key ingredient in proof)

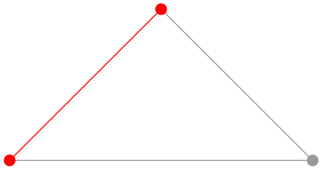
Let $\sim$ be an equivalence relation on the closed ball B^n so that

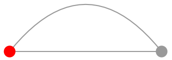
- all equivalence classes are contractible
- $S^{n-1}/\sim\, \cong S^{n-1}$
- if $x \sim y$ with $x \in S^{n-1}$, then $y \in S^{n-1}$
- if $x \sim y$ with $x \notin S^{n-1}$, then $y = x$

Then $B \cong B/\sim$

Example: $n = 3$

$v = (1, 2, 1)$, $p \in Y_{(1)}$, $f^{-1}_{(1,2,1)}(p)$ in red.





The Goal

Conjecture

Let $(i_1, \ldots, i_d)$ be a reduced word for $v \in W$, let $w \leq v$, and let $p \in Y_w^{\mathrm{o}}$. Then the strata of $f^{-1}_{(i_1,\ldots,i_d)}(p)$ are graphs of monotone maps, and hence this stratification is a regular cell decomposition of the fiber.

This holds in the cases $n = 3$ and $n = 4$, by computation

A few more examples: $n = 4$

$$f_{(1,3,2,1,3,2)} = \begin{bmatrix} 1 & t_1 + t_4 & (t_1 + t_4)t_6 + t_1 t_2 & t_1 t_3 t_5 \\ 0 & 1 & t_3 + t_6 & t_3 t_5 \\ 0 & 0 & 1 & t_2 + t_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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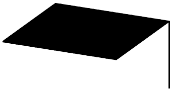
$$\begin{aligned} p &= (a, b, c, ab, 0, 0) \\ &\in Y^{\circ}_{(3,2,1)} = \{(x, y, z, xy, 0, 0) \mid x, y, z > 0, x + y + z = K\} \end{aligned}$$

$$\begin{aligned} f^{-1}(p) =& \{(t_1, t_2, 0, a - t_1, c - t_2, b) \mid 0 \le t_1 \le a, 0 \le t_2 \le c\} \\ & \cup \{(a, c, t_3, 0, 0, b - t_3) \mid 0 \le t_3 \le b\} \end{aligned}$$



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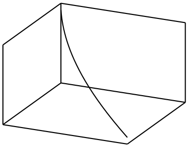
$$p = (a, b, 0, d, 0, 0)$$
$$\in Y^{o}_{(2,1,2)} = \{(x, y, 0, u, 0, 0) \mid x, y > 0, 0 < u < xy, x + y = K\}$$

$$f^{-1}(p) = \{(t_1, 0, \frac{ab-d}{a-t_1}, a - t_1, 0, \frac{d - t_1 b}{a - t_1}) \mid 0 \le t_1 \le d/b\}$$



Outline of Results

- Conjectured: strata of $f^{-1}_{(i_1,\dots,i_d)}(p)$ monotone for all $v, w \in S_n$, $w \le v$ ($v = (i_1, \dots, i_d)$, $p \in Y_w^{o}$).

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