Research Statement

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1. Overview

My work lies in the realms of real algebraic geometry and o-minimality. Real algebraic geometry has a flavor distinct from (complex) algebraic geometry. By stepping away from an algebraically closed field, we lose access to many of the tools ubiquitous there. However, by working over $\mathbb{R}$ (or more generally, a real closed field) rather than $\mathbb{C}$, we gain a linear order, expanding the range of questions we may ask. Real algebraic geometry thus centers around studying properties of semi-algebraic sets, which can be built up from polynomial equations and inequalities. I often consider more general sets, namely those definable in some o-minimal structure. O-minimal structures allow for the inclusion of certain functions which are not algebraic (examples include the exponential function or trigonometric functions restricted to bounded intervals), while still maintaining the strong finiteness properties inherent among semi-algebraic sets. More background on this and other topics can be found following this overview of my main results.

My most recent projects pertain to the role of symmetry (with respect to the action of some finite reflection group on $\mathbb{R}^n$) in the study of the topology of semi-algebraic sets. Various results suggest that one may leverage such symmetry when determining, for example, the Betti numbers of a semi-algebraic set (colloquially speaking, the number of holes of various dimension). Basu and Riener in [4] recently proved, in the case of the symmetric group $\mathfrak{S}_n$, results concerning the structure of the cohomology spaces of such sets, which yield an effective algorithm for computing their first l Betti numbers. I have strengthened those results by refining one of the tools used (the Gabrielov-Vorobjov construction) for the symmetric case. I now seek to extend the strategy of [4] to other types of symmetry.

The Gabrielov-Vorobjov construction, described in [9], is a technique for replacing an arbitrary set definable in some o-minimal structure over $\mathbb{R}$ with one that is compact, while preserving the first few homotopy and homology groups. The replacement set is defined by functions closely resembling those defining the original – an important facet when many complexity bounds make use of data such as the number of defining functions or (in the polynomial case) their degrees. When working with symmetry, though, one would like assurances that the group's action is preserved in translation. I have shown that, indeed, the maps involved in the Gabrielov-Vorobjov construction can be made equivariant.

Theorem 1. *Let G be a finite reflection group acting on $\mathbb{R}^n$ and say that $S \subset \mathbb{R}^n$ is a definable set described by continuous functions $\{h_1, \dots, h_s\}$. Say that S is symmetric and that the collection $\{h_1, \dots, h_s\}$ is invariant under the action of G. For an integer $m > 0$ and parameters $r > 0$ and $0 < \varepsilon_0, \delta_0, \dots, \varepsilon_m, \delta_m < 1$, we construct a symmetric closed and bounded set T described by functions of the form $h_i \pm \delta_j$ and $h_i \pm \varepsilon_j$ (along with the ball of radius r, $r^2 - (X_1^2 + \cdots + X_n^2)$). For sufficiently large r and $0 < \delta_0 \ll \varepsilon_0 \ll \dots \ll \varepsilon_m \ll \delta_m$, there exists an equivariant map $\psi : T \to S$ inducing (equivariant) isomorphisms*

$$\psi_{\#,k} : \pi_k(T, *) \to \pi_k(S, *') \qquad \text{and} \qquad \psi_{*,k} : H_k(T) \to H_k(S)$$

for $0 \le k \le m-1$ and epimorphisms for k=m.

For a set $S \subset \mathbb{R}^n$ defined by polynomials symmetric with respect to permutation of the variables, the action of $\mathfrak{S}_n$ allows one to decompose the cohomology spaces $H^i(S)$ as direct sums of copies of irreducible $\mathfrak{S}_n$-modules $\mathbb{S}^\lambda$ indexed over partitions λ of n. So, to understand $H^i(S) \cong_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} m_{i,\lambda}(S)\mathbb{S}^\lambda$, one need only calculate the mulitpicities $m_{i,\lambda}(S)$ with which the modules $\mathbb{S}^\lambda$ appear. Basu and Riener in [4] establish restrictions on the multiplicities appearing in the decomposition of a closed semi-algebraic set defined by symmetric polynomials. Via the equivariant Gabrielov-Vorobjov construction, though, one may replace an arbitrary symmetric semi-algebraic set by a closed and bounded one without altering the decomposition. Hence, the above theorem allows one to extend the results on multiplicities to arbitrary symmetric semi-algebraic sets. Basu and Riener's algorithm in [4] also now yields information about the mulitpicities.

Theorem 2. *Say that $S \subset \mathbb{R}^n$ is defined by symmetric polynomials $\{h_1, \dots, h_s\}$ having degree at most d (for some $d > 1$). Then $m_{i,\lambda}(S) = 0$ for any $\lambda \vdash n$ such that either* $\mathrm{length}(\lambda) \geq i + 2d - 1$ *or* $\mathrm{length}({}^t\lambda) \geq n - i + d + 1$*. (Compare to [4] Theorem 4 which requires that S be defined by closed conditions on the h_i's).*

Theorem 3. *There is an algorithm which takes as input a set of symmetric polynomials $\{h_1, \dots, h_s\}$ all having degree at most d and a formula in these polynomials describing a semi-algebraic $S \subset \mathbb{R}^n$, and for a chosen $l > 0$ computes the multiplicities $m_{i,\lambda}(S)$ for each $0 \leq i \leq l$ and $\lambda \vdash n$, as well as the Betti numbers $b_i(S)$ for $0 \leq i \leq l$. The complexity of this algorithm is bounded by $(snd)^{2^{O(d+l)}}$. (Compare to [4] Theorem 3, which only promises a computation of the first l Betti numbers).*

Though the equivariant Gabrielov-Vorobjov construction holds for any finite reflection group, Basu and Riener in [4] restrict their attention to the symmetric group (what is known as type A in certain classifications). I wish to extend their results to other types of symmetry. Currently, I am working in type B, where the symmetric group is replaced by the group of signed permutations. A special class of sets known as *Vandermonde varieties* plays a key role in the study of symmetric semi-algebraic sets, and so I am working to extend results on Vandermonde varieties to type B.

In type B, a Vandermonde variety is defined by the equations $X_1^{2m} + \cdots + X_n^{2m} = y_m$ for m between 1 and some d, with $y_m \in \mathbb{R}$. Basu and Riener demonstrate and then use that in type A, the intersection of a Vandermonde variety with a fundamental region of $\mathbb{R}^n$ is a topologically regular cell. I have now shown the same in type B.

Theorem 4. *The intersection of a type B Vandermonde variety in $\mathbb{R}^n$ with the set defined by $0 \leq X_1 \leq \cdots \leq X_n$ is either empty, a point, or a semi-algebraic regular cell of dimension $n - d$.*

I in fact showed something stronger: the interior of the above intersection is a *monotone* set. The concept of monotonicity was introduced by Basu, Gabrielov, and Vorobjov in [3] and [2]. Where applicable, it allows one to reduce the bulk of the work in proving that a set is topologically regular (typically a very touchy operation) to proving the connectedness of intersections with certain affine subspaces. I am interested in finding other instances in which to apply monotonicity.

I have already begun investigating monotonicity and totally nonnegative spaces. We say that a matrix with real entries is totally nonnegative if all minors are nonnegative. In type A, the totally nonnegative matrices of interest form a space with strata corresponding naturally to the elements of $\mathfrak{S}_n$. One typically considers subspaces $\mathrm{Lk}(u, v)$ (the *link* of u in v) for $u, v \in \mathfrak{S}_n$, which are compact and capture the strata "between" u and v. Fomin and Shapiro in [8] conjectured and Hersh in [12] proved that these spaces are regular CW complexes. However, it may be possible to offer a simpler proof via monotonicity.

Problem 5. *For each $u, v \in \mathfrak{S}_n$, are the cells in the stratification of* $\mathrm{Lk}(u, v)$ *monotone sets? (If so, then each* $\mathrm{Lk}(u, v)$ *is a regular CW complex).*

Motivated by further work of Davis, Hersh, and Miller in [7], I have studied the fibers of maps $f_{(i_1,\ldots,i_d)}$ indexed by words $(i_1,\ldots,i_d)$ for the elements of $\mathfrak{S}_n$. These fibers also admit a stratification, the regularity of whose cells implies the regularity of $\mathrm{Lk}(\mathrm{id},u)$ for various $u \in \mathfrak{S}_n$. I have shown monotonicity of certain fiber cells, but also encountered instances where monotonicity fails.

Theorem 6. *For certain classes of words $(i_1,\ldots,i_d)$, the fibers of $f_{(i_1,\ldots,i_d)}$ stratify into monotone cells. As a result, $\mathrm{Lk}(\mathrm{id},u)$ is a regular CW complex for all $u \in \mathfrak{S}_n$ for $n \leq 4$.*

After supplying details on their settings, I will elaborate on each of the three questions I have studied.

2. Background

O-minimality lies at the confluence of real algebraic geometry and model theory. Geometrically, a *structure* over $\mathbb{R}$ is a selection of subsets of $\mathbb{R}^n$ from each dimension n, which is closed under desired operations – these are called *definable* sets. Specifically, finite unions and intersections, complements, Cartesian products, and coordinate projections of definable sets should be definable, and the various diagonals $\{(x_1,\ldots,x_m) \in \mathbb{R}_m \mid x_i = x_j\}$ are always definable. We call a structure *o-minimal* if it contains the order relation and if the definable subsets of $\mathbb{R}$ are exactly the finite unions of points and intervals.

This last condition, that subsets of $\mathbb{R}$ should be as simple as possible given the presence of an ordering, provides the name, order minimal. More importantly, it forces the sets definable in an o-minimal structure to behave "nicely." As a consequence of a particular cell decomposition applicable to definable sets, we have that definable sets can be triangulated (given a compact definable set A, we can find a finite simplicial complex Λ definably homeomorphic to A). We also have certain triviality results for definable families (for example, for a definable family $\{X_\alpha\}_{\alpha\in A}$, there are only finitely many homeomorphism types among the sets X_α). For more background on o-minimal geometry, consult [14] or [6].

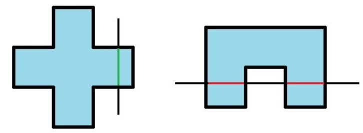
One may notice the symmetric group $\mathfrak{S}_n$ running as a common thread through all three of my projects. In each case, we think of $\mathfrak{S}_n$ as a Coxeter system, that is, a group together with a set of designated generators or *simple reflections*. For $\mathfrak{S}_n$, we will take these to be the adjacent transpositions $s_i = (i\ i+1)$ for $1 \leq i \leq n-1$. In the projects concerning symmetric sets, we interpret these geometrically as reflections through the hyperplanes $\{X_i = X_{i+1}\}$. In the setting of totally nonnegative spaces, we will focus instead on combinatorial aspects. In particular, for an element $u \in \mathfrak{S}_n$, we consider expressions $s_{i_1}\cdots s_{i_d}$ of generators multiplying to u. We say such an expression is reduced if it is of minimal length among expressions for u. From this, we obtain an ordering (called the *Bruhat ordering*) on the elements of $\mathfrak{S}_n$: we say $u \leq v$ if there is a reduced expression for u which is a subexpression of some reduced expression for v.

Coxeter systems in general are classified into certain types; type A_{n-1} refers to $\mathfrak{S}_n$ generated by the adjacent transpositions, while type B_n introduces the idea of signed permutations, and adds reflection through $X_1 = 0$ to the collection of Coxeter generators. Type A often serves as the most accessible case, but the possibility of generalizing to other types typically lurks within type A results.

Monotonicity was developed for sets definable in a fixed o-minimal structure over $\mathbb{R}$ by Basu, Gabrielov and Vorobjov. We say an open bounded $X \subset \mathbb{R}^n$ is *semimonotone* if for any affine coordinate subspace of the form

$$S = \{\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_{j_1} = c_1,\ldots,x_{j_m} = c_m\}$$

(with the c_i's various constants in $\mathbb{R}$), the intersection $S \cap X$ is connected. To address sets of higher codimension, Basu, Gabrielov, and Vorobjov introduce monotone maps. A map $\mathbf{f}$ from a semimonotone subset of $\mathbb{R}^n$ to $\mathbb{R}^k$ is *monotone* if it is *quasiaffine* (projections of the graph Γ to any n of the coordinate axes are injective whenever they maintain dimension) and the intersection of Γ with any affine coordinate subspace is connected. We will say that a set is monotone if it is either semimonotone or the graph of a monotone map.

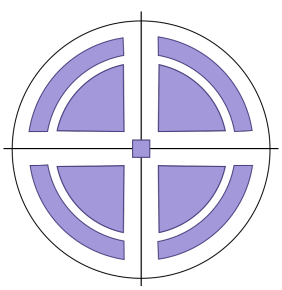


The left-hand set is semimonotone, while the right-hand set is not

For an open $U \subset \mathbb{R}^n$, we say that U is a regular cell if $(\overline{U}, U)$ is homeomorphic as a pair to $(\overline{B}^n, B^n)$, where B^n is the open ball of dimension n. Gabrielov, Basu, and Vorobjov in [2] show that any monotone set is a regular cell. Since to demonstrate monotonicity we only need to check a few conditions such as connectivity, this offers a useful route to proving regularity, which is often a delicate property to demonstrate directly.

3. Equivariant Gabrielov-Vorobjov Construction

Gabrielov and Vorobjov in [9] develop a means of approximating a given definable subset S of $\mathbb{R}^n$ by a closed and bounded set T. If S is defined by equalities and inequalities involving continuous definable functions $\{h_1, \ldots, h_s\}$, then T is defined by a similar collection of functions, and for a chosen integer $m > 0$, there are isomorphisms $\pi_k(T) \to \pi_k(S)$ and $H_k(T) \to H_k(S)$ for $1 \leq k \leq m-1$. Motivated by a recent paper of Basu and Riener which studies algorithms for computing the Betti numbers of sets defined by symmetric polynomials, I proved that one may construct an equivariant map $T \to S$, and thereby obtain equivariant maps of homotopy and homology.



$S_{\delta,\varepsilon}$ for $S = \{(x \neq 0 \wedge y \neq 0) \vee (x = 0 \wedge y = 0)\} \cap \{x^2 + y^2 \leq r^2\}$

A consequence of triviality allows one to intersect a definable set with a closed ball of sufficiently large radius r while maintaining homotopy equivalence. For a bounded set S, Gabrielov and Vorobjov's approximating set $T = S_{\delta_0,\varepsilon_0} \cup \cdots \cup S_{\delta_m,\varepsilon_m}$ is the union of a finite number of members selected from a family $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ of compact sets representing S. If we have that S is symmetric and that G applied to $\{h_1, \ldots, h_s\}$ is again $\{h_1, \ldots, h_s\}$, then the sets $S_{\delta,\varepsilon}$ from among which T is built are automatically symmetric, and hence so is T. The equivariance of the maps from the theorem, however, requires more attention.

Gabrielov and Vorobjov construct their maps of homotopy and homology groups by considering an intermediate set $V = V(\varepsilon_0, \delta_0, \ldots, \varepsilon_m, \delta_m)$. The construction of V is based on a triangulation of S in some larger compact set. In order to have any hope of symmetry for V, much less equivariance of the maps connecting this set to S and T, one must begin with a properly symmetric triangulation. The equivariant triangulation (in the spirit of the triangulation algorithm given by Coste in [6]) of symmetric definable sets is one pivotal new result.

Theorem 7. *Let A be a closed and bounded definable subset of $\mathbb{R}^n$ symmetric under the action of a finite reflection group G, and let $S_1, \ldots, S_l$ be symmetric sets which are subsets of A. There exists a triangulation (Λ, Φ) of A adapted to $S_1, \ldots, S_l$ such that the simplicial complex Λ is symmetric and $\Phi : |\Lambda| \to S$ is equivariant under the action of G.*

In the original paper, Gabrielov and Vorobjov could take S as connected without losing generality and thus ignore basepoint considerations when discussing homotopy groups. However, this assumption might easily destroy a set's symmetry. To account for basepoints, my symmetric version constructs not only maps on the level of homotopy and homology (as in the original), but also an equivariant map from T to S.

As described in the overview, equivariance in the Gabrielov-Vorobjov construction provides a strengthening of results of Basu and Riener in [4]. Here, one considers sets S described by symmetric polynomials of bounded degree. From the equivariance of the maps $H_i(T) \to H_i(S)$, one sees that $H^i(T)$ and $H^i(S)$ have the same structure as $\mathfrak{S}_n$-modules, and hence the same isotypic decomposition

$$\bigoplus_{\lambda \vdash n} m_{i,\lambda}(S)\mathbb{S}^\lambda \cong_{\mathfrak{S}_n} H^i(S) \cong_{\mathfrak{S}_n} H^i(T) \cong_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} m_{i,\lambda}(T)\mathbb{S}^\lambda$$

This means that each multiplicity $m_{i,\lambda}(S)$ is equal to $m_{i,\lambda}(T)$, and so any results obtained about the mulitpicities of T as a closed and bounded set hold for those of S as well.

4. Vandermonde Varieties

I would like to extend the results of Basu and Riener in [4] to more general classes of symmetry. Their theorem concerning the isotypic decomposition of the cohomology spaces of a set S defined by symmetric polynomials (ref. Thm 2 here) follows from a similar statement about Vandermonde varieties. I am currently working to prove an analogue of this statement in type B.

Vandermonde varieties were studied primarily in type A by Arnold ([1]), Givental ([11]), and Kostov ([13]). There, they appear as level sets of the first d weighted Newton power sums $p^{(n)}_{A,m} = w_1X_1^m + \cdots + w_nX_n^m$ (for some weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n_{>0}$). In the more general setting, one may define Vandermonde varieties using the first d generators of the ring of polynomials invariant under the action of the group in question. In type B, one can take the Newton power sums of even degree: $p^{(n)}_{b,m} = w_1X_1^{2m} + \cdots + w_nX_n^{2m}$. For a given $\mathbf{y} \in \mathbb{R}^d$, then, a Vandermonde variety in type B has the form $V = \{\mathbf{x} \in \mathbb{R}^n \mid p^{(n)}_{B,\mathbf{w},1}(\mathbf{x}) = y_1, \ldots, p^{(n)}_{B,\mathbf{w},d}(\mathbf{x}) = y_d\}$.

When all weights are equal, the Vandermonde variety's symmetry should allow one to understand the set as a whole simply by studying it on a fundamental region – here we will use the type B Weyl chamber defined by $0 \leq X_1 \leq \cdots \leq X_n$. To investigate (co)homology, we must pay particular attention to intersections with the walls of the Weyl chamber. Since these walls correspond to intersections of reflection hyperplanes, we may index them via subsets T of the Coxeter generators of the group. I have reworked a few results of Arnold and Kostov for type B in order to show that the intersection of a Vandermonde variety with the Weyl chamber is a regular cell. Denoting the intersection of V with the wall indexed by T as V^T, I have established the following regarding the walls.

Proposition 8. *Let $d \geq 2$, let $\mathbf{y} \in \mathbb{R}^d$, and let $T \subset \mathrm{Cox}_B(n)$. Then $H^i(V, V^T) = 0$ for all i and T satisfying either $i \leq \mathrm{card}(T) - 2d$ or $i \geq \mathrm{card}\, T + 1$.*

From this, following Basu and Riener's strategy from [4], I hope in the coming months to prove a type B analogue of Theorem 2, first for Vandermonde varieties and then for general symmetric sets. From there, I would like to move to the other Lie types, and perhaps beyond.

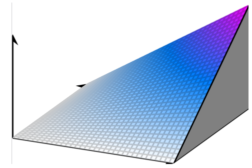
5. Totally Non-negative Spaces

I also hope to apply monotonicity as an alternate route to proving a conjecture concerning totally non-negative spaces. In type A_{n-1}, we consider those matrices in $SL(n, \mathbb{R})$ which are upper triangular, have 1's on the diagonal, and whose minors are all nonnegative. We can decompose this space into strata based on which minors are zero and which are strictly positive.

Example 9. The totally nonnegative part of $SL(3,\mathbb{R})$ is the space of matrices

$$\left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x \geq 0, y \geq 0, 0 \leq z \leq xy \right\}$$

Seeing this as a semi-algebraic subset of $\mathbb{R}^3$, the six strata of this space are the interior $\{(x,y,z) \in \mathbb{R}^3_{>0} \mid z < xy\}$, the surface $z = xy$ for $x, y > 0$, the positive xy-plane, the positive x-axis, the positive y-axis, and the origin.



The totally nonnegative part of $SL(3,\mathbb{R})$

The strata of this space correspond very naturally to the elements of $\mathfrak{S}_n$, and the containment ordering on strata precisely reflects the Bruhat ordering on $\mathfrak{S}_n$. Motivated by Björner's study of the connection between CW complexes and partially ordered sets (see [5]), Fomin and Shapiro in [8] propose totally nonnegative spaces as a source of naturally arising regular CW complexes whose face posets are intervals within the Bruhat order. For $u \leq v$ elements of $\mathfrak{S}_n$, they define spaces $\mathrm{Lk}(u,v)$ which inherit a stratification from the totally nonnegative part of $SL(n,\mathbb{R})$. They show that $\mathrm{Lk}(u,v)$ is indeed a CW complex whose face poset corresponds to the Bruhat interval $[u,v]$, and conjecture that each cell is regular. This was proved by Hersh in [12]. For the link of the identity in type A_{n-1}, regularity also follows as a corollary of results by Galashin, Karp, and Lam in [10]. However, we hope that monotonicity may offer an alternate, simpler proof.

My work thus far on this question has centered on further results by Davis, Hersh, and Miller. They interpret the zero links of our totally nonnegative spaces as the images of the standard simplex Δ_{d-1} under certain maps $f_{(i_1,\ldots,i_d)}$ indexed by elements $u = s_{i_1} \cdots s_{i_d}$ in $\mathfrak{S}_n$. In [7], they demonstrate that, if one can show that all fibers $f^{-1}_{(i_1,\ldots,i_d)}(p)$ are contractible, one may conclude that the zero links of the totally nonnegative spaces are regular cells.

I had hoped to demonstrate that the fibers $f^{-1}_{(i_1,\ldots,i_d)}(p)$ can be decomposed into monotone sets, which based on results of Davis, Hersh, and Miller would suffice to establish contractibility of the fibers. The techniques I developed, pertaining to the geometric interpretation of various 'moves' for transforming a non-reduced word to a reduced one, demonstrate monotonicity of many cells in the decompositions of the relevant fibers, including all those present for $n \leq 4$. Unfortunately, in certain examples in the $n = 5$ case, fibers appear which while regular are not monotone.

Though the application of monotonicity to the Fomin-Shapiro conjecture is thus not as straightforward as it originally appeared, there remain avenues to pursue. In the fiber case, changing the word defining the map $f_{(i_1,\ldots,i_d)}$ or applying some homeomorphisim might resolve the issue. I hope also to investigate more fully the monotonicity of the original links. Should this technique prove successful, I would turn to applying monotonicity to the analogous problem in the other Lie types.

6. Future Directions

While the theme of symmetry in all of these projects came about more or less by coincidence, it provides an intriguing thread to follow in future work. I would like to continue exploring and developing the role of symmetry and equivariance within real algebraic geometry. From my early days of studying monotonicity, I have hoped to continue searching for opportunities to apply the concept to questions of regularity. The commonalities between my projects also inspire my curiosity about real algebraic geometry in the context of group actions and representation theory. I hope to continue exploring and expanding these connections in the coming years.

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