Research Statement

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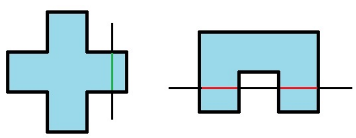
I work in the areas of o-minimality and real algebraic geometry. Though the real numbers may seem like safe and familiar territory, early in a real analysis class one begins to discover strange and counterintuitive properties. One cannot always draw a path connecting points within a connected set, and continuous functions can be so jagged that they are nowhere differentiable. The Cantor set acts as a counterexample to a host of reasonable-sounding analytical and topological claims. O-minimality lets us step away from this chaos and consider only 'nice' sets and functions. Real algebraic geometry specifically is the study of semialgebraic sets, which are subsets of $\mathbb{R}^n$ defined by polynomial equations and inequalities.

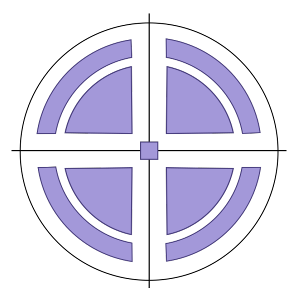
My own projects have incorporated the idea of symmetry in various forms. Recently, I revisited a method for replacing definable sets (any set we consider 'nice') with compact ones. If we begin with a symmetric set, I have shown that we may perform the replacement in such a way that certain related groups (the homology and homotopy groups) maintain the same structure relative to symmetry before and after the replacement. I have also studied a concept termed *monotonicity*. A definable set is monotone if intersections with lines, planes, etc. parallel to the coordinate axes are connected. If one can show that a set is monotone, one can conclude that it is topologically *regular*, i.e., that the set together with its boundary can be stretched and deformed to a ball of some dimension. I am studying how one might apply monotonicity to certain classes of sets (all with connections to symmetry) in order to prove regularity.



The left-hand set is monotone; the right-hand set is not

Symmetry and Compact Approximation

I have adapted a construction developed by Gabrielov and Vorobjov in [4] for replacing an arbitrary set definable in some o-minimal structure by one that is compact. Gabrielov and Vorobjov's approximation seeks to preserve the homotopy and homology groups of the set (colloquially speaking, these are two different ways to keep track of holes of various dimensions within a set). Many theorems about homology require a closed and bounded set, making this a useful tool. However, there has recently been interest in how we can leverage symmetry (relative to a finite reflection group) to determine information about homology, an aspect the original Gabrielov-Vorobjov construction does not consider.



A building block of T for $S = \{(x \neq 0 \wedge y \neq 0) \vee (x = 0 \wedge y = 0)\}$

Beginning with a definable set S, Gabrielov and Vorobjov provide a set T and maps $\psi_{\#,k} : \pi_k(T) \to \pi_k(S)$ and $\psi_{*,k} : H_k(T) \to H_k(S)$ between the homotopy groups π_k and homology groups H_k of S and T which are isomorphisms for k less than any chosen degree of accuracy m. I have shown that, if S is symmetric, there is an *equivariant* map $\psi : T \to S$ (reflecting has an analogous result in S and T) from which $\psi_{\#,k}$ and $\psi_{*,k}$ are derived.

Using the new construction, I have strengthened results by Basu and Riener in [2] regarding sets described by $\mathfrak{S}_n$-symmetric polynomials (interchanging variables does not affect the polynomial). I generalized their theorem on the structure of cohomology spaces to sets not necessarily closed and bounded. I also adapted their main algorithm for computing Betti numbers of symmetric semialgebraic sets to also supply this more detailed structural information.

Monotonicity

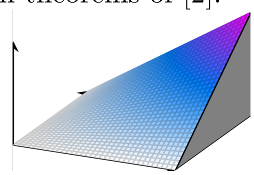
My other main research focus is monotonicity, as described by Basu, Gabrielov, and Vorobjov in [1]. We say an open bounded $X \subset \mathbb{R}^n$ is monotone if its intersection with any affine coordinate subspace (obtained by specifying any selection of coordinates to values in $\mathbb{R}$) is connected. If a set X is monotone, then it is a regular cell, i.e. $\overline{X}$ is homeomorphic to the closed ball $\overline{B}$ via a map that sends the interior X homeomorphically to B. Though monotonicity is a much stricter condition than regularity, it is much easier to check. Regularity is a sought-after property; for a CW complex (a topological space built up by gluing together balls of various dimensions), if we know that each cell is regular and we know how the cells fit together, then we understand the space completely, topologically speaking.

Monotonicity plays a role in my current project: expanding Basu and Riener's theorems and algorithms in [2] to other types of symmetry. There, sets known as Vandermonde varieties serve as models for more general symmetric sets. In my case of interest, these are defined by equations of the form $X_1^{2m} + \cdots + X_n^{2m} = c_m$ for various m and constants $c_m \in \mathbb{R}$. Via monotonicity, I have shown that the intersection of a Vandermonde variety with the set $0 \leq X_1 \leq \cdots \leq X_n$ is a regular cell complex. From this, I hope to develop analogues of Basu and Riener's main theorems of [2].

Monotonicity also inspired my investigation of totally nonnegative spaces. In the $n = 3$ case, the totally nonnegative part of $SL(3, \mathbb{R})$ is the space of matrices

$$\left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x \geq 0, y \geq 0, 0 \leq z \leq xy \right\}$$

If we plot the coordinates $(x, y, z) \in \mathbb{R}^3$ satisfying these inequalities, we obtain the surface shown.



The totally nonnegative part of $SL(3, \mathbb{R})$

There is a very natural decomposition of this space into cells which fit together in a manner that, surprisingly, echoes a particular ordering on the symmetric group $\mathfrak{S}_n$. Fomin and Shapiro in [3] conjectured that this gives us a regular CW complex. Hersh proved this conjecture in 2014 in [5]. However, it seems possible that these cells are not only regular but monotone. If I can demonstrate as much, it would offer a simpler proof of the Fomin-Shapiro conjecture.

Future Research Opportunities

I intend to continue investigating monotonicity both of totally nonnegative spaces and Vandermonde varieties beyond graduate school. In each case, there are generalizations and related questions which I would like to pursue. Monotonicity is relatively unknown, and so there are many other classes of sets for which monotonicity might reveal a simple regularity proof. In the longer term, I would like to continue to tease out the connections between real algebraic geometry and symmetry.

I appreciate my field because it allows me to interact with concepts from a wide variety of mathematical areas and yet work with concrete objects. These characteristics also mean that undergraduates with varied backgrounds and interests can share in my research. In the case of totally nonnegative spaces, a significant part of the work ahead of me involves computing examples to discover patterns within them. This requires little more than an understanding of matrix multiplication and a willingness to engage in a large amount of symbolic manipulation, but provides a platform to discover a number of fascinating concepts within algebra and topology. Students with an interest in programming would also find opportunities to build computational tools. I look forward to the chance to share both broad and specific aspects of what I have studied.

References

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