Safety Protocols

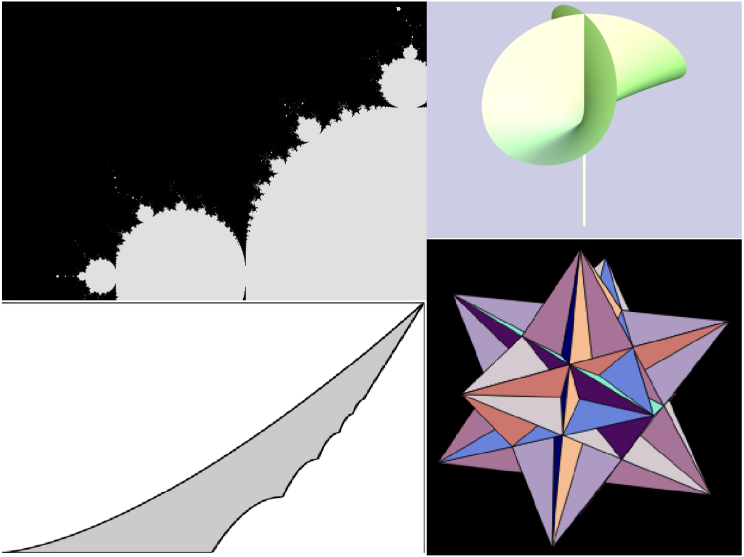
for the Cartography of Lands Unknown, Fantastic, and Symmetric

Alison Rosenblum

Purdue University

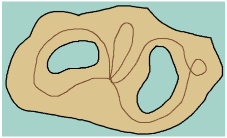
Student Colloquium
April 6, 2022

The Many Realms in $\mathbb{R}^n$



Tools of the Trade

Homotopy ($\pi_k(S)$ the k-th homotopy group of the space S)



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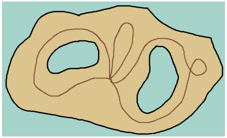
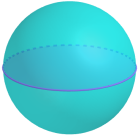
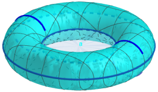


Homology ($H_k(S)$: kth (singular) homology group of S)





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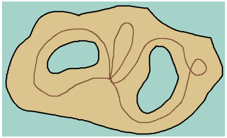

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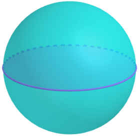
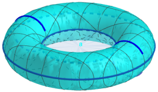
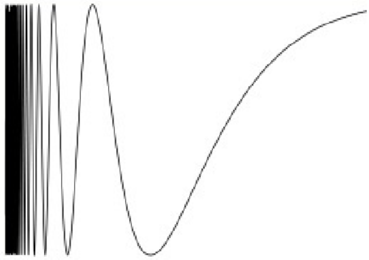
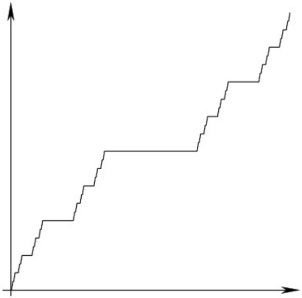
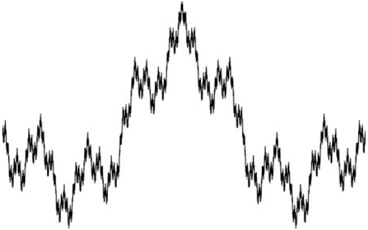
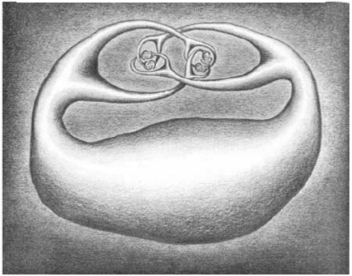




The kth Betti number of S is $b_k(S) = \text{rank}(H_k(S))$

- the number of "holes" of dimension k in S

The Dangers of $\mathbb{R}^n$

Protocol 1

O-minimal Structures

Building a Structure

Definition

A structure $\mathcal{S}$ on $\mathbb{R}$ is a collection of subsets of $\mathbb{R}^n$ for each n which contains

1. Unions, intersection, and complements of sets in $\mathcal{S}$
2. Cartesian products $(A \times B)$ of sets in $\mathcal{S}$
3. Coordinate projections of sets in $\mathcal{S}$ ($\pi(A)$ where $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ takes $(x_1, \dots, x_m, x_{m+1}) \mapsto (x_1, \dots, x_m)$).
4. Each set $\{(x_1, \dots, x_m) \in \mathbb{R}_m \mid x_i = x_j\}$

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An o-minimal structure is a structure with

5. $\{(x, y) \in \mathbb{R}^2 \mid x < y\}$ in $\mathcal{S}$
6. All subsets of $\mathbb{R}^1$ in $\mathcal{S}$ are finite unions of points and intervals

O-minimal Structures

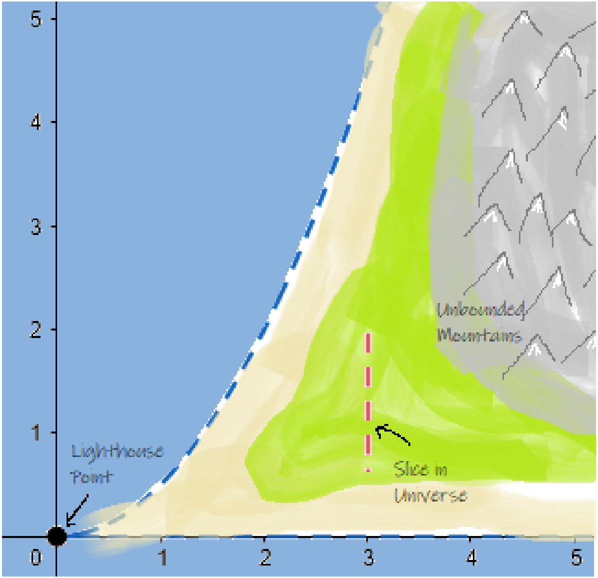
Example O-minimal structures

- Semi-algebraic sets
  - defined by polynomial equalities and inequalities
- Structure generated by e^x
- Structure generated by $\sin(x)$ defined on e.g. $-\pi/2 \leq x \leq \pi/2$

Advantages

- Avoid infinite(simal) traps
- Decompose definable sets into simple components

The Dangers that Remain

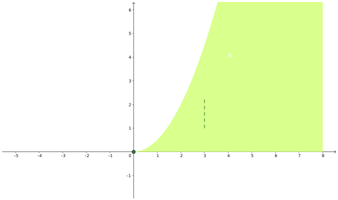


Protocol 27-i

The Gabrielov-Vorobjov Construction

The Goal

Example



$$S = \{(x, y) \mid x > 0, 0 < y < \frac{1}{2}x^2\}$$
$$\setminus \{(3, y) \mid 1 \le y \le 2.5\} \cup \{(0, 0)\}$$

The Goal

- Replace S with a closed and bounded set T

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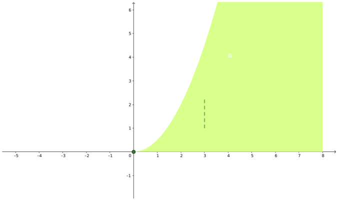

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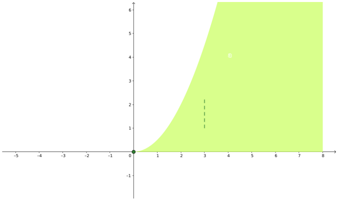

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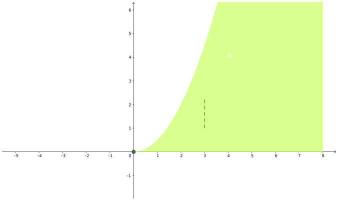
The Goal

- Replace S with a closed and bounded set T
- Retain information about homotopy and homology
- T described by functions similar to those defining S

Example



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Bounds of Exploration

Definable sets: eventually boring

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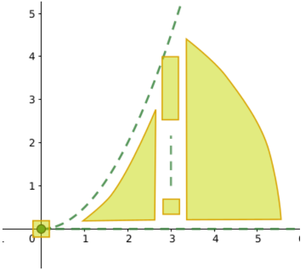
Theorem (Conic Structure at Infinity)

Let $S \subset \mathbb{R}^n$ be a definable set. Then there exists a $0 < r \in \mathbb{R}$ such that S is definably homotopy equivalent to $S \cap \overline{B(0,r)}$.

Exploring S: can stay inside ball of radius r and not miss anything interesting

Building Blocks

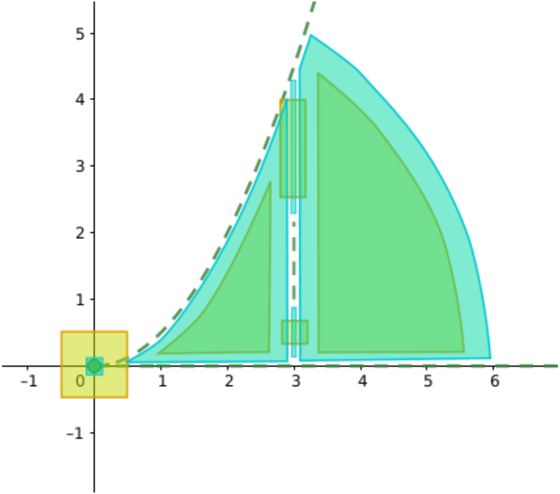
Let $\varepsilon, \delta > 0$
$S_{\delta,\varepsilon}$: closed and bounded set based on S



$$\begin{aligned} S_{\delta,\varepsilon} =& \{-\varepsilon \le x \le \varepsilon, -\varepsilon \le y \le \varepsilon\} \cup \{\delta \le x \le 3-\delta, \delta \le y \le \frac{1}{2}x^2 - \delta\} \\ & \cup \{3-\varepsilon \le x \le 3+\varepsilon, \delta \le y \le 1-\delta\} \cup \{3-\varepsilon \le x \le 3+\varepsilon, 2.5+\delta \le y \le \frac{1}{2}x^2 - \delta\} \\ & \cup \{3+\delta \le x, \delta \le y \le \frac{1}{2}x^2 - \delta, x^2+y^2 \le \frac{1}{\delta}\} \end{aligned}$$

The Replacement Set

T: Layer several $S_{\delta,\varepsilon}$'s (m: number of extra layers)



Here, $T = S_{\delta_0,\varepsilon_0} \cup S_{\delta_1,\varepsilon_1}$

The Official Theorem

And what about homotopy and homology?

Theorem (Gabrielov and Vorobjov)

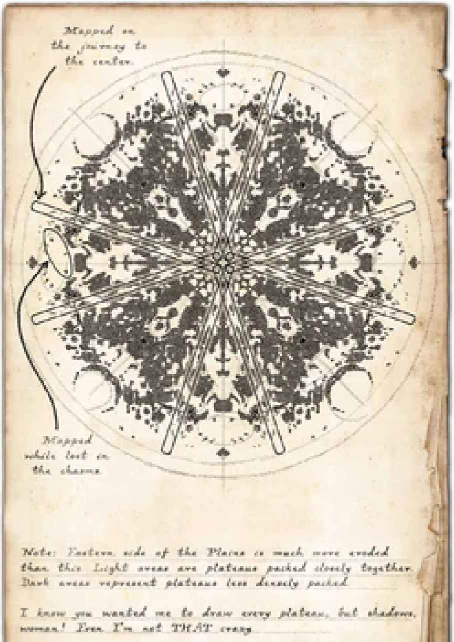
For any choice of $m > 0$ and small enough ε_i's and δ_i's, we have surjective maps

$$\psi_k : \pi_k(T) \to \pi_k(S)$$
$$\phi_k : H_k(T) \to H_k(S)$$

which are isomorphisms for $0 \leq k \leq m - 1$.

The Symmetric Lands

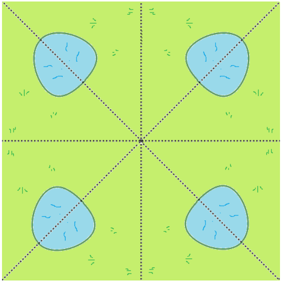
Advantage of Symmetry



Exploring the Symmetric Lands

Symmetry: action of dihedral group D_4

- Group elements reflect or rotate
- Set: eight copies of fundamental region



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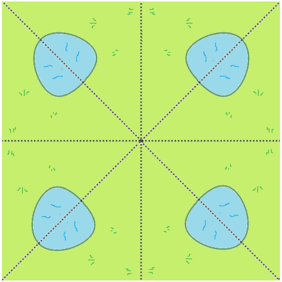

So what about homotopy and homology?

The Setting

Symmetry: action of symmetric group $\mathfrak{S}_n$

- Group elements interchange variables
- $\mathbb{R}^2$: reflection symmetry over line $y = x$

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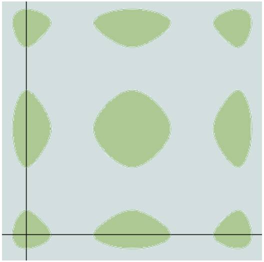
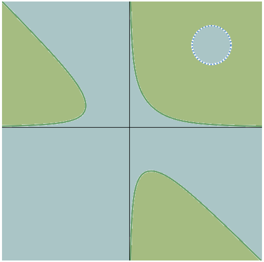


$x^2(x-1)^2(x-2)^2 + y^2(y-1)^2(y-2)^2 \leq 0.1$



$x^2y + xy^2 \geq 1$ and $(x-3)^2 + (y-3)^2 > \frac{1}{2}$

Using Symmetry's Advantages

Setting: S defined by symmetric polynomials in n variables

- s: number of polynomials
- d: largest degree

Theorem (Basu and Riener)

There is an algorithm computing each Betti number $b_k(S)$ of S for $0 \leq k \leq l$, with complexity bounded by

$$(snd)^{2^{O(d+l)}}$$

(Without symmetry, best known algorithm for computing all Betti numbers has doubly exponential complexity in n)

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Strategy: use Gabrielov-Vorobjov construction (and do a lot of other things)

Gabrielov-Vorobjov Construction Revisited

Reminder

- Replace S with closed and bounded set T
- Isomorphisms $\psi_k : \pi_k(T) \to \pi_k(S)$ and $\phi_k : H_k(T) \to H_k(S)$ for $0 \leq k \leq m - 1$

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Advantages:

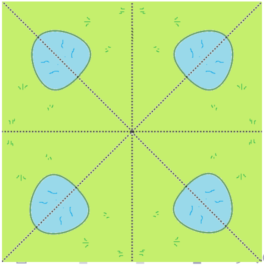
- S defined by symmetric polynomials $\Rightarrow$ so is T

Question:

- Are ψ_k and ϕ_k equivariant?

Definition

Let $f : X \to Y$ with G a group acting on both X and Y. Then f is equivariant if for all $g \in G$, $f(g(x)) = g(f(x))$



Equivariant Gabrielov-Vorobjov Construction

Good news: can adapt Gabrielov-Vorobjov construction so maps are equivariant.

Theorem (Basu and R., 2022+)

Let T be as described in the Gabrielov-Vorobjov construction, $m > 0$ and ε_i's and δ_i's small enough. Then we have equivariant surjective maps

$$\psi_k : \pi_k(T) \to \pi_k(S)$$
$$\phi_k : H_k(T) \to H_k(S)$$

which are isomorphisms for $0 \leq k \leq m - 1$.

Implications

Basu and Riener: decompose cohomology spaces

$$H^k(S) \simeq_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} m_{k,\lambda}(S)\mathbb{S}^\lambda$$

- $\mathbb{S}^\lambda$ components of the decomposition
- $m_{k,\lambda}$ number of times each component appears

If S is closed and bounded, can effectively compute each mulitpicity $m_{k,\lambda}$.

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With equivariance: decomposition the same for S and replacement T
$\Rightarrow$ can effectively compute each $m_{k,\lambda}$ for arbitrary semialgebraic S.

References I

[1] Saugata Basu, Richard Pollack, and Marie-Françoise Coste-Roy. *Algorithms in Real Algebraic Geometry*. eng. Vol. 10. Algorithms and Computation in Mathematics. Berlin, Heidelberg: Springer Berlin / Heidelberg, 2006. ISBN: 3642069649.

[2] Andrei Gabrielov and Nicolai Vorobjov. "Approximation of definable sets by compact families, and upper bounds on homotopy and homology". In: *Journal of the London Mathematical Society* 80.1 (2009), pp. 35–54.

[3] Saugata Basu and Cordian Riener. "Vandermonde varieties, mirrored spaces, and the cohomology of symmetric semi-algebraic sets". In: *Foundations of Computational Mathematics* (2021), pp. 1–68.

Isotypic Decomposition

Assume $S \subset \mathbb{R}^n$ symmetric (so, $\mathfrak{S}_n$ acts on $H^i(S)$)

Isotypic decomposition

$$H^k(S) \simeq_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} m_{k,\lambda}(S)\mathbb{S}^\lambda$$

- λ partition of n
- $\mathbb{S}^\lambda$ Specht module (dimension computed via "hook length formula")
- $m_{k,\lambda}$ mulitpicity of $\mathbb{S}^\lambda$ in $H^k(S)$

So $b_k(S) = \sum_{\lambda \vdash n} m_{k,\lambda} \dim(\mathbb{S}^\lambda)$

Multiplicities

$$H^k(S) \simeq_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} m_{k,\lambda}(S)\mathbb{S}^\lambda$$

Problem: number of partitions grows exponentially with n

Theorem (Basu and Riener)

S $\mathcal{P}$-closed (and bounded) semialgebraic set, $\mathcal{P} = \{h_1, \ldots, h_s\} \subset \mathbb{R}[X_1, \ldots, X_n]^{\mathfrak{S}_n}_{\leq d}$. Then

(a) $m_{i,\lambda} = 0$ *for* $\mathrm{length}(\lambda) \geq 1 + 2d - 1$

(b) $m_{i,\lambda} = 0$ *for* $\mathrm{length}({}^t\lambda) \geq n - i + d + 1$

With equivariance in Gabrielov and Vorobjov construction, theorem holds for S not necessarily closed and bounded