A Tourist's Guide to O-Minimality

Alison Rosenblum

Purdue University

Student Colloquium
March 20, 2019

The Real Numbers

Remember that the real line is not the safe and simple locale that people assume at first glance; it's a wild jungle. You never know when you might stumble upon a compact, uncountable, totally disconnected, nowhere dense set of measure zero just as it starts to accumulate everywhere.

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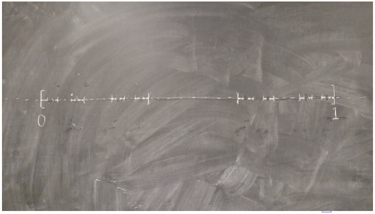
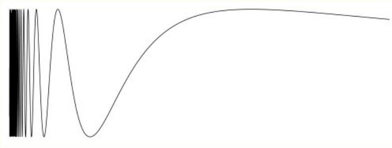
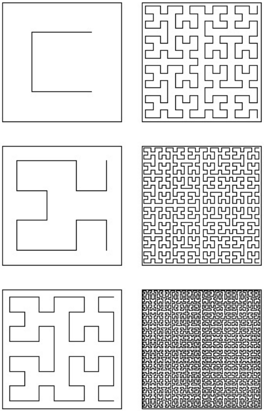
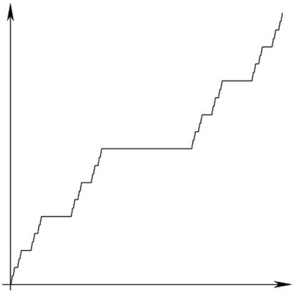
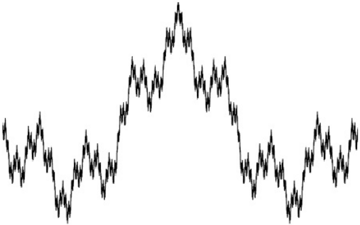


Within $\mathbb{R}^n$ lurks...









Welcome to O-Minimality

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2. If $A \in \mathcal{S}_m$, then $A \times \mathcal{R}, \mathcal{R} \times A \in \mathcal{S}_{m+1}$
3. If $A \in \mathcal{S}_{m+1}$, then $\pi(A) \in \mathcal{S}_m$ (where $\pi : \mathcal{R}^{m+1} \to \mathcal{R}^m$ is the projection onto the first m coordinates).

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5. $\{(x, y) \in \mathcal{R}^2 \mid x < y\} \in \mathcal{S}_2$
6. The sets in $\mathcal{S}_1$ are exactly the finite unions of intervals and points

Examples

- Semialgebraic Sets:

$$\{\mathbf{x} \in \mathbb{R}^n \mid f_1(\mathbf{x}) = \ldots = f_k(\mathbf{x}) = 0, \\ g_1(\mathbf{x}) > 0, \ldots, g_l(\mathbf{x}) > 0\}$$

  for f_i, g_j polynomials
- Slightly more boring: Semilinear Sets
- Really boring: Structure generated by $<$
- Structure generated by $\exp : \mathbb{R} \to \mathbb{R}$
- Globally subanalytic sets

"Definable ________"

- If a set $A \subset \mathcal{S}_m$ for some m, we call A definable.
- A function $f : A \to B$ ($A \subset \mathcal{R}^m$, $B \subset \mathcal{R}^n$) is definable if its graph $\Gamma(f) \subset \mathcal{R}^{m+n}$ is definable
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- Similarly, definably connected/path connected, etc.

A few definable things

- $\{r\}$ for $r \in \mathcal{R}$
- Interiors and closures of definable sets
- Inverses, and compositions of definable functions (also images and preimages of and restrictions to definable sets)
- If $\mathcal{R} = \mathbb{R}$ and addition and mulitpication are definable: sums, products, limits, and derivatives of definable functions

As you set out...

What to expect

What not to expect

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- Infinite subsets of $\mathcal{R}$ contain an interval

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- $\mathbb{Z}$

Our Itinerary

- Monotonicity Theorem
- Cell Decomposition
- Curve Selection Lemma
- Connected $\Leftrightarrow$ Path Connected

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- Cell Decomposition
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- All Groups are Abelian (ish)

Monotonicity Theorem

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Let $f : (a, b) \to \mathcal{R}$ be a definable function. Then there are points $a = a_0 < a_1 < \ldots < a_k < a_{k+1} = b$ such that on each subinterval (a_j, a_{j+1}), either f is constant or f is strictly monotone and continuous.

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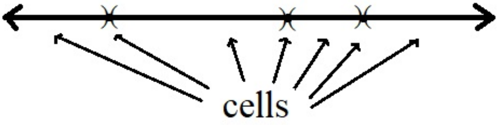
Outline of Proof.

The proof follows from three claims:

1. There is a subinterval of I on which f is constant or injective.
2. If f is injective, then f is strictly monotone on a subinterval of I.
3. if f is strictly monotone, then f is continuous on a subinterval of I.

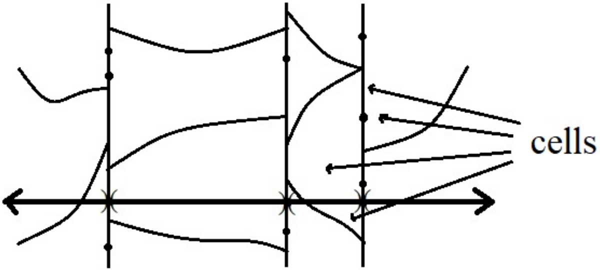
■

Cell Decomposition of $\mathcal{R}$



(0)-cells: points
(1)-cells: intervals

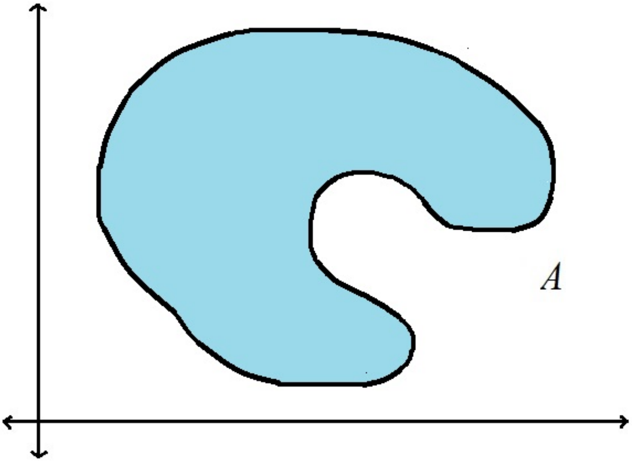
Cell Decomposition of $\mathcal{R}^2$



(0,0)-cells: points
(0,1)-cells: vertical intervals
(1,0)-cells: graphs of continuous definable functions on an interval
(1,1)-cells: "bands" between two (1,0) cells

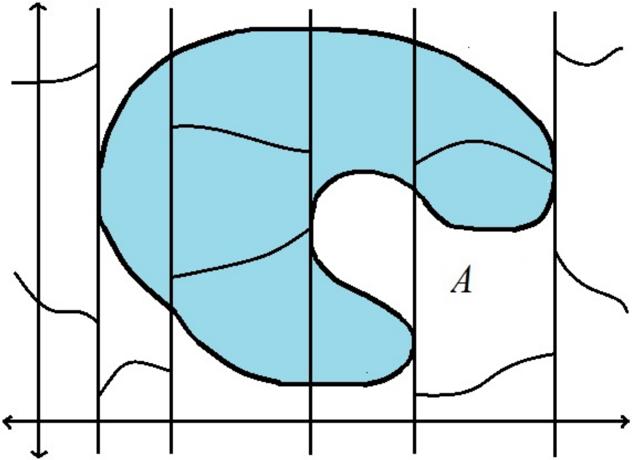
Cell Decomposition Adapted to a Set

Let $A \subset \mathcal{R}^n$ be definable



Cell Decomposition Adapted to a Set

Can write A as a union of cells of $\mathcal{C}$



Curve Selection

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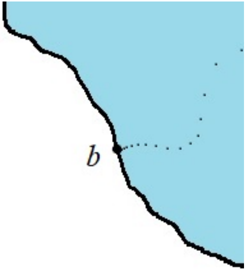
Let $A \subset \mathcal{R}^n$ be definable, and let $b \in \overline{A}$. Then there exists a continuous definable map $\gamma : [0,1) \to \mathcal{R}^n$ such that $\gamma(0) = b$ and $\gamma((0,1)) \subset A$.

Without o-minimality



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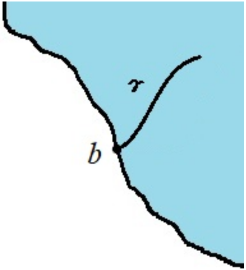

Connected $\Leftrightarrow$ Path Connected

Theorem

Let $A \subset \mathcal{R}^n$ be definable. Then we may write $A = A_1 \sqcup \ldots \sqcup A_k$ (with $A_i \cap A_j = \emptyset$ for $i \neq j$), where each A_i is nonempty, open and closed in A, and definably path connected. Furthermore, this partition is unique.

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4. Build A_is from equivalence classes of $\sim$

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When $\mathcal{R} = \mathbb{R}$, if A is definable, A connected $\Rightarrow$ A definably connected $\Leftrightarrow$ A definably path connected $\Rightarrow$ A path connected.

O-Minimal Ordered Groups

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Not an Ordered Group

- $(\mathbb{R} \setminus \{0\}, \cdot)$

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Lemma
The only definable subsets of $\mathcal{R}$ that are also subgroups are $\{e\}$ and $\mathcal{R}$

References

[1] [2]

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Lou van den Dries. *Tame Topology and O-minimal Structures*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1998. ISBN: 0521598389.