A Tourist's Guide to O-Minimality

Alison Rosenblum

Purdue University

September 25, 2021

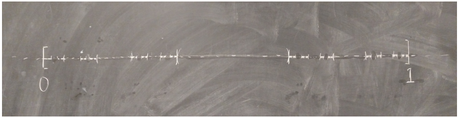
The Real Numbers

> Remember that the real line is not the safe and simple locale that people assume at first glance; it's a wild jungle. You never know when you might stumble upon a compact, uncountable, totally disconnected, nowhere dense set of measure zero just as it starts to accumulate everywhere.
>
> ---
>
> Dr. Mel Friske
> Professor Emeritus
> Wisconsin Lutheran College

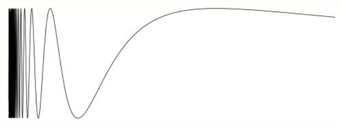
The Real Numbers

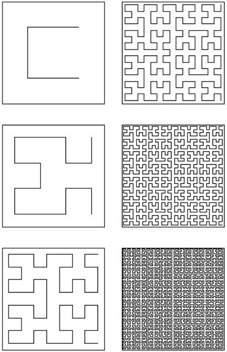
Remember that the real line is not the safe and simple locale that people assume at first glance; it's a wild jungle. You never know when you might stumble upon a compact, uncountable, totally disconnected, nowhere dense set of measure zero just as it starts to accumulate everywhere.

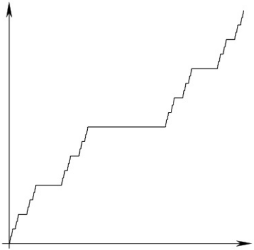
Dr. Mel Friske
Professor Emeritus
Wisconsin Lutheran College

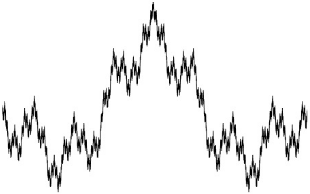


Within $\mathbb{R}^n$ lurks...









Welcome to O-Minimality

Welcome to O-Minimality

Our Itinerary:

- Welcome Center (definitions and examples)
- A few nice theorems
- Cell Decomposition

Welcome to O-Minimality

Our Itinerary:

- Welcome Center (definitions and examples)
- A few nice theorems
- Cell Decomposition
- All Groups are Abelian

Welcome to O-Minimality

Our Itinerary:

- Welcome Center (definitions and examples)
- A few nice theorems
- Cell Decomposition
- All Groups are Abelian (ish)

O-Minimal Structures

Let $(\mathcal{R}, <)$ be a nonempty dense linearly ordered set without endpoints (or, let $\mathcal{R} = \mathbb{R}$).

Definition

A structure $\mathcal{S}$ on $\mathcal{R}$ is made up of $\mathcal{S}_n \subset \mathcal{P}(\mathcal{R}^n)$ for each $n \in \mathbb{N}$, with

O-Minimal Structures

Let $(\mathcal{R}, <)$ be a nonempty dense linearly ordered set without endpoints (or, let $\mathcal{R} = \mathbb{R}$).

Definition

A structure $\mathcal{S}$ on $\mathcal{R}$ is made up of $\mathcal{S}_n \subset \mathcal{P}(\mathcal{R}^n)$ for each $n \in \mathbb{N}$, with

1. $\mathcal{S}_m$ closed under $\cup$, $\cap$, and complements

O-Minimal Structures

Let $(\mathcal{R}, <)$ be a nonempty dense linearly ordered set without endpoints (or, let $\mathcal{R} = \mathbb{R}$).

Definition

A structure $\mathcal{S}$ on $\mathcal{R}$ is made up of $\mathcal{S}_n \subset \mathcal{P}(\mathcal{R}^n)$ for each $n \in \mathbb{N}$, with

1. $\mathcal{S}_m$ closed under $\cup$, $\cap$, and complements
2. If $A \in \mathcal{S}_m$, then $A \times \mathcal{R}, \mathcal{R} \times A \in \mathcal{S}_{m+1}$
3. If $A \in \mathcal{S}_{m+1}$, then $\pi(A) \in \mathcal{S}_m$ (where $\pi : \mathcal{R}^{m+1} \to \mathcal{R}^m$ is the projection onto the first m coordinates).

O-Minimal Structures

Let $(\mathcal{R}, <)$ be a nonempty dense linearly ordered set without endpoints (or, let $\mathcal{R} = \mathbb{R}$).

Definition

A structure $\mathcal{S}$ on $\mathcal{R}$ is made up of $\mathcal{S}_n \subset \mathcal{P}(\mathcal{R}^n)$ for each $n \in \mathbb{N}$, with

1. $\mathcal{S}_m$ closed under $\cup$, $\cap$, and complements
2. If $A \in \mathcal{S}_m$, then $A \times \mathcal{R}, \mathcal{R} \times A \in \mathcal{S}_{m+1}$
3. If $A \in \mathcal{S}_{m+1}$, then $\pi(A) \in \mathcal{S}_m$ (where $\pi : \mathcal{R}^{m+1} \to \mathcal{R}^m$ is the projection onto the first m coordinates).
4. $\{(x_1, \dots, x_m) \in \mathcal{R}^m \mid x_i = x_j\} \in \mathcal{S}_m$ $(i, j \in \{1, \dots, m\})$

O-Minimal Structures

Definition

A structure $\mathcal{S}$ on $\mathcal{R}$ is made up of $\mathcal{S}_n \subset \mathcal{P}(\mathcal{R}^n)$ for each $n \in \mathbb{N}$, with

1. $\mathcal{S}_m$ closed under $\cup$, $\cap$, and complements
2. If $A \in \mathcal{S}_m$, then $A \times \mathcal{R}, \mathcal{R} \times A \in \mathcal{S}_{m+1}$
3. If $A \in \mathcal{S}_{m+1}$, then $\pi(A) \in \mathcal{S}_m$ (where $\pi : \mathcal{R}^{m+1} \to \mathcal{R}^m$ is the projection onto the first m coordinates).
4. $\{(x_1, \ldots, x_m) \in \mathcal{R}^m \mid x_i = x_j\} \in \mathcal{S}_m$ $(i, j \in \{1, ..., m\})$

Definition

An o-minimal structure is a structure $\mathcal{S}$ with

O-Minimal Structures

Definition

A structure $\mathcal{S}$ on $\mathcal{R}$ is made up of $\mathcal{S}_n \subset \mathcal{P}(\mathcal{R}^n)$ for each $n \in \mathbb{N}$, with

1. $\mathcal{S}_m$ closed under $\cup$, $\cap$, and complements
2. If $A \in \mathcal{S}_m$, then $A \times \mathcal{R}, \mathcal{R} \times A \in \mathcal{S}_{m+1}$
3. If $A \in \mathcal{S}_{m+1}$, then $\pi(A) \in \mathcal{S}_m$ (where $\pi : \mathcal{R}^{m+1} \to \mathcal{R}^m$ is the projection onto the first m coordinates).
4. $\{(x_1, \dots, x_m) \in \mathcal{R}^m \mid x_i = x_j\} \in \mathcal{S}_m$ $(i, j \in \{1, ..., m\})$

Definition

An o-minimal structure is a structure $\mathcal{S}$ with

5. $\{(x, y) \in \mathcal{R}^2 \mid x < y\} \in \mathcal{S}_2$

O-Minimal Structures

Definition

A structure $\mathcal{S}$ on $\mathcal{R}$ is made up of $\mathcal{S}_n \subset \mathcal{P}(\mathcal{R}^n)$ for each $n \in \mathbb{N}$, with

1. $\mathcal{S}_m$ closed under $\cup$, $\cap$, and complements
2. If $A \in \mathcal{S}_m$, then $A \times \mathcal{R}, \mathcal{R} \times A \in \mathcal{S}_{m+1}$
3. If $A \in \mathcal{S}_{m+1}$, then $\pi(A) \in \mathcal{S}_m$ (where $\pi : \mathcal{R}^{m+1} \to \mathcal{R}^m$ is the projection onto the first m coordinates).
4. $\{(x_1, \ldots, x_m) \in \mathcal{R}^m \mid x_i = x_j\} \in \mathcal{S}_m$ $(i, j \in \{1, ..., m\})$

Definition

An o-minimal structure is a structure $\mathcal{S}$ with

5. $\{(x, y) \in \mathcal{R}^2 \mid x < y\} \in \mathcal{S}_2$
6. The sets in $\mathcal{S}_1$ are exactly the finite unions of intervals and points

Examples

- Semialgebraic Sets: assembled from

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = 0\} \text{ and } \{\boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) > 0\}$$

  for f, g polynomials

Examples

- Semialgebraic Sets: assembled from

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = 0\} \text{ and } \{\boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) > 0\}$$

  for f, g polynomials
- Slightly more boring: Semilinear Sets
- Really boring: Structure generated by $<$

Examples

- Semialgebraic Sets: assembled from

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = 0\} \text{ and } \{\boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) > 0\}$$

  for f, g polynomials
- Slightly more boring: Semilinear Sets
- Really boring: Structure generated by $<$
- Structure generated by e^x or by $\sin(x)$ where eg $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

"Definable ________"

- If a set $A \subset \mathcal{S}_m$ for some m, we call A definable.

"Definable ________"

- If a set $A \subset \mathcal{S}_m$ for some m, we call A definable.
- A function $f : A \to B$ $(A \subset \mathcal{R}^m,\ B \subset \mathcal{R}^n)$ is definable if its graph $\Gamma(f) \subset \mathcal{R}^{m+n}$ is definable

"Definable _________"

- If a set $A \subset \mathcal{S}_m$ for some m, we call A definable.
- A function $f : A \to B$ ($A \subset \mathcal{R}^m$, $B \subset \mathcal{R}^n$) is definable if its graph $\Gamma(f) \subset \mathcal{R}^{m+n}$ is definable
- etc...

Model Theory Connection

$$\cap,\ \cup,\ \text{complement}$$
$$A \times \mathcal{R},\ \mathcal{R} \times A$$
$$\pi : \mathcal{R}^{n+1} \to \mathcal{R}^n$$
$$\{(x_1, \ldots, x_n) \in \mathcal{R}^n \mid x_i = x_j\}$$
$$\{(x, y) \in \mathcal{R}^2 \mid x < y\}$$

Model Theory Connection

$$\begin{array}{ccc} \cap,\ \cup,\ \text{complement} & \leftrightarrow & \wedge,\ \vee,\ \neg \\ A \times \mathcal{R},\ \mathcal{R} \times A & & \\ \pi : \mathcal{R}^{n+1} \to \mathcal{R}^n & & \\ \{(x_1, \ldots, x_n) \in \mathcal{R}^n \mid x_i = x_j\} & & \\ \{(x, y) \in \mathcal{R}^2 \mid x < y\} & & \end{array}$$

Model Theory Connection

$$\cap,\ \cup,\ \text{complement} \quad \leftrightarrow \quad \wedge,\ \vee,\ \neg$$

$$A \times \mathcal{R},\ \mathcal{R} \times A$$

$$\pi : \mathcal{R}^{n+1} \to \mathcal{R}^n$$

$$\{(x_1, \ldots, x_n) \in \mathcal{R}^n \mid x_i = x_j\}$$

$$\{(x, y) \in \mathcal{R}^2 \mid x < y\}$$

Model Theory Connection

$\cap$, $\cup$, complement	$\leftrightarrow$	$\wedge$, $\vee$, $\neg$
$A \times \mathcal{R}$, $\mathcal{R} \times A$		
$\pi : \mathcal{R}^{n+1} \to \mathcal{R}^n$	$\leftrightarrow$	$\exists$
$\{(x_1, \ldots, x_n) \in \mathcal{R}^n \mid x_i = x_j\}$		
$\{(x, y) \in \mathcal{R}^2 \mid x < y\}$		

Model Theory Connection

$$
\begin{array}{ccc}
\cap,\ \cup,\ \text{complement} & \leftrightarrow & \wedge,\ \vee,\ \neg \\
A \times \mathcal{R},\ \mathcal{R} \times A & & \\
\pi : \mathcal{R}^{n+1} \to \mathcal{R}^n & \leftrightarrow & \exists \\
\{(x_1, \ldots, x_n) \in \mathcal{R}^n \mid x_i = x_j\} & \leftrightarrow & = \\
\{(x, y) \in \mathcal{R}^2 \mid x < y\} & &
\end{array}
$$

Model Theory Connection

$\cap$, $\cup$, complement	$\leftrightarrow$	$\wedge$, $\vee$, $\neg$
$A \times \mathcal{R}$, $\mathcal{R} \times A$		
$\pi : \mathcal{R}^{n+1} \to \mathcal{R}^n$	$\leftrightarrow$	$\exists$
$\{(x_1, \ldots, x_n) \in \mathcal{R}^n \mid x_i = x_j\}$	$\leftrightarrow$	$=$
$\{(x, y) \in \mathcal{R}^2 \mid x < y\}$	$\leftrightarrow$	$<$

Model Theory Connection

$$
\begin{array}{rcc}
\cap,\ \cup,\ \text{complement} & \leftrightarrow & \wedge,\ \vee,\ \neg \\
A \times \mathcal{R},\ \mathcal{R} \times A & & \\
\pi : \mathcal{R}^{n+1} \to \mathcal{R}^n & \leftrightarrow & \exists \\
\{(x_1, \ldots, x_n) \in \mathcal{R}^n \mid x_i = x_j\} & \leftrightarrow & = \\
\{(x, y) \in \mathcal{R}^2 \mid x < y\} & \leftrightarrow & <
\end{array}
$$

New definable sets

$$\{\boldsymbol{x} \in \mathcal{R}^n \mid \text{ statement involving definable things and } \uparrow\}$$

Model Theory Connection

$$\begin{array}{ccc}
\cap,\ \cup,\ \text{complement} & \leftrightarrow & \wedge,\ \vee,\ \neg \\
A \times \mathcal{R},\ \mathcal{R} \times A & & \\
\pi : \mathcal{R}^{n+1} \to \mathcal{R}^n & \leftrightarrow & \exists \\
\{(x_1, \ldots, x_n) \in \mathcal{R}^n \mid x_i = x_j\} & \leftrightarrow & = \\
\{(x, y) \in \mathcal{R}^2 \mid x < y\} & \leftrightarrow & <
\end{array}$$

New definable sets

$$\{\boldsymbol{x} \in \mathcal{R}^{\cancel{n}\,m} \mid \text{statement involving definable things and } \uparrow\}$$

for $m \geq n$

A Few Definable Things

- $\{r\}$ for $r \in \mathcal{R}$
- Interiors and closures of definable sets
- Inverses, and compositions of definable functions (also images and preimages of and restrictions to definable sets)
- If $\mathcal{R} = \mathbb{R}$ and addition and mulitpication are definable: sums, products, limits, and derivatives of definable functions

As you set out...

What to expect

What not to expect

As you set out...

What to expect

- Infinite subsets of $\mathcal{R}$ contain an interval

What not to expect

As you set out...

What to expect

- Infinite subsets of $\mathcal{R}$ contain an interval
- Uniform bounds

What not to expect

As you set out...

What to expect

- Infinite subsets of $\mathcal{R}$ contain an interval
- Uniform bounds

What not to expect

- Too much 'infiniteness'

As you set out...

What to expect

- Infinite subsets of $\mathcal{R}$ contain an interval
- Uniform bounds

What not to expect

- Too much 'infiniteness'
- $\mathbb{Z}$

Some Theorems

Curve Selection

The Curve Selection Lemma

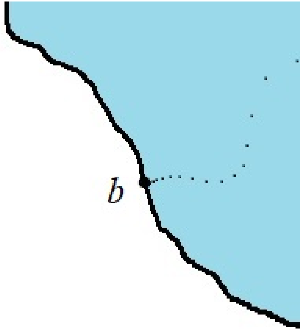
Let $A \subset \mathcal{R}^n$ be definable, and let $b \in \overline{A}$.

Curve Selection

The Curve Selection Lemma

Let $A \subset \mathcal{R}^n$ be definable, and let $b \in \overline{A}$.

Without o-minimality

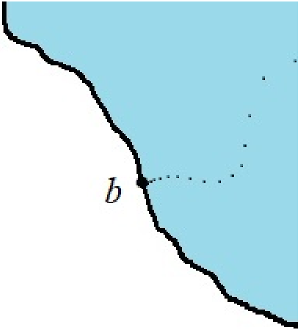


Curve Selection

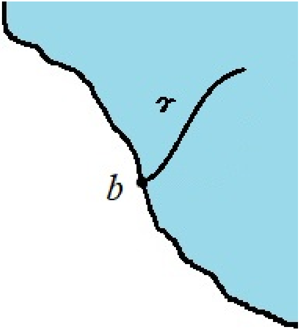
The Curve Selection Lemma

Let $A \subset \mathcal{R}^n$ be definable, and let $b \in \overline{A}$. Then there exists a continuous definable map $\gamma : [0,1) \to \mathcal{R}^n$ such that $\gamma(0) = b$ and $\gamma((0,1)) \subset A$.

Without o-minimality



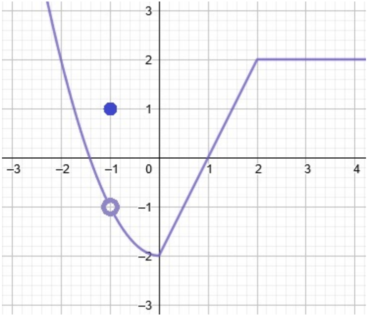
With o-minimality



Monotonicity Theorem

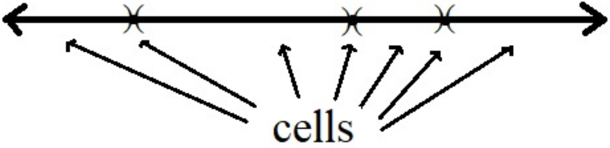
The Monotonicity Theorem

Let $f : (a, b) \to \mathcal{R}$ be a definable function. Then there are points $a = a_0 < a_1 < \ldots < a_k < a_{k+1} = b$ such that on each subinterval (a_j, a_{j+1}), either f is constant or f is strictly monotone and continuous.



Cell Decomposition of $\mathcal{R}$

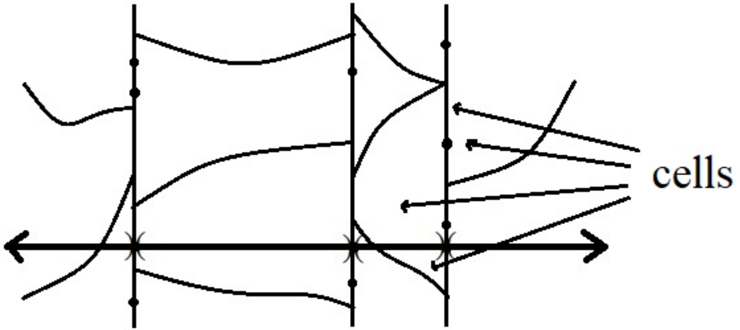
Cell Decomposition of $\mathcal{R}$



Cells:

1. points
2. intervals

Cell Decomposition of $\mathcal{R}^2$

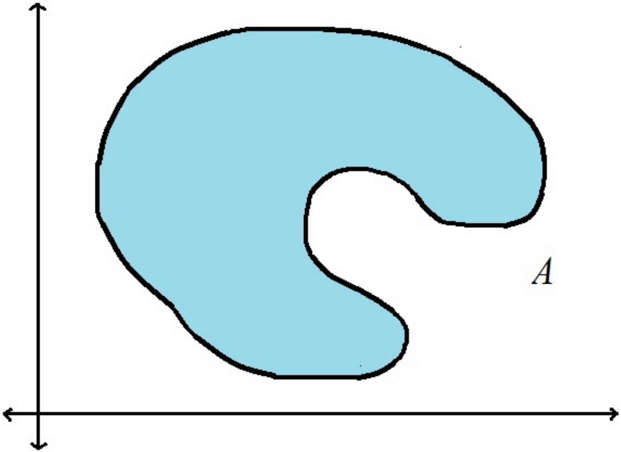


Cells:

1. points
2. vertical "intervals"
3. graphs of continuous definable functions on an interval
4. "bands" between two graphs

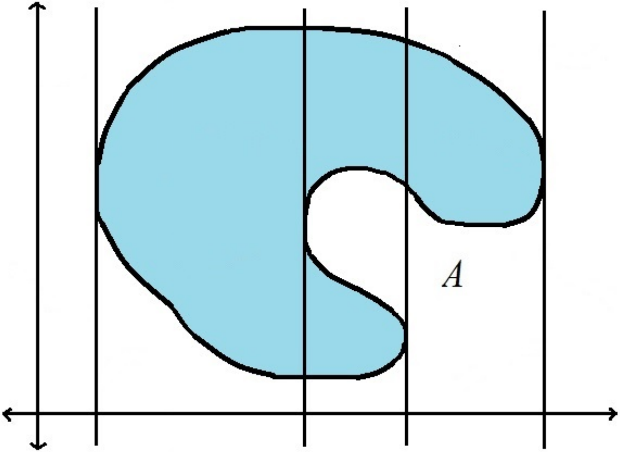
Cell Decomposition Adapted to a Set

Let $A \subset \mathcal{R}^n$ be definable.



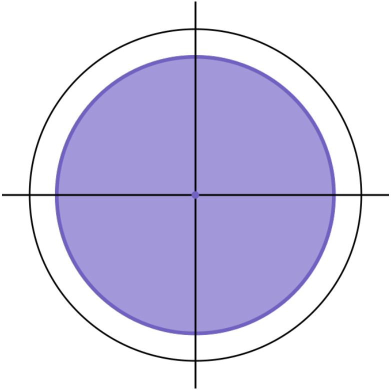
Cell Decomposition Adapted to a Set

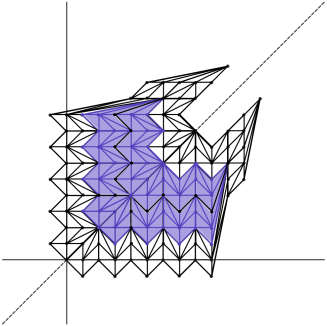
Let $A \subset \mathcal{R}^n$ be definable. Can decompose $\mathcal{R}^n$ to write A as a union of cells.



Consequences of Cell Decomposisiton

1. Intuitive concept of dimension
2. Definably connected ⇒ definably path connected
3. Triangulation of definable sets





Ordered Groups

O-Minimal Ordered Groups

Definition

An ordered group is a group G with a linear order $<$ such that for all $x, y, z \in G$,

$$x < y \Rightarrow zx < zy \text{ and } xz < yz$$

O-Minimal Ordered Groups

Definition

An ordered group is a group G with a linear order $<$ such that for all $x, y, z \in G$,

$$x < y \Rightarrow zx < zy \text{ and } xz < yz$$

Examples

Ordered Groups

- $(\mathbb{R}, +)$
- $(\mathbb{R}_{>0}, \cdot)$

O-Minimal Ordered Groups

Definition

An ordered group is a group G with a linear order $<$ such that for all $x, y, z \in G$,

$$x < y \Rightarrow zx < zy \text{ and } xz < yz$$

Examples

Ordered Groups

- $(\mathbb{R}, +)$
- $(\mathbb{R}_{>0}, \cdot)$

Not an Ordered Group

- $(\mathbb{R} \setminus \{0\}, \cdot)$

O-Minimal Ordered Groups

Theorem

All groups are Abelian

O-Minimal Ordered Groups

~~Theorem~~

~~All groups are Abelian~~

Theorem

Let $\mathcal{S}$ be an o-minimal structure on an ordered group $\mathcal{R}$, and say $\cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is definable in $\mathcal{S}$. Then $\mathcal{R}$ is Abelian.

O-Minimal Ordered Groups

~~Theorem~~

~~All groups are Abelian~~

Theorem

Let $\mathcal{S}$ be an o-minimal structure on an ordered group $\mathcal{R}$, and say $\cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is definable in $\mathcal{S}$. Then $\mathcal{R}$ is Abelian.

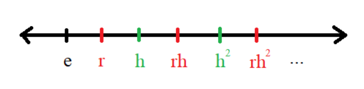
Lemma

The only definable subsets of $\mathcal{R}$ that are also subgroups are $\{e\}$ and $\mathcal{R}$

Proof Sketch

Proof (lemma).

- Let $\{e\} \neq H \subset \mathcal{R}$ be a definable subgroup. Then $H = (s^{-1}, s)$ or $H = [s^{-1}, s]$.
  Assume not, and say $e < r < h$ for $h \in H$ and $r \notin H$

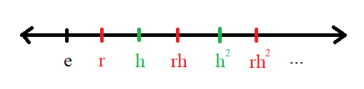


- $s = \infty$

Proof Sketch

Proof (lemma).

- Let $\{e\} \neq H \subset \mathcal{R}$ be a definable subgroup. Then $H = (s^{-1}, s)$ or $H = [s^{-1}, s]$.
  Assume not, and say $e < r < h$ for $h \in H$ and $r \notin H$



- $s = \infty$

■

Proof ($\mathcal{R}$ is Abelian).

$r \in \mathcal{R}$: consider $C(r) = \{s \in \mathcal{R} \mid sr = rs\}$, a definable subgroup of $\mathcal{R}$
Since $r \in C(r)$, $C(r) \neq \{e\}$, so $C(R) = \mathcal{R}$ for all $r \in \mathcal{R}$ ■

References

Michel Coste. *An introduction to o-minimal geometry*. Istituti editoriali e poligrafici internazionali Pisa, 2000.

LOU AUTOR VAN DEN DRIES, Lou Van den Dries, et al. *Tame topology and o-minimal structures*. Vol. 248. Cambridge university press, 1998.