A Tourist's Guide to O-Minimality

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You thought you knew $\mathbb{R}$ ...

Subsets of the real numbers

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Subsets of the real numbers

- $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$

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- Points and intervals: $(-2, 0)$, $[-1, 1]$, $(-8.76, \pi]$, $\{\frac{1}{3}\}$, $[50, \infty)$, etc.

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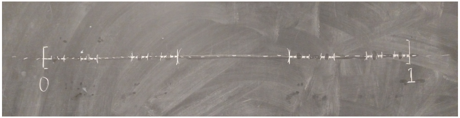
- $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$
- Points and intervals: $(-2, 0)$, $[-1, 1]$, $(-8.76, \pi]$, $\{\frac{1}{3}\}$, $[50, \infty)$, etc.
- The Cantor Set



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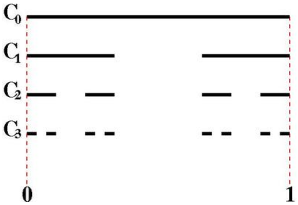

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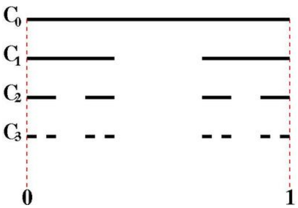

- Bounded

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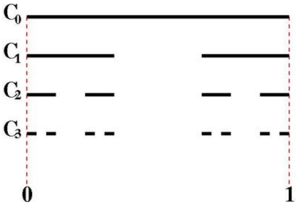

- Bounded
- Measure zero

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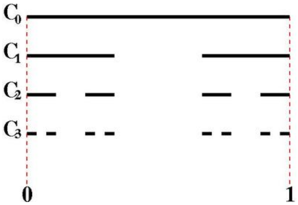

- Bounded
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- Uncountably infinite

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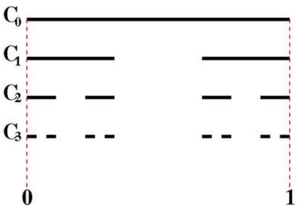

- Bounded
- Measure zero
- Uncountably infinite
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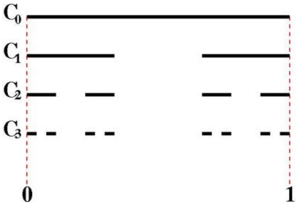

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- Nowhere dense
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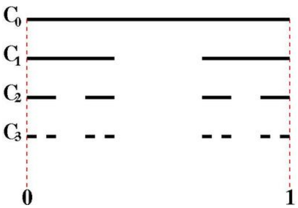

- Bounded
- Measure zero
- Uncountably infinite
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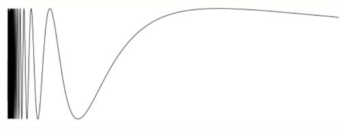
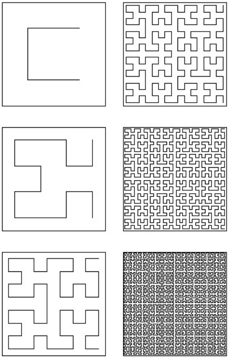
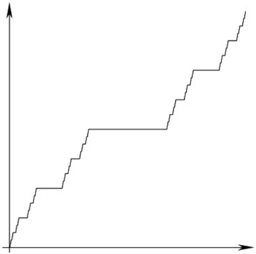
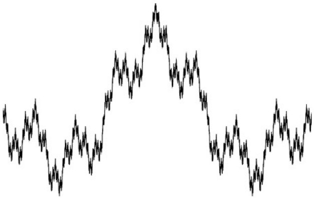
- Accumulates everywhere
- Totally disconnected

More Monsters of $\mathbb{R}^n$









Welcome to O-Minimality

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Our Itinerary:

- Welcome Center (definitions and examples)
- Walk to the Cell Decomposition Theorem
- Vista from Cell Decomposition

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O-Minimal Structures

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2. Cartesian products $(A \times B)$ of sets in $\mathcal{S}$
3. Coordinate projections of sets in $\mathcal{S}$ ($\pi(A)$ where $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ takes $(x_1, \ldots, x_m, x_{m+1}) \mapsto (x_1, \ldots, x_m)$).

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An o-minimal structure is a structure with

5. $\{(x, y) \in \mathbb{R}^2 \mid x < y\}$ in $\mathcal{S}$
6. All subsets of $\mathbb{R}^1$ in $\mathcal{S}$ are finite unions of points and intervals

Examples

- Semialgebraic Sets: assembled from

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = 0\} \text{ and } \{\boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) > 0\}$$

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  for f, g polynomials
- Slightly more boring: Semilinear Sets
- Structure generated by e^x or by $\sin(x)$ where e.g. $0 \leq x \leq 2\pi$
- Globally subanalytic sets

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- etc...

A Few Definable Things

- $\{r\}$ for $r \in \mathbb{R}$
- Interiors and closures of definable sets
- Inverses and compositions of definable functions (also images and preimages of and restrictions to definable sets)
- Addition and mulitpication definable: sums, products, limits, and derivatives of definable functions

As you set out...

What to expect

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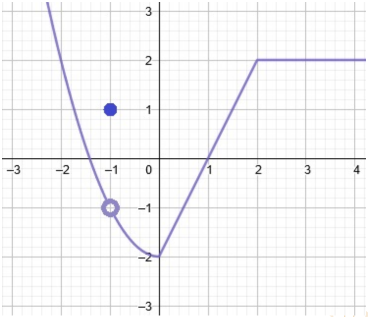
- $\mathbb{Z}$

To the Cell Decomposition Theorem

Monotonicity Theorem

The Monotonicity Theorem

$f : (a, b) \to \mathbb{R}$ definable function:
Can find points $a = a_0 < a_1 < \ldots < a_k < a_{k+1} = b$ such that on each subinterval (a_j, a_{j+1}), either f is constant or f is strictly monotone and continuous.



Cell Decomposition of $\mathbb{R}$

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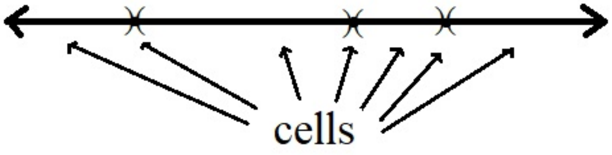
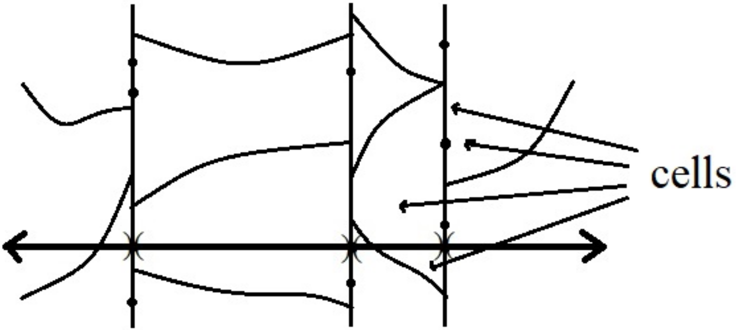


Cells:

1. points
2. intervals

Cell Decomposition of $\mathbb{R}^2$

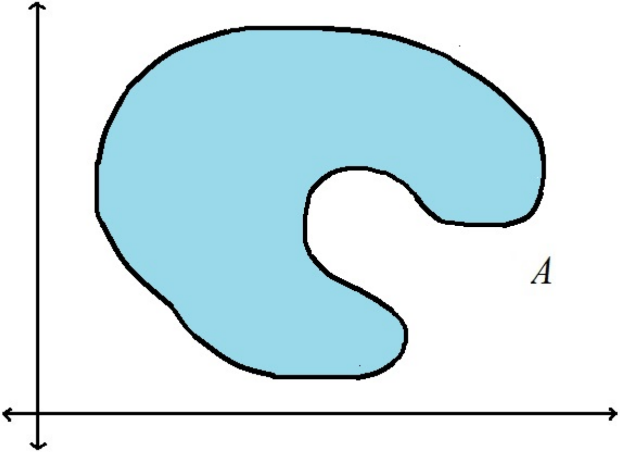


Cells:

1. points
2. vertical "intervals"
3. graphs of continuous definable functions on an interval
4. "bands" between two graphs

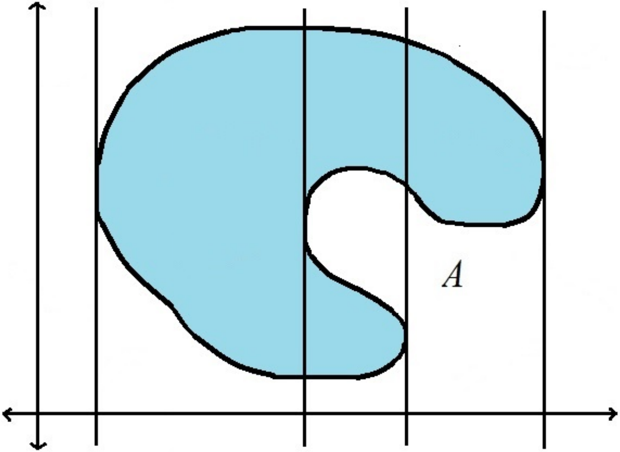
Cell Decomposition Adapted to a Set

$A \subset \mathbb{R}^n$ definable:



Cell Decomposition Adapted to a Set

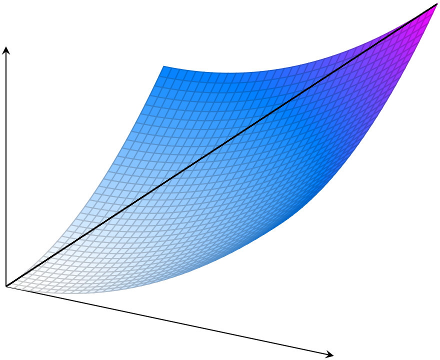
$A \subset \mathbb{R}^n$ definable:
Can decompose $\mathbb{R}^n$ to write A as a finite union of cells.



Vista from Cell Decomposition

Dimension

Dimension of a definable set: biggest cell dimension present



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$A \subset \mathbb{R}^n$ definable, $b \in \overline{A}$.

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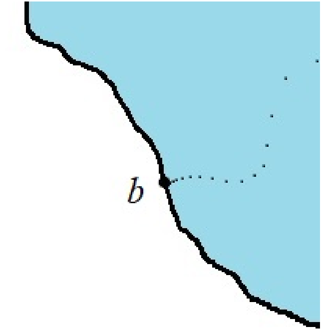
Without o-minimality



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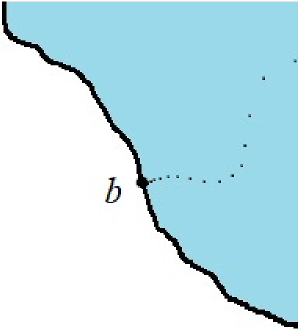
$A \subset \mathbb{R}^n$ definable, $b \in \overline{A}$. Then there exists a continuous definable map $\gamma : [0,1) \to \mathbb{R}^n$ such that $\gamma(0) = b$ and $\gamma((0,1)) \subset A$.

Without o-minimality



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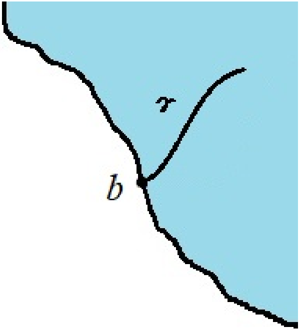
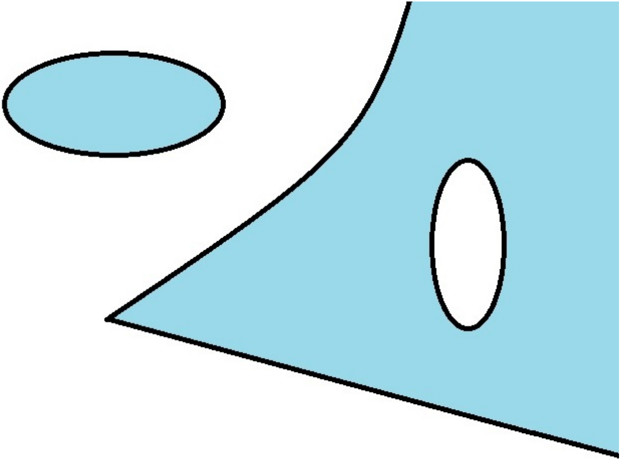


Connectedness

Definable subsets of $\mathbb{R}$: connected $\Leftrightarrow$ (definably) path connected



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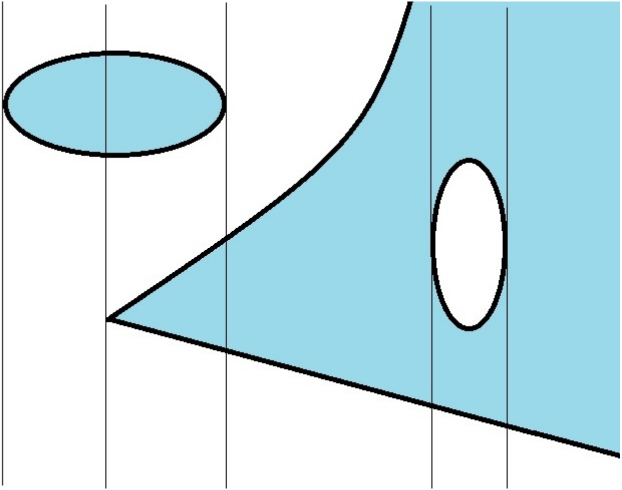

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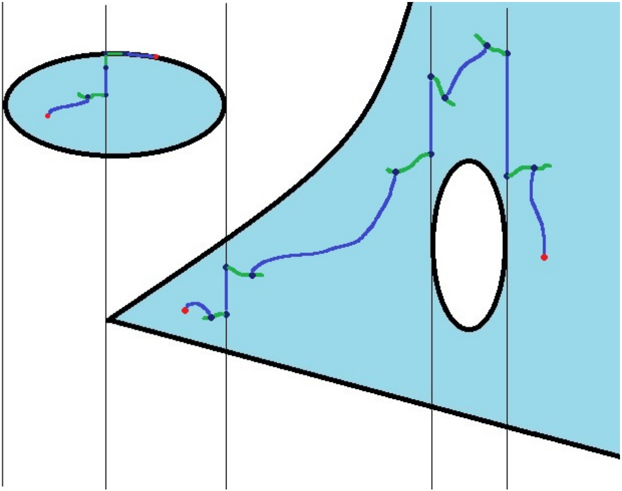
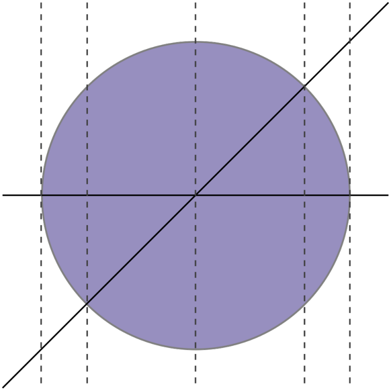
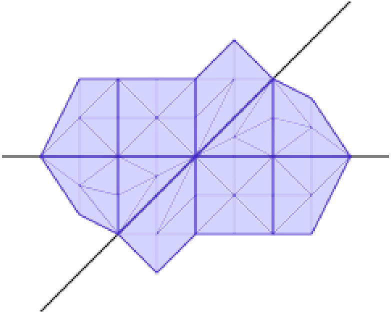


Triangulation

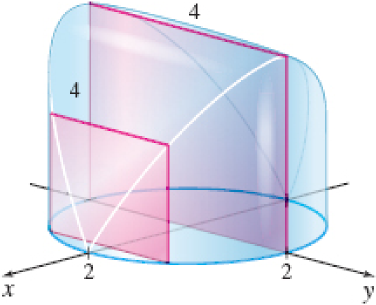
Another decomposition of definable sets...

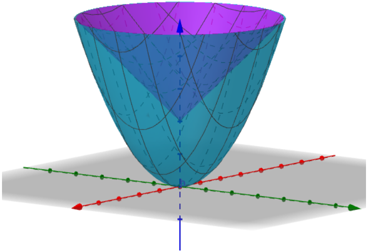




Triviality

Definable family: $\{A_x\}_{x\in\mathcal{I}}$ definable sets with $A = \bigcup_{x\in\mathcal{I}}(x, A_x)$ also definable.





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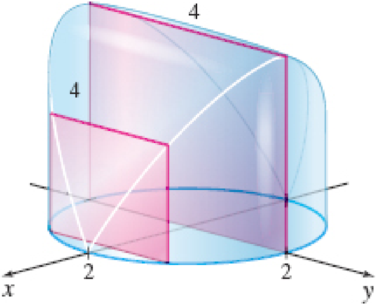

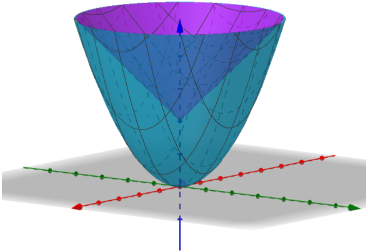

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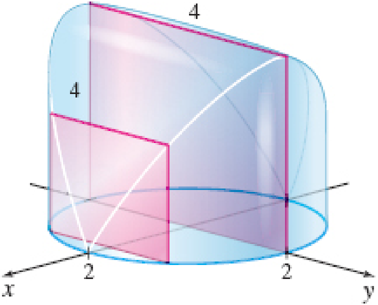

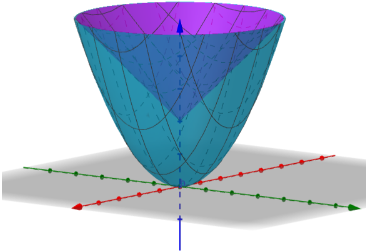

Only finitely many "types" of sets among $\{A_x\}$.

Also means "interesting" features of definable sets all clustered in a bounded region

Ordered Groups

Groups

Group: a set with an operation that behaves (sort of) like addition

Example ((Possibly Unhelpful) Groups)

- $\mathbb{R}$, operation: addition
- $\mathbb{R}_{>0}$, operation: mulitpication
- $n \times n$ invertible matrices, operation: matrix multiplication

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Identity element e:

- $(\mathbb{R}, +)$, identity: 0
- $(\mathbb{R}_{>0}, \cdot)$, identity: 1
- 2×2 invertible matrices, identity: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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A subgroup is a subset that is a group under the same operation

- Entire group, $\{e\}$ always subgroups
- $(\mathbb{Z}, +)$ a subgroup of $(\mathbb{R}, +)$

Abelian Groups

Definition

A group is Abelian if its operation is commutative

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e.g. matrix mulitpication is not commutative:

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \cdot \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} = \begin{bmatrix}5 & 4\\11 & 10\end{bmatrix} \text{ but } \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}7 & 10\\5 & 8\end{bmatrix}$$

Souvenier Theorem

Theorem

All groups are Abelian

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Theorem

Let $\mathcal{S}$ be an o-minimal structure on an ordered group $\mathcal{R}$, and say $+ : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is definable in $\mathcal{S}$. Then $\mathcal{R}$ is Abelian.

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Lemma

The only definable subsets of $\mathcal{R}$ that are also subgroups are $\{e\}$ and $\mathcal{R}$

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An ordered group is a group G with a linear order $<$ such that for all $x, y, z \in G$,

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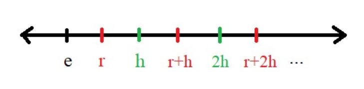
Not an Ordered Group

- $(\mathbb{R} \setminus \{0\}, \cdot)$

Proof Sketch

Proof (lemma).

- Let $\{e\} \neq H \subset \mathcal{R}$ be a definable subgroup. Then $H = (-s, s)$ or $H = [-s, s]$.
  Assume not, and say $e < r < h$ for $h \in H$ and $r \notin H$



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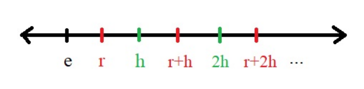

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□

Proof ($\mathcal{R}$ is Abelian).

$e \neq r \in \mathcal{R}$: consider $C(r) = \{s \in \mathcal{R} \mid s + r = r + s\}$, a definable subgroup of $\mathcal{R}$
Since $r \in C(r)$, $C(r) \neq \{e\}$, so $C(R) = \mathcal{R}$ for all $r \in \mathcal{R}$ □

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