Vandermonde Varieties in Type B

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Motivation

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Central topics:

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Betti Numbers

$$b_i(S) = \dim_{\mathbb{Q}} H_i(S, \mathbb{Q})$$

Central topics:

- Bounds on $b_i(S)$ e.g. in terms of n, s, i, $d = \max\{\deg(f_j)\}$
  - singly exponential in n
- Algorithms computing $b_i(S)$
  - current best complexity for all Betti numbers doubly exponential in n

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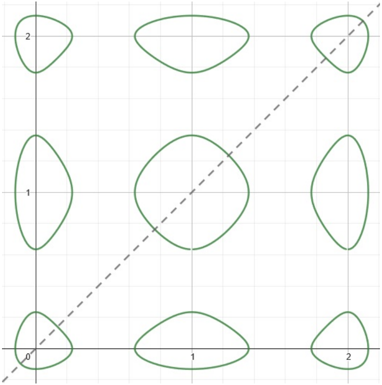

$$x^2(x-1)^2(x-2)^2 + y^2(y-1)^2(y-2)^2 = 0.1$$

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What about other finite reflection groups?

Vandermonde Varieties

See work of Arnold (1986), Givental (1987), and Kostov (1989)

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Weighted Newton power sums

$$p^{(n)}_{A,\mathbf{w},m} = w_1 X_1^m + \cdots + w_n X_n^m$$

for $\mathbf{w} = (w_1, \ldots, w_n) \in \mathcal{R}^n_{>0}$ weight vector

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Vandermonde variety

$$V^{(n)}_{A,\mathbf{w},d,\mathbf{y}} = \{\mathbf{x} \in \mathcal{R} \mid p^{(n)}_{A,\mathbf{w},1}(\mathbf{x}) = y_1, \ldots, p^{(n)}_{A,\mathbf{w},d}(\mathbf{x}) = y_d\}$$

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Study $Z^{(n)}_{A,\mathbf{w},d,\mathbf{y}} = V^{(n)}_{A,\mathbf{w},d,\mathbf{y}} \cap \mathcal{W}^{(n)}_A$

Type B: New Results

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Theorem

For every $\mathbf{w} \in \mathcal{R}^n_{>0}$, $0 \leq d \leq n$, and $\mathbf{y} \in \mathcal{R}^d$, $Z^{(n)}_{B,\mathbf{w},d,\mathbf{y}}$ is either empty, a point, or a semi-algebraic regular cell of dimension $n - d$.

Theorem

Let $d \geq 2$, $\mathbf{y} \in \mathcal{R}^d$. If $T \subset \mathrm{Cox}_B(n)$, then

$$H^i\left(Z^{(n)}_{B,d,\mathbf{y}}, Z^{(n)}_{B,d,\mathbf{y}} \cap \left(\bigcup_{s \in T} \mathcal{W}^{(n)}_{B,s}\right)\right) = 0$$

for all (i, T) satisfying either $i \leq \mathrm{card}(T) - 2d$ or $i \geq \mathrm{card}(T) + 1$

Type B Symmetry

Lie Types

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- *Dihedral groups*

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- Type A_n ($\leftrightarrow$ symmetric group)
- Type B_n ($\leftrightarrow$ signed permutations)
- (Type C_n)
- Type D_n
- *Dihedral groups*
- Exceptional types (e.g. E_6, E_7, E_8, F_4, G_2)

Coxeter Systems

Group W together with set of generators satisfying certain properties (e.g. $s^2 = e$)

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$W_A = \mathfrak{S}_n$

Generators $\mathrm{Cox}_A(n) = \{s_1 = (1\ 2), \ldots, s_{n-1} = (n-1\ n)\}$

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$W_B = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$

Generators $\mathrm{Cox}_B(n) = \{s_0\} \cup \mathrm{Cox}_A(n)$

$s_0 \leftrightarrow$ reflection through $X_1 = 0$

Weyl Chamber

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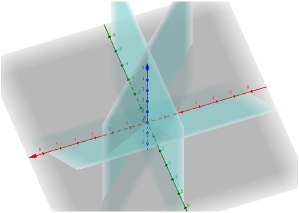

Figure: $\mathcal{W}_B^{(n)}$ and walls

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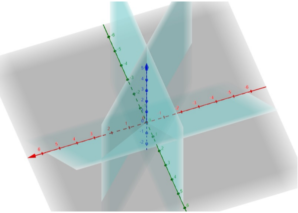

Figure: $\mathcal{W}_B^{(n)}$ and walls

Walls $\mathcal{W}_{s_j}^{(n)} = \mathcal{W}^{(n)} \cap \{X_j = X_{j+1}\}$
For $T \subset \mathrm{Cox}(n)$,

$$\mathcal{W}_T^{(n)} = \bigcap_{s \in T} \mathcal{W}_s^{(n)} \qquad \mathcal{W}^{(n,T)} = \bigcup_{s \in T} \mathcal{W}_s^{(n)}$$

Vandermonde Varieties

$V^{(n)}_{A,d,\mathbf{y}}$ intersection of level sets of first d generators of invariant ring of $\mathfrak{S}_n$

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Weighted Vandermonde varieties: $\mathbf{w} \in \mathcal{R}^n_{>0}$

$$p^{(n)}_{B,\mathbf{w},m} = w_1 X_1^{2m} + \cdots + w_n X_n^{2m}$$

$$V^{(n)}_{B,\mathbf{w},d,\mathbf{y}} = \{p^{(n)}_{B,\mathbf{w},1} = y_1, \ldots, p^{(n)}_{B,\mathbf{w},d} = y_d\}$$

Some Vandermonde Varieties

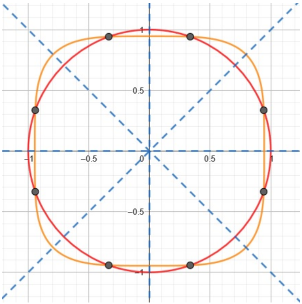

Figure: $V^{(2)}_{B,2,\mathbf{y}}$ for $\mathbf{y} = (1, 0.8)$

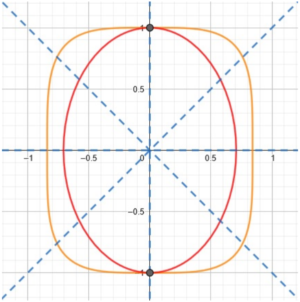

Figure: $V^{(2)}_{B,\mathbf{w},2,\mathbf{y}}$ for $\mathbf{w} = (2, 1)$, $\mathbf{y} = (1, 1)$

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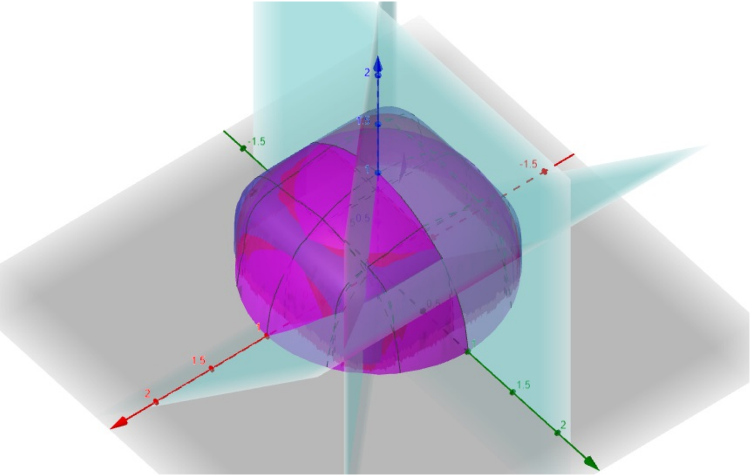

Figure: $V^{(3)}_{B,3,\mathbf{y}}$ for $\mathbf{y} = (1, 0.5, 0.27)$

Monotonicity

Monotone Sets and Maps

Assume sets and functions definable in some fixed o-minimal structure over $\mathbb{R}$

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Theorem (Basu, Gabrielov, Vorobjov; 2013)

If X is monotone, then X is a regular cell, i.e. $(\overline{X}, X)$ is homeomorphic as a pair to $(\overline{B}^k, B^k)$

Monotonicity of Vandermonde Varieties

Lemma (Basu, Riener; 2021)

$Z^{(n)}_{A,\mathbf{w},d,\mathbf{y}}$ is either empty, a single point, or the closure of a monotone cell of dimension $n-d$.

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Note $Z^{(n)}_{B,\mathbf{w},d,\mathbf{y}}$ is homeomorphic to $Z^{(n)}_{A,\mathbf{w},d,\mathbf{y}} \cap \mathcal{R}^n_{\geq 0}$

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Lemma

$Z^{(n)}_{B,\mathbf{w},d,\mathbf{y}}$ is either empty, a single point, or the s.a.-homeomorphic image of the closure of a monotone cell of dimension $n - d$.

Consequences and Goals

Consequences of Regularity

Notice: for $T \subset \mathrm{Cox}_B(n)$, $V^{(n)}_{B,d,\mathbf{y}} \cap \mathcal{W}^{(n)}_{B,T}$ is again a weighted Vandermonde variety

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$\Rightarrow Z^{(n)}_{B,d,\mathbf{y}}$ a regular cell complex

$\Rightarrow$ vanishing of various cohomologies of certain sets of the form $V^{(n)}_{B,d,\mathbf{y}} \cap \mathcal{W}^{(n,T)}_{B}$ for $T \subset \mathrm{Cox}_B(n)$

Decomposition of Homology

S is G-symmetric: induced action of G on $H_*(S)$, $H^*(S)$

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Assume $d \geq 2$

Know $H^i\left(Z^{(n)}_{B,d,\mathbf{y}}, Z^{(n)}_{B,d,\mathbf{y}} \cap \mathcal{W}^{(n,T)}_B\right) = 0$ for all (i, T) satisfying either $i \leq \mathrm{card}(T) - 2d$ or $i \geq \mathrm{card}(T) + 1$

Decomposition of Homology

S is G-symmetric: induced action of G on $H_*(S)$, $H^*(S)$

Assume $d \geq 2$

$$H_*(V^{(n)}_{B,d,\mathbf{y}}) \simeq_{W_B(n)} \bigoplus_{T \subset \mathrm{Cox}_B(n)} H_*(Z^{(n)}_{B,d,\mathbf{y}}, Z^{(n)}_{B,d,\mathbf{y}} \cap \mathcal{W}_B^{(n,T)}) \otimes \Psi^{(n)}_{B,T}$$

where $\Psi^{(n)}_{B,T}$ is the Solomon module in type B_n indexed by T

Know $H^i\left(Z^{(n)}_{B,d,\mathbf{y}}, Z^{(n)}_{B,d,\mathbf{y}} \cap \mathcal{W}_B^{(n,T)}\right) = 0$ for all (i, T) satisfying either $i \leq \mathrm{card}(T) - 2d$ or $i \geq \mathrm{card}(T) + 1$

Next Steps

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$$H^i(S) \cong_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} m_{i,\lambda}(S)\, \mathbb{S}^\lambda$$

(λ a partition of n, S^λ Specht module associated to λ, $m_{i,\lambda}(S)$ multiplicity of S^λ in $H^i(S)$)

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Theorem (Basu, Riener; 2021)

For $d \geq 2$, $m_{i,\lambda}(V^{(n)}_{B,d,\mathbf{y}}) = 0$ if either $i \leq \text{length}(\lambda) - 2d + 1$ or $i \geq n - \text{length}({}^t\lambda) + 1$

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References I

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