The Distribution of Values of the Riemann Zeta Function inside the Critical Strip

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References

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Overview of the Talk

1. Introduction
   - The Limiting Distribution of $\log \zeta(s)$
   - Euler Product for the Riemann Zeta Function
   - The Random Model for the Riemann Zeta Function
2. Main Results
   - Discrepancy for $\log \zeta$
   - The Characteristic Function
3. Sketch of the Proof
   - Approximating Characteristic Functions
   - Approximation $\log \zeta$ by a Dirichlet polynomial
   - Relating $R_Y(\sigma + it)$ to $R_Y(\sigma, X)$
   - Relating $R_Y(\sigma + it)$ to $R_Y(\sigma, X)$

Notation and Preliminaries

We use the following notation to emphasize the probabilistic ideas in this area:

- We view $[T, 2T]$ as a probability space with the normalized Lebesgue measure, which we denote by $\mathbb{P}_T(\cdot)$.

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- We will use $\mathbb{P}(\cdot)$ for probability and $\mathbb{E}(\cdot)$ for expectation associated with other sources of randomness.
- Any limit of random variables in this talk will be in the sense of convergence in distribution.

How is $\log \zeta(\sigma + it)$ distributed for large t?

Fix $\sigma \in \mathbb{R}$. Then the map

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This question amounts to asking what is the distributional limit as $T \to \infty$ of the random variables

$$\{t \mapsto \log \zeta(\sigma + it) : t \in [T, 2T]\}_{T>0},$$

if it exists.

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The case $\sigma = 1/2$ was considered by Selberg, who proved his Central Limit Theorem: loosely, it says that $\operatorname{Re} \log \zeta(1/2 + it)$ and $\operatorname{Im} \log \zeta(1/2 + it)$ are both normally distributed with mean 0 and variance $\frac{1}{2} \log \log T$. We will not discuss this further today (it will come up in a later talk in the series).

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We will thus restrict ourselves to $1/2 < \sigma \leq 1$.

The distribution of $\log \zeta(\sigma + it)$ for $\sigma > 1/2$

Bohr and Jessen [BJ30] proved the following (paraphrased) theorem:

Theorem

Let $\sigma > 1/2$ be fixed. Then, the sequence of $\mathbb{C}$-valued random variables

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converges in distribution as $T \to \infty$. Furthermore, the limiting distribution is continuous.

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For simplicity of exposition, we will not consider $\sigma = 1$ in this talk, although the same ideas apply, and are treated in [LLR19].

Euler Product for the Riemann Zeta Function

For $\sigma > 1$, we have the following convergent product formula for the Riemann zeta function due to to Euler:

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Putting $s = \sigma + it$, and rearranging a bit, we get that

$$\zeta(\sigma + it) = \prod_p \left(\frac{1}{1 - \frac{p^{-it}}{p^\sigma}} \right)$$

The Behaviour of p^{-it}

For $t \in \mathbb{R}$, we have that $n^{-it} \in \mathbb{T} \subseteq \mathbb{C}$, for $n \in \mathbb{N}$. n^{-it} is clearly distributed uniformly on $\mathbb{T}$ for every n.

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  In general there is no reason to expect m^{-it} and n^{-it} to show any sort of relationship when m and n are coprime.

The Behaviour of p^{-it}

The heuristic for why m^{-it} and n^{-it} should behave independently when $(m, n) = 1$ comes from the following theorem from harmonic analysis:

Theorem (Kronecker-Weyl)

Let $\theta_1, \cdots, \theta_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then the set

$$\{(e(\theta_1 x), \cdots, e(\theta_n x) : x \in \mathbb{R}\}$$

is equidistributed on $\mathbb{T}^n$, where $e(\cdot) = e^{2\pi i(\cdot)}$ as usual.

Note that $\{\log p : p \text{ prime}\}$ is $\mathbb{Q}$-linearly independent – this is the fundamental theorem of arithmetic. We conclude that any finite subset of pair-wise coprime integers should behave independently.

The Definition of the Random Model $\zeta(\sigma, X)$

This behaviour of p^{-it} as approximately uniform and i.i.d. random variables on $[T, 2T]$ leads to the following definition:

Definition (Random Model for ζ)

Let X be random variable uniformly taking values in $\mathbb{T}^\infty$, indexed by the primes. In other words, $X = \{X(p)\}_p$ is a family of independent random variables uniformly distributed on the unit circle in $\mathbb{C}$, indexed by the primes. We define the $\mathbb{C}$-valued random variable $\zeta(\sigma, X)$ as follows

$$\zeta(\sigma, X) = \prod_p \left(\frac{1}{1 - \frac{X(p)}{p^\sigma}} \right)$$

The Random Model $\zeta(\sigma, X)$

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It can be shown using probabilistic techniques (e.g., the Kolmogorov 3-series theorem or Chernoff-style concentration bounds) that the above product converges almost surely for $\sigma > 1/2$.

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Furthermore, for $\sigma > 1/2$, $\zeta(\sigma, X)$ is a $\mathbb{C}$-valued random variable with a continuous distribution, and Bohr-Jessen's result [BJ30] is essentially that $\{\log \zeta(\sigma + it)\}_{t \in [T, 2T]} \to \log \zeta(\sigma, X)$ as $T \to \infty$.

The Discrepancy Between $\log \zeta(\sigma + it)$ and $\log \zeta(\sigma, X)$

The limit $\{\log \zeta(\sigma + it)\}_{t \in [T, 2T]} \to \log \zeta(\sigma, X)$ as $T \to \infty$ naturally leads one to the question of how large the discrepancy between the distributions of true $\log \zeta$ and the the random model get for a fixed T.

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Definition (Discrepancy)

Let $\sigma > 1/2$ be fixed, and T be large. Then,

$$\mathcal{D}_\sigma(T) = \sup_{\mathcal{R}} \left| \mathbb{P}_T \left(\log \zeta(\sigma + it) \in \mathcal{R} \right) - \mathbb{P} \left(\log \zeta(\sigma, X) \in \mathcal{R} \right) \right|$$

where the supremum runs over all axis-parallel rectangles $\mathcal{R} \subseteq \mathbb{C}$.

The Bohr-Jessen result is $D_\sigma(T) = o(1)$.

Lamzouri, Lester and Radziwiłł prove the following bound in [LLR19]:

Theorem

Let $1/2 < \sigma < 1$ *be fixed. Then*

$$D_\sigma(T) \ll_\sigma \frac{1}{(\log T)^\sigma}.$$

This improves on an earlier bound by Harman and Matsumoto.

The Characteristic Function of a Random Variable

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If $F(u) = \mathbb{P}(\xi \leq u)$ is the distibution function of ξ on $\mathbb{R}$, then clearly,

$$\Phi_\xi(x) = \int_{-\infty}^{\infty} e^{ixu}\, dF(u)$$

and so Φ_ξ is just the Fourier transform of the measure dF.

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When working with complex random variables ξ, the domain is extended to $z \in \mathbb{C}$, and the definition is changed to

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Taking $z = u + iv$, and thinking of Φ_ξ as a function of two real variables, this is the same as saying

$$\Phi_\xi(u, v) = \mathbb{E}\left(e^{i(u \operatorname{Re} \xi + v \operatorname{Im} \xi)}\right)$$

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Definition (Characteristic Function of $\log \zeta(\sigma, X)$)

Let $\sigma > 1/2$ be fixed. Then, we define

$$\Phi^r_\sigma(u, v) = \mathbb{E}\left(\exp\left(iu \operatorname{Re}\log\zeta(\sigma, X) + iv \operatorname{Im}\log\zeta(\sigma, X)\right)\right).$$

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Definition (Characteristic Function of $\log \zeta(\sigma + it)$)

Let $\sigma > 1/2$ and T large be fixed. Then, we define

$$\Phi_{\sigma,T}(u, v) = \mathbb{E}_T\left(\exp\left(iu \operatorname{Re}\log\zeta(\sigma + it) + iv \operatorname{Im}\log\zeta(\sigma + it)\right)\right)$$

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Let $\sigma > 1/2$ and T large be fixed. Then, we define

$$\begin{aligned}\Phi_{\sigma,T}(u,v) &= \mathbb{E}_T\left(\exp\left(iu \operatorname{Re}\log \zeta(\sigma+it) + iv \operatorname{Im}\log \zeta(\sigma+it)\right)\right)\\ &= \frac{1}{T}\int_T^{2T} \exp\left(iu \operatorname{Re}\log \zeta(\sigma+it) + iv \operatorname{Im}\log \zeta(\sigma+it)\right)\, dt.\end{aligned}$$

Motivation: Lévy's Convergence Theorem

The motivation for considering the characteristic function of $\log \zeta$ comes from the following theorem from probability:

Theorem (Lévy's Convergence Theorem)

Let X_n be a sequence of $\mathbb{R}^n$-valued random variables, and X be an $\mathbb{R}^n$-valued random variable, with corresponding characteristic functions Φ_n and Φ. Then,

$$X_n \to X \text{ in distribution } \iff \Phi_n \to \Phi \text{ pointwise.}$$

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$$X_n \to X \text{ in distribution} \iff \Phi_n \to \Phi \text{ pointwise.}$$

Hence, to find the distributional discrepancy in $\log \zeta$, one looks for pointwise estimates for the characteristic functions.

Approximating $\Phi_{\sigma,T}$ by Φ_σ^r

We have the following theorem from [LLR19] that tells us that these characteristic functions are not too far apart:

Theorem

Let $1/2 < \sigma < 1$ and $A \geq 1$ be fixed. There exists a constant $b = b(\sigma, A)$ such that for all $|u|, |v| \leq b(\log T)^\sigma$, we have

$$\Phi_{\sigma,T}(u,v) = \Phi_\sigma^r(u,v) + \mathcal{O}\left(\frac{1}{(\log T)^A}\right).$$

Approximating $\Phi_{\sigma,T}$ by Φ_σ^r: High Level Proof Idea

Proof Idea.

Let $Y \geq 0$ be a real number, and define the Dirichlet polynomial $R_Y(\sigma + it)$ by

$$R_Y(\sigma + it) = \sum_{n \leq Y} \frac{\Lambda(n)}{n^{\sigma+it} \log n} = \sum_{p^k \leq Y} \frac{1}{kp^{k(\sigma+it)}}$$

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Correspondingly, we define the random Dirichlet polynomial by

$$R_Y(\sigma, X) = \sum_{n \leq Y} \frac{\Lambda(n)X(n)}{n^\sigma \log n} = \sum_{p^k \leq Y} \frac{X(p)^k}{kp^{k\sigma}}$$

□

Approximating $\Phi_{\sigma,T}$ by Φ_σ^r: High Level Proof Idea

Proof Idea.

Then, one can show that for $u, v \ll_{\sigma,A} (\log T)^\sigma$,

$$\Phi_{\sigma,T}(u,v) = \mathbb{E}_T\left(\exp\left(iu \operatorname{Re}\log\zeta(\sigma+it) + iv \operatorname{Im}\log\zeta(\sigma+it)\right)\right)$$

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$$\begin{aligned}\Phi_{\sigma,T}(u,v) &= \mathbb{E}_T\left(\exp\left(iu \operatorname{Re}\log\zeta(\sigma+it) + iv\operatorname{Im}\log\zeta(\sigma+it)\right)\right)\\ &\approx \mathbb{E}_T\left(\exp\left(iu\operatorname{Re} R_Y(\sigma+it) + iv\operatorname{Im} R_Y(\sigma+it)\right)\right)\end{aligned}$$

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&= \Phi_\sigma^r(u,v).
\end{aligned}$$

Here $\approx$ means up to an acceptable error.

□

$\log \zeta(\sigma + it) \approx R_Y(\sigma + it)$

Lemma

Assume RH. Let $1/2 < \sigma \leq 1$ *be fixed and* $1 \ll Y \ll T$*. For* $t \in [T, 2T]$*, we have*

$$\begin{aligned}\log \zeta(\sigma + it) &= R_Y(\sigma + it) + \mathcal{O}(Y^{-(\sigma - 1/2)/2} \log^3 T) \\ &= \sum_{n \leq Y} \frac{1}{kp^{k(\sigma + it)}} + \mathcal{O}(Y^{-(\sigma - 1/2)/2} \log^3 T).\end{aligned}$$

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The above equality will hold as long as one stays away from any potential zeroes of $\zeta(s)$. For the application to characteristic functions, one can use a zero-density estimate to remove the need for RH.

$\log \zeta(\sigma + it) \approx R_Y(\sigma + it)$

Proof Sketch.

By Perron's formula, one has for $c = 1 - \sigma + \frac{1}{\log Y}$,

$$\frac{1}{2\pi i} \int_{c-iY}^{c+iY} \log \zeta(\sigma + it + w) \frac{Y^w}{w} \, dw =$$

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We can now pull the contour left until a vertical line $\operatorname{Re} w = \sigma'$ with $1/2 < \sigma' + \sigma < \sigma$. Because we are assuming no zeroes, the integrand is regular except when $w = 0$, which gives $\log \zeta(\sigma + it)$.

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Thus, using bounds for $\log \zeta$ far away from zeroes, the contribution of the horizontal segments and the new vertical segment can be bounded, giving the desired error.

$\square$

$R_Y(\sigma + it) \approx R_Y(\sigma, X)$ on the Fourier side

Lemma

Let $\frac{1}{2} < \sigma < 1$ *and* $A \geq 1$ *be fixed. Let* $Y = (\log T)^A$. *There exists a constants* $b = b(\sigma, A) > 0$ *such that for all complex numbers* z_1, z_2 *with* $|z_1|, |z_2| \ll_{\sigma,A} (\log T)^\sigma$ *we have*

$$\frac{1}{T}\int_{\mathcal{A}(T)} \exp\left(z_1 R_Y(\sigma+it) + z_2\overline{R_Y(\sigma+it)}\right)dt$$

$$= \mathbb{E}\left(\exp\left(z_1 R_Y(\sigma, X) + z_2\overline{R_Y(\sigma, X)}\right)\right) + O\left(\exp\left(-b_6\frac{\log T}{\log\log T}\right)\right),$$

where

$$\mathcal{A}(T) = \{t \in [T, 2T] : |R_Y(\sigma+it)| \leq (\log T)^{1-\sigma}/\log\log T\}.$$

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Putting $z_1 = i(u - iv)/2$ and $z_2 = i(u + iv)/2$, we get (approximately) the characteristic function on both sides.

$R_Y(\sigma + it) \approx R_Y(\sigma, X)$ on the Fourier side

Proof Sketch.

For simplicitly, let's set $z_1 = z$ and $z_2 = 0$. By Taylor's theorem,

$$\exp t \approx \sum_{j \leq N} \frac{t^j}{j!}.$$

This approximation works well when $N \gg |t|$.

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Taking $t = zR_Y(\sigma + it)$, we find that t is a Dirichlet polynomial of length $Y = (\log T)^A$, and $t^j = [zR_Y(\sigma + it)]^j$ is of length Y^j.

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Thus, $\exp t$ is approximately a Dirichlet polynomial of length Y^N for some $N \gg |t|$.

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We can compute moments of "short" Dirichlet polynomials – say of length $T^{1/3}$. Hence, we can compute the moment of $\exp t$ provided $Y^N \le T^{1/3}$.

$R_Y(\sigma + it) \approx R_Y(\sigma, X)$ on the Fourier side

Proof Sketch.

Taking log, this translates to

$$N \log Y \leq \frac{\log T}{3},$$

and hence we need $N \ll_A \frac{\log T}{\log \log T}$ with a sufficiently small implicit constant.
Thus, if we take $|z| \ll_A (\log T)^\sigma$ (again with a sufficiently small constant), the constraint that

$$|R_Y(\sigma + it)| \leq \frac{(\log T)^{1-\sigma}}{\log \log T},$$

gives that

$$t = zR_Y(\sigma + it) \ll \frac{\log T}{\log \log T}.$$

$R_Y(\sigma + it) \approx R_Y(\sigma, X)$ on the Fourier side

Proof Sketch.

Thus, choosing the constants carefully, the problem reduces to showing that

$$\frac{1}{T}\int_{\mathcal{A}(T)} R_Y(\sigma+it)^j\overline{R_Y(\sigma+it)^\ell}\,dt \approx \mathbb{E}(R_Y(\sigma,X)^j\overline{R_Y(\sigma,X)^\ell}),$$

for $j+\ell \leq N \ll_A \frac{\log T}{\log\log T}$. At this scale, n^{it} is a good harmonic oscillator, and so this can be done. □

Thank You!