Moments of the Hurwitz zeta function on the critical line

Anurag Sahay

University of Rochester

anuragsahay@rochester.edu

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(partly joint with Winston Heap and Trevor Wooley)

Overview of the Talk

1. What is the Hurwitz zeta function?
2. Moments of the Hurwitz zeta function for rational shifts
3. Moments of the Hurwitz zeta function for irrational shifts

The zeta functions of Hurwitz and Riemann

Let $s = \sigma + it \in \mathbb{C}$, and $0 < \alpha \leqslant 1$. Then, for $\sigma > 1$, the Hurwitz zeta function is defined by

$$\zeta(s, \alpha) = \sum_{n \geqslant 0} \frac{1}{(n+\alpha)^s},$$

for $\sigma > 1$. This is the shifted integer analogue for the (usual) zeta function of Riemann, $\zeta(s) = \zeta(s, 1)$, given by

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s},$$

for $\sigma > 1$.

Similarities between $\zeta(s)$ and $\zeta(s, \alpha)$

There are many similarities between $\zeta(s)$ and $\zeta(s, \alpha)$:

- They both converge absolutely in $\sigma > 1$, and uniformly on $\sigma \geqslant \sigma_0 > 1$, thereby defining a holomorphic function on $\sigma > 1$.
- They both extend to meromorphic functions on $\mathbb{C}$ with a simple pole at $s = 1$, with residue 1.
- They both have "trivial" zeros on the negative real line, but are zero-free in the region $\sigma \geqslant 1 + \alpha$ (Spira, 1976).
- They both satisfy a "functional equation".

The functional equation of $\zeta(s, \alpha)$

Let $P(s, \alpha)$ be the analytic continuation of

$$\sum_{n \geqslant 1} \frac{e(n\alpha)}{n^s}.$$

Then,

$$\zeta(1 - s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\pi i s/2} P(s, \alpha) + e^{\pi i s/2} P(s, -\alpha) \right).$$

Putting $\alpha = 1$, we recover Riemann's functional equation,

$$\zeta(1 - s) = \frac{\Gamma(s)}{2^{s-1}\pi^s} \cos(\tfrac{\pi s}{2})\zeta(s).$$

These can both be viewed as manifestations of the Poisson summation formula.

Differences between $\zeta(s)$ and $\zeta(s, \alpha)$

We have that $\zeta(s, 1) = \zeta(s)$ and $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$. Other than these cases, there are considerable differences. For the following, we assume that the shift parameter $\alpha \neq 1, \frac{1}{2}$.

- The Riemann zeta function has an Euler product,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

  for $\sigma > 1$. The Hurwitz zeta function $\zeta(s, \alpha)$ does not.
- For any $\delta > 0$, $\zeta(s, \alpha)$ has infinitely many zeroes in the strip $1 < \sigma < 1 + \delta$. In particular, the strip $1 < \sigma < 1 + \alpha$ is not zero-free! (Davenport–Heilbronn, 1936 for rational and transcendental shifts; Cassels, 1961 for algebraic irrational shifts).
- The same is true in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ for rational shifts (Voronin, 1976) and transcendental shifts (Gonek, 1979). This is open for algebraic irrationals!

Moments of $\zeta(s)$

For $k > 0$,

$$M_k(T) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} \, dt,$$

are called the moments of $\zeta(s)$. Estimates for $M_k(T)$ are useful in several problems in analytic number theory; in particular,

$$\zeta(\tfrac{1}{2} + it) \ll_\epsilon |t|^\epsilon \iff M_k(T) \ll_{k,\epsilon} T^{1+\epsilon},$$

where the left hand side here is the Lindelöf hypothesis.

What is known about $M_k(T)$?

It is a folklore conjecture that

$$M_k(T) \sim c_k T(\log T)^{k^2}$$

for some $c_k > 0$.

- (Hardy–Littlewood, 1916) proved this for $k = 1$ with $c_1 = 1$.
- (Ingham, 1926) proved this for $k = 2$ with $c_2 = \frac{1}{2\pi^2}$.
- (Conrey–Ghosh, 1998) gave a conjecture for c_3 using a number theoretic approach.
- (Conrey–Gonek, 2001) gave a conjecture for c_4 using a different number theoretic approach.
- (Keating–Snaith, 2000) gave a conjecure for c_k for every $k > 0$ by the analogy with random matrix theory.

Moments of $\zeta(s, \alpha)$

In analogy, we define

$$M_k(T; \alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^{2k} \, dt.$$

One might expect that

$$M_k(T; \alpha) \sim c_k(\alpha) T (\log T)^{k^2}.$$

We will justify this expectation for rational α.

Things are more complicated for irrational α – if time permits, we will return to this later.

What is known about $M_k(T;\alpha)$?

The classical mean-square methods for $\zeta(s)$ apply also to $\zeta(s,\alpha)$. (Rane, 1980) showed that uniformly for all $0 < \alpha \leqslant 1$,

$$M_1(T) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^2 \, dt$$
$$= T \log T + B(\alpha) T - \frac{1}{\alpha} + O\left(\frac{T^{1/2} \log T}{\alpha^{1/2}} \right),$$

for an explicit constant $B(\alpha)$. The error term has been improved a few times; the best error is due to (Zhan, 1993).

The uniformity in α here is perhaps a coincidence – more on this later.

Moments of $\zeta(s, \frac{a}{q})$

We now specialize to rational α. We can clearly assume that $\alpha = a/q$ with $q \geqslant 3$, $1 \leqslant a < q$, and $(a, q) = 1$. The main focus of this talk is our following conjecture:

Conjecture (S., 2021+)

Let $k \geqslant 0$ and $\alpha = a/q$ be as above. Then,

$$\int_T^{2T} \left|\zeta\left(\tfrac{1}{2} + it, \alpha\right)\right|^{2k} dt \sim c_k(\alpha) T (\log T)^{k^2},$$

as $T \to \infty$ where $c_k(\alpha)$ is given by

$$c_k(\alpha) = c_k \frac{q^k}{\varphi(q)^{2k-1}} \prod_{p|q} \left\{ \sum_{m=0}^{\infty} \binom{m+k-1}{k-1}^2 p^{-m} \right\}^{-1}.$$

Here $c_k = c_k(1)$ is the usual proportionality constant for moments of $\zeta(s)$.

Note that $c_k(\alpha)$ does not depend on a!

Reduction to mean-square of $\mathcal{L}^\ell(s)$

By orthogonality of Dirichlet characters, we have for $\alpha = a/q$ and $\sigma > 1$,

$$\begin{aligned}\zeta(s,\alpha) &= \sum_{n\geqslant 0} \frac{1}{(n+\alpha)^s} = \sum_{n\geqslant 0} \frac{q^s}{(qn+a)^s} \\ &= \sum_{\substack{m\geqslant a \\ m\equiv a \pmod q}} \frac{q^s}{m^s} \\ &= q^s \sum_{m\geqslant 1} \frac{1}{m^s} \left(\frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a)\chi(m) \right) \\ &= \frac{q^s}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) L(s,\chi).\end{aligned}$$

By analytic continuation, this holds everywhere in $\mathbb{C}$.

Reduction to mean-square of $\mathcal{L}^{\boldsymbol{\ell}}(s)$

Thus, by the multinomial theorem,

$$\begin{aligned}\zeta(s,\alpha)^k &= \left(\frac{q^s}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)L(s,\chi)\right)^k \\ &= \frac{q^{ks}}{\varphi(q)^k}\sum_{|\boldsymbol{\ell}|=k}\binom{k}{\boldsymbol{\ell}}\prod_{\chi}\left\{\overline{\chi}(a)L(s,\chi)\right\}^{\ell_\chi} \\ &= \frac{q^{ks}}{\varphi(q)^k}\sum_{|\boldsymbol{\ell}|=k}\binom{k}{\boldsymbol{\ell}}\mathcal{L}^{\boldsymbol{\ell}}(s)\left\{\prod_{\chi}\overline{\chi}(a)^{\ell_\chi}\right\},\end{aligned}$$

where $\binom{k}{\boldsymbol{\ell}} = \frac{k!}{\prod_\chi \ell_\chi!}$ are the multinomial coefficients.

Reduction to mean-square of $\mathcal{L}^{\boldsymbol{\ell}}(s)$

Using $|\zeta(s,\alpha)|^{2k} = \zeta(s,\alpha)^k \overline{\zeta(s,\alpha)}^k$,

$$|\zeta(s,\alpha)|^{2k} = \frac{q^{2k\sigma}}{\varphi(q)^{2k}} \sum_{\substack{|\boldsymbol{\ell}^{(1)}|=k \\ |\boldsymbol{\ell}^{(2)}|=k}} \binom{k}{\boldsymbol{\ell}^{(1)}} \binom{k}{\boldsymbol{\ell}^{(2)}} \mathfrak{s}(a;\boldsymbol{\ell}^{(1)},\boldsymbol{\ell}^{(2)}) \mathcal{L}^{\boldsymbol{\ell}^{(1)}}(s) \overline{\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(s)},$$

where $\mathfrak{s}(a;\boldsymbol{\ell}^{(1)},\boldsymbol{\ell}^{(2)})$ is a complex number of magnitude 1. If we now put $s = \frac{1}{2} + it$ and integrate over $t \in [T, 2T]$, we get $M_k(T;\alpha)$ on the left, while on the right we expect terms with $\boldsymbol{\ell}^{(1)} \neq \boldsymbol{\ell}^{(2)}$ to oscillate very fast. Further, note that $\mathfrak{s}(a;\boldsymbol{\ell},\boldsymbol{\ell}) = 1$.

Reduction to mean-square of $\mathcal{L}^{\ell}(s)$

This gives, heuristically,

$$M_k(T;\alpha) \approx \frac{q^k}{\varphi(q)^{2k}} \sum_{|\ell|=k} \binom{k}{\ell}^2 \int_T^{2T} \left| \mathcal{L}^{\ell}\left(\tfrac{1}{2}+it\right) \right|^2 dt.$$

whence the problem reduces to studying the mean-square of $\mathcal{L}^{\ell}(s)$.

Note that the right hand side does not depend on a, as predicted in our conjecture!

Previous results on products of L-functions

We highlight two previous works on products of L-functions:

- (Heap, 2021): among other things, he modifies the recipe of Conrey–Farmer–Keating–Rubinstein–Snaith to give a conjecture for moments of products of L-functions in the Selberg class satisfying Selberg's orthonormality conjecture.
- (Milinovich–Turnage-Butterbaugh, 2014): under the generalized Riemann hypothesis (GRH) for the relevant L-functions, they prove upper bounds of almost the right order of magnitude (up to $(\log T)^{\epsilon}$) for moments of products of automorphic L-functions.

Since Dirichlet L-functions fall in both these classes, their results apply also to $\mathcal{L}^{\ell}(s)$ (and, in fact, also to the Dedekind zeta functions $\zeta_K(s)$ of a Galois number field K).

The main theorem

Theorem (S., 2021+)

Under some reasonable conjectures[a] *we have that for any* ℓ,

$$\frac{1}{T}\int_T^{2T} |\mathcal{L}^{\boldsymbol{\ell}}(\tfrac{1}{2}+it)|^2\,dt \sim_{q,k} c_{\boldsymbol{\ell}}(q)\left\{\prod_{\chi}(\log q^*(\chi)T)^{\ell_\chi^2}\right\},$$

where $c_{\boldsymbol{\ell}}(q)$ *is given by*

$$\prod_p\left\{\left(1-\frac{1}{p}\right)^{\lambda}\sum_{m=0}^{\infty}\frac{|d_{\boldsymbol{\ell}}(p^m)|^2}{p^m}\right\}\prod_{\chi}\frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)}.$$

Here $\lambda=\sum_\chi \ell_\chi^2$, $G(\cdot)$ *is the Barnes* G*-function and* $q^*(\chi)$ *is the conductor of* $L(s,\chi)$.

[a] To be described; based on the approach of (Gonek–Hughes–Keating, 2007) instead of the CFKRS recipe.

Theorem $\implies$ Conjecture

Note that if $|\boldsymbol{\ell}| = k$, then $\lambda < k^2$ unless $\boldsymbol{\ell} = k\boldsymbol{\delta}^{\chi}$ for some character χ. Thus,

$$M_k(T;\alpha) \approx \frac{q^k}{\varphi(q)^{2k}} \sum_{|\boldsymbol{\ell}|=k} \binom{k}{\boldsymbol{\ell}}^2 \int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}\left(\tfrac{1}{2}+it\right)\right|^2 dt$$

$$\approx \frac{q^k}{\varphi(q)^{2k}} \sum_{\chi} \int_T^{2T} \left|L(\tfrac{1}{2}+it,\chi)\right|^{2k} dt.$$

Thus, our main conjecture follows from this theorem after some book-keeping.

Hybrid Euler-Hadamard product

The first step is proving a hybrid Euler-Hadamard product for $\mathcal{L}^{\ell}(s)$, a tool originally developed for $\zeta(s)$ by (Gonek–Hughes–Keating, 2007). Informally, it says that

$$\mathcal{L}^{\ell}(s) \approx \mathcal{P}_X^{\ell}(s)\mathcal{Z}_X^{\ell}(s),$$

where

$\mathcal{P}_X^{\ell}(s)$	approximate Euler product	Primes	$p \leqslant X$
$\mathcal{Z}_X^{\ell}(s)$	approximate Hadamard product	Zeroes	$\lvert\rho - t\rvert \leqslant \frac{1}{\log X}$

The splitting conjecture

One expects $\mathcal{P}_X^{\boldsymbol{\ell}}(s)$ and $\mathcal{Z}_X^{\boldsymbol{\ell}}(s)$ to behave like independent random variables for $X = o(T)$.

Conjecture (Splitting)

Let $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$. Then, for any tuple of nonnegative integers $\boldsymbol{\ell}$ indexed by characters modulo q, we have for $s = 1/2 + it$,

$$\frac{1}{T}\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}(s)\right|^2 dt \sim \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{P}_X^{\boldsymbol{\ell}}(s)\right|^2 dt\right) \times \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{Z}_X^{\boldsymbol{\ell}}(s)\right|^2 dt\right).$$

Mean-square of $\mathcal{P}_X^{\boldsymbol{\ell}}(s)$

Theorem (S., 2021+)

For integer $\ell_\chi \geqslant 0$ *such that* $|\boldsymbol{\ell}| = \sum_\chi \ell_\chi = k$*, further, suppose that* $2 \leqslant X \ll_\epsilon (\log T)^{2-\epsilon}$.

$$\frac{1}{T}\int_T^{2T} |\mathcal{P}_X^{\boldsymbol{\ell}}(\tfrac{1}{2}+it)|^2\,dt = b(\boldsymbol{\ell})F_X(\boldsymbol{\ell})\left(1 + \mathcal{O}_{q,k,\epsilon}\left(\frac{1}{\log X}\right)\right)$$

where $b(\boldsymbol{\ell})$ *is an explicit Euler product independent of* X*, and*

$$F_X(\boldsymbol{\ell}) = (e^\gamma \log X)^\lambda \prod_p \left(1 - \frac{1}{p}\right)^{\lambda - |d_{\boldsymbol{\ell}}(p)|^2}.$$

Here γ *is the Euler-Mascheroni constant,* $d_{\boldsymbol{\ell}}(n)$ *is the coefficient of* n^{-s} *in the Dirichlet series for* $\mathcal{L}^{\boldsymbol{\ell}}(s)$*, and* $\lambda = \sum_\chi \ell_\chi^2$.

Mean-square of $\mathcal{Z}_X^{\ell}(s)$

The random matrix theory analogy gives us the following conjecture

Conjecture

Suppose that $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$. Then, for $\boldsymbol{\ell}$ as before,

$$\frac{1}{T}\int_T^{2T} |\mathcal{Z}_X^{\boldsymbol{\ell}}(\tfrac{1}{2}+it)|^2\, dt \sim \prod_{\chi}\left[\frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)}\left(\frac{\log q^*(\chi)T}{e^\gamma \log X}\right)^{\ell_\chi^2}\right],$$

where $G(\cdot)$ is the Barnes G-function, and $q^*(\chi)$ is the conductor of χ.

Here $L(s,\chi)$ forms a unitary family in the t-aspect, and so we model each $Z(s,\chi)$ in $\mathcal{Z}_X^{\boldsymbol{\ell}}(s)$ by unitary matrices chosen independently and uniformly with respect to the Haar measure. The size of the unitary matrices are chosen appropriately so that it matches the approximate mean density of the zeroes.

Other evidence for the conjecture

The previous three slides form the basis of our conjecture. Here's some more evidence for our conjecture:

1. We show that $T(\log T)^{k^2} \ll_{k,\alpha} M_k(T;\alpha) \ll_{k,\alpha,\epsilon} T(\log T)^{k^2+\epsilon}$. The upper bound is conditional on GRH for all Dirichlet L-functions mod q, while the lower bound is unconditional.
2. We prove some small ($|\boldsymbol{\ell}| \leqslant 2$) cases of the splitting and random matrix theory conjectures using standard techniques.
3. We verify that our conjectural constants match up in all the cases where asymptotics are known.

Conjectures for irrational shifts

In ongoing work, we have made the following conjecture.

Conjecture (Heap–S., 2022++)

Let $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ be an irrational number. Then for algebraic α of degree $d \geqslant k$ and almost all transcendental α we have

$$M_k(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^{2k} dt \sim k!\, T(\log T)^k$$

as $T \to \infty$.

As mentioned earlier, note that $1^2 = 1$, and so the main term of $M_1(T;\alpha)$ is uniform in $0 < \alpha \leqslant 1$.

Note that this conjecture suggests Gaussian behaviour!

Pseudomoments of $\zeta(s,\alpha)$, $\alpha \notin \mathbb{Q}$

To gain some insight into this conjecture, we first consider the case that α is transcendental and consider the pseudomoments $M_k'(N;\alpha)$ defined by

$$\begin{aligned} M_k'(N;\alpha) &= \lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \left| \sum_{0\leqslant n\leqslant N} \frac{1}{(n+\alpha)^{\frac{1}{2}+it}} \right|^{2k} dt \\ &= \sum_{\substack{0\leqslant n_1,\cdots,n_k,m_1,\cdots,m_k\leqslant N \\ (n_1+\alpha)\cdots(n_k+\alpha)=(m_1+\alpha)\cdots(m_k+\alpha)}} \frac{1}{(n_1+\alpha)\cdots(n_k+\alpha)}. \end{aligned}$$

Pseudomoments for transcendental shift parameters

This leads us to investigating the solutions to

$$\prod_{j=1}^{k}(n_j+\alpha)=\prod_{j=1}^{k}(m_j+\alpha).$$

Since α is transcendental, this can only happen if $\{n_1,\cdots,n_k\}=\{m_1,\cdots,m_k\}$. Thus,

$$M_k'(N;\alpha)\sim k!\sum_{\substack{0\leqslant n_j\leqslant N\\1\leqslant j\leqslant k}}\frac{1}{(n_1+\alpha)\cdots(n_k+\alpha)}$$

$$=k!\Bigg(\sum_{0\leqslant n\leqslant N}\frac{1}{n+\alpha}\Bigg)^k\sim k!(\log N)^k.$$

Pseudomoments for algebraic irrational shift parameters

Now, suppose that α is algebraic of degree $d \geqslant 2$. If $k \leqslant d$, then the argument for transcendental α goes through to give

$$M_k'(N; \alpha) \sim k!(\log N)^k.$$

For $k > d$, however, we can now have solutions to

$$\prod_{j=1}^{k}(n_j + \alpha) = \prod_{j=1}^{k}(m_j + \alpha).$$

with $\{n_1, \cdots, n_k\} \neq \{m_1, \cdots, m_k\}$.

We thus need to understand a fairly complicated problem of a logarithmic weight count of integer points in a variety. It is still unclear how these weighted counts behave for large k, but something can be said if we drop the logarithmic weights $1/(n_j + \alpha)$.

The unweighted integer-point counting problem

Theorem (Heap–S.–Wooley, 2022; independently Bourgain–Garaev–Konyagin–Shparlinski, 2014)

Let $k \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $\alpha \in \mathbb{C}$ is algebraic of degree d over $\mathbb{Q}$ where $k > d$. Then, one has

$$\sum_{\nu \in \mathbb{Z}[\alpha]} \tau_k(\nu; X, \alpha)^2 = \sum_{\substack{1 \leqslant n_1, \cdots, n_k, m_1, \cdots, m_k \leqslant N \\ (n_1+\alpha)\cdots(n_k+\alpha) = (m_1+\alpha)\cdots(m_k+\alpha)}} 1$$

$$= T_k(X) + O_{k,\alpha,\epsilon}(X^{k-d+1+\epsilon}).$$

Here $T_k(X) = k! X^k + O_k(X^{k-1})$ is the number of pairs $(\boldsymbol{n}, \boldsymbol{m})$ with $1 \leqslant n_j, m_j \leqslant X$, $1 \leqslant j \leqslant k$ and $\boldsymbol{n}$ is a permutation of $\boldsymbol{m}$.
The error term here may be omitted if, instead, $k \leqslant d$, or if α is transcendental.

The fourth moment of $\zeta(s, \alpha)$ for irrational α

Theorem (Heap–S, 2022++)

Let $0 < \alpha < 1$ be an irrational number. Then, under certain Diophantine conditions[a]*, we have that*

$$M_2(T; \alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^4 dt \sim 2T(\log T)^2,$$

as $T \to \infty$.

[a] WIP

In particular, our Diophantine conditions appear to be satisfied by almost all α, so this verifies the previous conjecture for $k = 2$.

Why do we need a Diophantine criterion?

After applying an approximate functional equation, and focusing just on the first piece, which terms contribute depend on our control over the harmonics

$$\int_T^{2T} \left[\frac{(n_1+\alpha)(n_2+\alpha)}{(n_3+\alpha)(n_4+\alpha)}\right]^{it} dt.$$

It's hard to rule out a main contribution arising from an off-diagonal term $\{n_1, n_2\} \neq \{n_3, n_4\}$ with

$$(n_1+\alpha)(n_2+\alpha) \approx (n_3+\alpha)(n_4+\alpha),$$

or, in other words, from terms with

$$\alpha \approx \frac{n_1 n_2 - n_3 n_4}{n_1 + n_2 - n_3 - n_4}.$$

Our Diophantine assumptions let us show that this does not happen frequently enough to give a main term.

Thank You!