The value distribution of the Hurwitz zeta function with irrational shifts

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(joint with Winston Heap)

Value distribution of $\zeta(s)$

Let $s = \sigma + it \in \mathbb{C}$. Questions of value distribution of $\zeta(s)$ naturally resolve into three regimes:

- $\sigma > 1$: here the Dirichlet series is absolutely convergent, making things straightforward.
- $1/2 < \sigma < 1$: here one is far away from the zeroes; zero-density estimates let us ignore the contribution from zeroes, and the value distribution is essentially determined by the primes.
- $\sigma = 1/2$: here things are most complicated, and the contributions from zeroes may require care. This includes questions about moments, pseudomoments, and Selberg's central limit theorem.

Bohr–Jessen theory I: a random model for $\zeta(s)$

Let X be a Steinhaus random multiplicative function. That is, $\{X(p)\}_p$ is a family of independent random variables that are each uniformly distributed on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, with

$$X(n) = \prod_{p^\alpha \| n} X(p)^\alpha.$$

Then, the random variable

$$\zeta(\sigma, X) = \prod_p \left(1 - \frac{X(p)}{p^\sigma}\right)^{-1}$$

is well defined (converges almost surely) for $\sigma > 1/2$ and can be viewed as a random model for $\zeta(s)$.

The main result of Bohr–Jessen theory can be stated as

Theorem (Bohr–Jessen)

Viewing $t \in [T, 2T] \mapsto \log \zeta(\sigma + it)$ as a $\mathbb{C}$-valued random variable, we have that for fixed $1/2 < \sigma < 1$,

$$\{\log \zeta(\sigma + it)\}_{t \in [T,2T]} \xrightarrow{\textit{in distribution}} \log \zeta(\sigma, X)$$

as $T \to \infty$. Here the random variable $\zeta(\sigma, X)$ is absolutely continuous.

Key idea: Kronecker–Weyl

The key fact that lets us replace n^{-it} by $X(n)$ for the purposes of value distribution is the Kronecker-Weyl theorem from harmonic analysis:

Theorem (Kronecker–Weyl)

Let $\theta_1, \cdots, \theta_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then the set

$$\{(e(\theta_1 t), \cdots, e(\theta_n t) : t \in \mathbb{R}\}$$

is equidistributed on $\mathbb{T}^n$, where $e(\cdot) = e^{2\pi i(\cdot)}$ as usual.

In particular, due to the fundamental theorem of arithmetic, $\{-\log p : p \text{ prime}\}$ is linearly independent over $\mathbb{Q}$, giving that p^{-it} all behave like i.i.d. uniform random variables on $\mathbb{T}$. The same principle also underlies the proof of Voronin's universality theorem.

Moments of $\zeta(s)$

For $\sigma = 1/2$, the key quantity of concern is the moments of $\zeta(s)$. For $k > 0$, these are defined by

$$M_k(T) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} \, dt.$$

It is a folklore conjecture that

$$M_k(T) \sim c_k T (\log T)^{k^2}$$

for some $c_k > 0$.

- (Hardy–Littlewood, 1916) proved this for $k = 1$ with $c_1 = 1$.
- (Ingham, 1926) proved this for $k = 2$ with $c_2 = \frac{1}{2\pi^2}$.
- Much more is known due to work by several authors – however, these are the only asymptotic formulae known unconditionally.

Pseudomoments for $\zeta(s)$

The "pseudomoments" of $\zeta(s)$, introduced by Conrey and Gamburd is defined as follows:

$$M'_k(N) = \lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \left| \sum_{n\leqslant N} \frac{1}{n^{\frac{1}{2}+it}} \right|^{2k} dt.$$

They showed that for $k \in \mathbb{N}$, $M_k(N) \sim c'_k (\log N)^{k^2}$, where c'_k is a constant with the same arithmetic part as c_k, but a different geometric part.

Generalizations of this to $\zeta(s)^m$ have been studied by Bondarenko–Heap–Seip, Bondarenko–Brevig–Saksman–Seip–Zhao, Heap, Brevig–Heap, Brevig–Heap, Gerspach, Gerspach–Lamzouri, with various results for different regimes of $k, m > 0$.

Pseudomoments and random multiplicative functions

Pseudomoments of $\zeta(s)^m$ can be related to moments of random multiplicative functions. For $k \in \mathbb{N}$, this follows from the orthogonality identities:

$$\mathbb{E}[X(n_1)\cdots X(n_k)\overline{X(m_1)}\cdots\overline{X(m_k)}] = \mathbb{1}(n_1\cdots n_k = m_1\cdots m_k)$$

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T}\left(\frac{n_1\cdots n_k}{m_1\cdots m_k}\right)^{-it}dt = \mathbb{1}(n_1\cdots n_k = m_1\cdots m_k)$$

In particular, this gives that

$$M_k'(N) = \mathbb{E}\left[\left|\sum_{n\leqslant N}\frac{X(n)}{n^{1/2}}\right|^{2k}\right].$$

The Hurwitz zeta function

Let $0 < \alpha \leqslant 1$. Then, for $\sigma > 1$, the Hurwitz zeta function is defined by

$$\zeta(s,\alpha) = \sum_{n \geqslant 0} \frac{1}{(n+\alpha)^s},$$

This is a shifted integer analogue for the (usual) zeta function of Riemann, which is given by $\zeta(s) = \zeta(s,1)$.

Trichotomy for the Hurwitz zeta function

There is usually a trichotomy for $\zeta(s, \alpha)$, with techniques of different flavor and results of differing strength depending on the nature of α:

- Rational: one usually reduces things to a problem on Dirichlet L-functions.
- Transcendental: one can use ideas like Kronecker-Weyl to prove things directly.
- Algebraic irrational: in this case, which is typically the hardest, one needs some input from algebraic number theory.

Similarities between $\zeta(s)$ and $\zeta(s,\alpha)$

There are many similarities between $\zeta(s)$ and $\zeta(s,\alpha)$:

- They both converge absolutely in $\sigma > 1$, and uniformly on $\sigma \geqslant \sigma_0 > 1$, thereby defining a holomorphic function on $\sigma > 1$.
- They both extend to meromorphic functions on $\mathbb{C}$ with a simple pole at $s = 1$, with residue 1.
- They both have "trivial" zeros on the negative real line, but are zero-free in the region $\sigma \geqslant 1 + \alpha$.
- They both satisfy a "functional equation".

Differences between $\zeta(s)$ and $\zeta(s, \alpha)$

We have that $\zeta(s, 1) = \zeta(s)$ and $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$. Other than these cases, there are considerable differences. For the following, we assume that the shift parameter $\alpha \neq 1, \frac{1}{2}$.

- The Riemann zeta function has an Euler product,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

  for $\sigma > 1$. The Hurwitz zeta function $\zeta(s, \alpha)$ does not.
- For any $\delta > 0$, $\zeta(s, \alpha)$ has infinitely many zeroes in the strip $1 < \sigma < 1 + \delta$. In particular, the strip $1 < \sigma < 1 + \alpha$ is not zero-free! (Davenport–Heilbronn, 1936 for rational and transcendental shifts; Cassels, 1961 for algebraic irrational shifts).
- The same is true in the strip $1/2 < \sigma_1 < \sigma < \sigma_2 < 1$ for rational shifts (Voronin, 1976) and transcendental shifts (Gonek, 1979). This is open for algebraic irrationals!

Rational case: reduction to Dirichlet L-functions

By orthogonality of Dirichlet characters, we have for $\alpha = a/q$ and $\sigma > 1$,

$$\begin{aligned}\zeta(s,\alpha) &= \sum_{n\geqslant 0}\frac{1}{(n+\alpha)^s} = \sum_{n\geqslant 0}\frac{q^s}{(qn+a)^s} \\ &= \sum_{\substack{m\geqslant a \\ m\equiv a \pmod q}} \frac{q^s}{m^s} \\ &= \frac{q^s}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)L(s,\chi).\end{aligned}$$

By analytic continuation, this holds everywhere in $\mathbb{C}$. This reduces almost any problem about $\zeta(s,\alpha)$ into a question about linear combinations of Dirichlet L-functions instead.

e.g. Moments of $\zeta(s, \frac{a}{q})$

For example, I made the following conjecture:

Conjecture (S., 2022)

Let $k \in \mathbb{N}$ and $\alpha = a/q$ with $q \geqslant 3$ and $(a, q) = 1$. Then,

$$\int_T^{2T} \left|\zeta\left(\tfrac{1}{2} + it, \alpha\right)\right|^{2k} dt \sim c_k(\alpha) T (\log T)^{k^2},$$

as $T \to \infty$ where $c_k(\alpha)$ is given by

$$c_k(\alpha) = c_k \frac{q^k}{\varphi(q)^{2k-1}} \prod_{p|q} \left\{ \sum_{m=0}^{\infty} \binom{m+k-1}{k-1}^2 p^{-m} \right\}^{-1}.$$

Here $c_k = c_k(1)$ is the usual proportionality constant for moments of $\zeta(s)$.

This was based on heuristics modelling Dirichlet L-functions by independent random matrices.

Bohr–Jessen theory III: a random model for $\zeta(s, \alpha)$

For most of the rest of the talk, we will assume that $\alpha \notin \mathbb{Q}$. If α is transcendental, then

$$\{-\log(n+\alpha) : n \in \mathbb{N}\}$$

is linearly independent over $\mathbb{Q}$, as a non-trivial dependence would give a non-trivial algebraic relation for α. This suggests that $(n+\alpha)^{-it}$ can be replaced by uniform i.i.d. Steinhaus variables $X(n)$ [which are *not* multiplicative in n]. This suggests the random model

$$\zeta(\sigma, \alpha, X) = \sum_{n \geqslant 0} \frac{X(n)}{(n+\alpha)^{\sigma}}$$

for the Hurwitz zeta function with transcendental shifts. A similar model works for algebraic irrational shifts, but now there may be non-trivial dependencies between the $X(n)$ [more on this later].

Bohr–Jessen theory IV: limit theorems

The following theorem is a corollary of a general result of Masahiro Mine:

Theorem (Mine, 2020)

Let α be transcendental. Viewing $t \in [T, 2T] \mapsto \zeta(\sigma + it, \alpha)$ as a $\mathbb{C}$-valued random variable, we have that for fixed $1/2 < \sigma < 1$,

$$\{\zeta(\sigma + it, \alpha)\}_{t\in[T,2T]} \xrightarrow{\text{in distribution}} \zeta(\sigma, \alpha, X)$$

as $T \to \infty$. Here the random variable $\zeta(\sigma, \alpha, X)$ is absolutely continuous.

Note the lack of logarithms unlike the classical Bohr–Jessen type result!

A version of this result is known also when α is algebraic irrational due to Garunkštis–Laurinčikas, but in that case it is not known that the limit distribution is absolutely continuous. The Lithuanian school and the Japanese school have studied variants of this type of phenomenon for irrational shifts (e.g. Garunkštis, Laurinčikas, Matsumoto, Mine...).

Pseudomoments for $\zeta(s, \alpha)$

We now consider the pseudomoments for $\zeta(s, \alpha)$, defined by

$$M'_k(N; \alpha) = \lim_{T \to \infty} \frac{1}{T} \int_T^{2T} \left| \sum_{0 \leqslant n \leqslant N} \frac{1}{(n+\alpha)^{\frac{1}{2}+it}} \right|^{2k} dt$$

$$= \sum_{\substack{0 \leqslant n_1, \cdots, n_k, m_1, \cdots, m_k \leqslant N \\ (n_1+\alpha)\cdots(n_k+\alpha) = (m_1+\alpha)\cdots(m_k+\alpha)}} \frac{1}{(n_1+\alpha)\cdots(n_k+\alpha)}.$$

As with pseudomoments of $\zeta(s)$, one can relate this to moments of Steinhaus (non-multiplicative) random functions X; when α is transcendental, the values $\{X(n)\}_{n \geqslant 0}$ are i.i.d., while when α is algebraic irrational there might be non-trivial dependencies.

Pseudomoments for $\zeta(s, \alpha)$ with transcendental shifts

This leads us to investigating the solutions to

$$\prod_{j=1}^{k}(n_j + \alpha) = \prod_{j=1}^{k}(m_j + \alpha).$$

When α is transcendental, this can only happen if $\{n_1, \cdots, n_k\} = \{m_1, \cdots, m_k\}$. Thus,

$$M_k'(N; \alpha) \sim k! \sum_{\substack{0 \leqslant n_j \leqslant N \\ 1 \leqslant j \leqslant k}} \frac{1}{(n_1 + \alpha) \cdots (n_k + \alpha)}$$

$$= k! \Bigg(\sum_{0 \leqslant n \leqslant N} \frac{1}{n + \alpha} \Bigg)^k \sim k! (\log N)^k.$$

Gaussian behaviour of $\zeta(s, \alpha)$ with transcendental shifts

The pseudomoments are the same as the moments of a complex Gaussian with mean 0 and variance $\sqrt{\log N}$. Another way to see this is by the fact that these pseudomoments are essentially the moments of

$$\sum_{0 \leqslant n \leqslant N} \frac{X(n)}{(n+\alpha)^{1/2}},$$

which are Gaussian as $N \to \infty$ due to the usual central limit theorem.

The difficulty of the algebraic irrational case

If α is algebraic and irrational, then it is no longer clear whether

$$\{-\log(n+\alpha) : n \in \mathbb{N}\}$$

is linearly independent over $\mathbb{Q}$. In fact, a question of Drungilas and Dubickas from 2003 essentially asks if there is even a single example of an algebraic irrational α for which this set is $\mathbb{Q}$-linearly independent.[1] This is still open, though it is known for several classes of α (e.g. quadratic irrationals) that this set is $\mathbb{Q}$-linearly *dependent*.

[1] I expect the answer is no.

The difficulty of the algebraic irrational case

Many results for the algebraic irrational case uses input from algebraic number theory to avoid this issue – for example, in Cassels' paper proving the vanishing of $\zeta(s, \alpha)$ for $\sigma > 1$, he uses algebraic number theory techniques to show that $> 50\%$ of the set

$$\{-\log(n + \alpha) : n \leqslant X\}$$

is linearly independent.

Even in the regime of absolute convergence, $\sigma > 1$, easy-to-ask questions about $\zeta(s, \alpha)$ can be hard for algebraic irrational shifts!

There is a nice (unpublished) article by Johan Andersson[2] which discusses some of these issues.

[2] On questions of Cassels and Drungilas-Dubickas, arXiv:1606.02524

Pseudomoments for $\zeta(s, \alpha)$ with algebraic irrational shifts

Recall that the pseudomoments for $\zeta(s, \alpha)$ satisfy

$$M'_k(N; \alpha) = \sum_{\substack{0 \leqslant n_1, \cdots, n_k, m_1, \cdots, m_k \leqslant N \\ (n_1+\alpha)\cdots(n_k+\alpha)=(m_1+\alpha)\cdots(m_k+\alpha)}} \frac{1}{(n_1+\alpha)\cdots(n_k+\alpha)},$$

and are thus connected to the Diophantine solutions of

$$\prod_{j=1}^{k}(n_j+\alpha) = \prod_{j=1}^{k}(m_j+\alpha).$$

If α is algebraic irrational with degree d, and $d > k$, then the argument from the transcendental case tells us that $\{n_1, \cdots, n_k\} = \{m_1, \cdots, m_k\}$. On the other hand, if $d \leqslant k$, we could potentially have many solutions with $\{n_1, \cdots, n_k\} \neq \{m_1, \cdots, m_k\}$.

Pseudomoments for $\zeta(s,\alpha)$ with algebraic irrational shifts

However, the following theorem tells us this cannot happen *too* frequently:

Theorem (Heap–S.–Wooley, 2022; independently Bourgain–Garaev–Konyagin–Shparlinski, 2014)

Let $k \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $\alpha \in \mathbb{C}$ is algebraic of degree d over $\mathbb{Q}$ where $k > d$. Then, one has

$$\sum_{\substack{1 \leqslant n_1, \cdots, n_k, m_1, \cdots, m_k \leqslant N \\ (n_1+\alpha)\cdots(n_k+\alpha) = (m_1+\alpha)\cdots(m_k+\alpha)}} 1 = T_k(N) + O_{k,\alpha,\epsilon}(N^{k-d+1+\epsilon}).$$

Here $T_k(N) = k! N^k + O_k(N^{k-1})$ is the number of pairs $(\boldsymbol{n}, \boldsymbol{m})$ with $1 \leqslant n_j, m_j \leqslant N$, $1 \leqslant j \leqslant k$ and $\boldsymbol{n}$ is a permutation of $\boldsymbol{m}$. The error term may be omitted if, instead, $k \leqslant d$, or if α is transcendental.

Unfortunately, it's not so clear what this implies for the pseudomoments given the weights $\prod_j 1/(n_j + \alpha)$.

Moments for irrational shifts

Motivated by the previous discussion, in ongoing work, we have made the following conjecture.

Conjecture (Heap–S., 2022+)

Let $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ be an irrational number. Then for algebraic α of degree $d \geqslant k$ and almost all transcendental α we have

$$M_k(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^{2k} dt \sim k!\, T(\log T)^k$$

as $T \to \infty$.

This conjecture suggests Gaussian behaviour for $\zeta(s, \alpha)$! (without taking logarithms, unlike Selberg's central limit theorem)

Diophantine approximation

The *irrationality exponent* of α, denoted by $\mu(\alpha)$ is defined as the supremum over all μ such that the inequality

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^{\mu}},$$

has infinitely many solutions in rational numbers a/q. Note that:

- If α is rational, $\mu(\alpha) = 1$. On the other hand, by Dirichlet's approximation theorem, $\mu(\alpha) \geqslant 2$ if α is irrational.
- By Roth's theorem, $\mu(\alpha) = 2$ if α is algebraic irrational.
- There are transcendental numbers with arbitrarily large $\mu(\alpha)$. In particular, the Liouville numbers have $\mu(\alpha) = \infty$.
- It is known that the set of α with $\mu(\alpha) > 2$ has null Lesbegue measure, but full Hausdorff dimension.

The fourth moment of $\zeta(s, \alpha)$ for irrational α

We can now state our partial progress towards our conjecture.

Theorem (Heap–S, 2022+)

Let $0 < \alpha < 1$ *be an irrational number. Then, for* α *having* $\mu(\alpha) < 3$*, we have that*

$$M_2(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^4 dt \sim 2T(\log T)^2,$$

as $T \to \infty$*.*

In particular, $\mu(\alpha) < 3$ for all algebraic numbers and almost all transcendental numbers, verifying our conjecture for $k = 2$.

Why do we need a Diophantine criterion?

After applying an approximate functional equation, and focusing just on the first piece, which terms contribute depend on our control over the harmonics

$$\int_T^{2T} \left[\frac{(n_1+\alpha)(n_2+\alpha)}{(n_3+\alpha)(n_4+\alpha)}\right]^{-it} dt.$$

It's hard to rule out a contribution arising from an off-diagonal term $\{n_1, n_2\} \neq \{n_3, n_4\}$ with

$$(n_1+\alpha)(n_2+\alpha) \approx (n_3+\alpha)(n_4+\alpha),$$

or, in other words, from terms with

$$\alpha \approx \frac{n_1 n_2 - n_3 n_4}{n_1 + n_2 - n_3 - n_4}.$$

Our Diophantine assumptions let us show that this does not happen frequently enough to give a main term contribution.

Thank You!

Appendix: the functional equation of $\zeta(s, \alpha)$

Let $P(s, \alpha)$ be the analytic continuation of

$$\sum_{n \geqslant 1} \frac{e(n\alpha)}{n^s}.$$

Then,

$$\zeta(1-s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\pi i s/2} P(s, \alpha) + e^{\pi i s/2} P(s, -\alpha) \right).$$

Putting $\alpha = 1$, we recover Riemann's functional equation,

$$\zeta(1-s) = \frac{\Gamma(s)}{2^{s-1}\pi^s} \cos(\tfrac{\pi s}{2}) \zeta(s) = \chi(1-s)\zeta(s).$$

These can both be viewed as manifestations of the Poisson summation formula.

Appendix: main tools

The basic idea is to use an approximate functional equation to write

$$\zeta(s,\alpha) = S_1 + S_2 = \sum_{n \ll \sqrt{T}} \frac{1}{(n+\alpha)^s} + \chi(s) \sum_{n \ll \sqrt{T}} \frac{e(-n\alpha)}{n^s}.$$

Taking $|\cdot|^4$, we need to understand the integral over terms like $|S_1|^4$, $|S_2|^4$, $|S_1|^2|S_2|^2$, $S_1^2\overline{S_2}^2$, $S_1^2\overline{S_1 S_2}$, …

The main terms come from the diagonal contributions of the non-oscillating terms. The off-diagonal terms are bounded using classical bounds (Kruse, 1967) on sums of the form

$$\sum_{n \leqslant N} \frac{1}{\|n\alpha\|^{1/2}}.$$

Showing that the non-oscillating terms do not contribute a main term involves methods from the standard toolkit such as stationary phase.

Thank You (x2)!